An introduction to interval observers and some recent developments

Speaker

Thach Ngoc Dinh

November 07th, 2016
An introduction to interval observers and some recent developments
## Educational Background

<table>
<thead>
<tr>
<th>Year</th>
<th>Qualification</th>
<th>Institution</th>
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<tbody>
<tr>
<td>2006 – 2011</td>
<td>Diplôme d’ingénieur</td>
<td>INSA de Lyon</td>
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<tr>
<td></td>
<td>Electrical Engineering</td>
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<td>2010 – 2011</td>
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<td>Automated Systems Engineering</td>
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<td>Physics</td>
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## Past Research Experience

Young professional with 5 years of experience in Control Theory

<table>
<thead>
<tr>
<th>Time</th>
<th>Research Focus</th>
<th>Institution</th>
<th>Collaborators</th>
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<tr>
<td>03.2011-08.2011</td>
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<td>@ LAGEP</td>
<td>w/ V. Andrieu, M. Nadri, U. Serres</td>
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<td>Interval observer, positive observer &amp; robustness</td>
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# Past Research Experience

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*Advertisement: JSPS Fellowship*
## Editorial Activities

### Occasional reviewer of Journals
- Automatica
- European Journal of Control
- IEEE Transactions on Automatic Control
- IEEE Transactions on Industrial Electronics
- ISA Transactions

### Conferences
- The 26th Chinese Control and Decision Conference
- The 19th World Congress of the IFAC
- The 54th IEEE Conference on Decision and Control
- The 2016 American Control Conference
- The SICE Annual Conference 2016
- International Conference on System Theory, Control and Computing

*An introduction to interval observers and some recent developments*
Collaborators

1. Andrieu Vincent, Université de Lyon, France
2. Bonnabel Silvère, Mines ParisTech, France
3. Ito Hiroshi, Kyushu Institute of Technology, Japan
4. Malisoff Michael, Louisiana State University, USA
5. Mazenc Frédéric, INRIA Saclay, France
6. Nadri Madiha, Université de Lyon, France
7. Niculescu Silviu-Iulian, L2S CNRS-CentraleSupélec, France
8. Sepulchre Rodolphe, University of Cambridge, UK
9. Serres Ulysse, Université de Lyon, France

⋯ Hope to be able to extend my research network!

An introduction to interval observers and some recent developments
More seriously......

Plan

- What are interval observers?
  - Standard observer
  - Interval observer
    - Classical design
    - Recent developments

- On flexibility in designing iISS interval observers for nonlinear control systems
What are interval observers?
Standard observer: rough definition

A spider-man would like to know a group of people ... 

Know one, know all?

Rough definition of observer

"observer" ... an observation of what’s going inside dynamical systems based on limited set of possible measurements in our hand
Standard observer: motivation

• Studies of physical and biological phenomena
  ✓ Mathematical model

• Consider model (continuous-time version):
  \[
  \dot{x}(t) = f(t, x(t))
  \]

• Problem: from the model and information (sensor measure) \(y(t) = h(t, x(t))\), **how to know the all state** \(x(t)\) **of the system?**

• A solution: using an estimation algorithm in real-time (**an observer**)
Standard observer: mathematical view

This algorithm provides an estimate \( \hat{x} \) by integrating a dynamical system depending on \( y \) of the form:

\[
\dot{z}(t) = \theta(t, z(t), y(t)), \quad \hat{x}(t) = \psi(t, z(t), y(t))
\]

\( \Rightarrow \) problem is to find the functions \( \theta \) and \( \psi \) in order to ensure an asymptotic stability of \( \hat{x}(t) \) to \( x(t) \)
Interval observer: motivation from Robotics

In Drevelle and Bonnifait 2013 · · · localization problem

Knowledge of localization uncertainties is of prime importance when the navigation of intelligent vehicles such as ground unmanned vehicles, has to deal with safety issues.
Interval observer: motivation from Biology

Estimation for biological systems (Monod model, Droop model) such as in Bernard and Gouzé 2004, Goffaux et al. 2009

\[
\begin{align*}
\dot{x} &= f(x, u, p) + w(t) \\
y &= h(x, u, p) + \delta(t)
\end{align*}
\]

Large uncertainties ··· how to develop state estimators?
Interval observer: Gouzé et al. 2000

- An important application of positive dynamics ··· Interval observers
- All times estimation ··· ∀t
  cf. Standard observers = Asymptotic estimate ··· t → ∞
Interval observer: Gouzé et al. 2000

- An important application of positive dynamics \( \cdots \) Interval observers
- All times estimation \( \cdots \) \( \forall t \)
  
  cf. Standard observers = Asymptotic estimate \( \cdots t \to \infty \)

Estimation of the state \textit{at each instant} under assumptions of knowing bounds on uncertain terms and initial condition

![Diagram showing interval observers](image)
Interval observer: Gouzé et al. 2000

- An important application of positive dynamics \( \cdots \) Interval observers
- All times estimation \( \cdots \forall t \)
  
  cf. Standard observers = Asymptotic estimate \( \cdots t \to \infty \)

Estimation of the state *at each instant* under assumptions of knowing bounds on uncertain terms and initial condition

Frames + Their convergence as \( t \to \infty \) = An interval observer
Interval observer: Gouzé et al. 2000

- An important application of positive dynamics \( \cdots \) Interval observers
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Estimation of the state *at each instant* under assumptions of knowing bounds on uncertain terms and initial condition

Frames + Their convergence as \( t \to \infty \) = An interval observer

Gouzé, Bernard, Moisan, Raïssi, Mazenc, Niculescu, Fridman, Efimov, Zolghadri, Kieffer, Walter, Dinh......
## Interval observer: Advantages

<table>
<thead>
<tr>
<th>Standard observer</th>
<th>Interval observer</th>
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<tbody>
<tr>
<td>✓</td>
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<tr>
<td>Purpose of feedback control (e.g., stabilization, tracking)</td>
<td>✓</td>
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<tr>
<td>×</td>
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<tr>
<td>Purpose of monitoring</td>
<td>✓</td>
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<tr>
<td>×</td>
<td>✓</td>
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<tr>
<td>Estimation with guarantees at all time</td>
<td>✓</td>
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<tr>
<td>×</td>
<td>✓</td>
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<tr>
<td>Guarantees in the presences of uncertain terms</td>
<td>✓</td>
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</table>
Classical design: Prerequisites

Definition (Metzler matrix)
A square matrix $A$ is said to be Metzler if its off-diagonal entries are nonnegative

$$A = \begin{pmatrix} \star & + & + \\ + & \star & + \\ + & + & \star \end{pmatrix}$$

Theorem (Positive system)
A system $\dot{x} = Ax + b$ is positive (i.e. $x_0 \geq 0 \Rightarrow \forall t \geq 0, x(t, x_0) \geq 0$) iff $A$ is Metzler and $b \geq 0$. The operator $\geq$ is understood as a set of inequalities applied component by component.
Classical design

We consider:

\[ \dot{x} = Ax + w \]

\( w(t) \): unknown: \( w^-(t) \leq w(t) \leq w^+(t), \quad \forall t \geq 0. \) If \( A \in \mathbb{R}^{n \times n} \) is Metzler and Hurwitz, then:

\[
\begin{aligned}
\dot{z}^+ &= Az^+ + w^+, \quad z_0^+ = x_0^+ \\
\dot{z}^- &= Az^- + w^-, \quad z_0^- = x_0^- \\
x^+ &= z^+ \\
x^- &= z^-
\end{aligned}
\]

exponentially stable interval observer
Classical design: Proof

Proof:

• Positive systems · · · FP (framer property):

\[
\begin{align*}
\dot{z}^+ - \dot{x} &= A(z^+ - x) + \underbrace{w^+ - w}_{\text{Metzler} \geq 0} \\
\dot{x} - \dot{z}^- &= A(x - z^-) + \underbrace{w - w^-}_{\text{Metzler} \geq 0}
\end{align*}
\]

• When \( w^+ = w^- = 0 \), system \( z^+ - z^- = A(z^+ - z^-) \) GAS · · · SP (stability property)
Classical design: Discrete-time case

For a family of linear discrete-time system $x_{k+1} = Ax_k + w_k$ with $w_k^- \leq w_k \leq w_k^+$, $\forall k \in \mathbb{N}$, if $A$ is positive\(^1\) and Schur stable\(^2\), then an interval observer can be constructed.

\(^1\) $A = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix} \cdots$ FP (framer property)

\(^2\) Spectral radius $\rho(A) < 1 \cdots$ SP (stability property)
Recent developments: Discrete-time systems

1. Constructed interval observer for a family of nonlinear discrete-time systems which possess specific stability and monotonicity properties

\[ x_{k+1} = \mathcal{F}(y_k, x_k, w_k), \quad k \in \mathbb{N}, \]
\[ y_k = \mathcal{H}(x_k), \]

2. Constructed interval observer for a fundamental family of linear time-invariant discrete-time systems

\[ x_{k+1} = Ax_k + w_k, \quad k \in \mathbb{N}, \]

when A is ONLY \textbf{Schur stable}

Consider \( \alpha_{k+1} = -\frac{1}{2} \alpha_k \) \( \cdots \) Change of coordinates \( a_k = (-1)^k \alpha_k \) \( \cdots \)

\[ a_{k+1} = (-1)^{k+1} \alpha_{k+1} = (-1)^{k+1} \left( -\frac{1}{2} \right) \alpha_k = \]
\[ (-1)^{k+1} \left( -\frac{1}{2} \right) (-1)^k a_k = \frac{1}{2} a_k \]
Recent developments: Robust interval observer

3. Constructed interval observer for a family of nonlinear discrete-time systems with input and output

\[
\begin{align*}
    x_{k+1} &= \left[ A + A_d(x_k) \right] x_k + Bu_k + \Phi(y_k) + \nu_k, \\
    y_k &= Cx_k,
\end{align*}
\]

where \( A_d : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) is an unknown bounded nonlinear function, \( \Phi : \mathbb{R}^p \to \mathbb{R}^n \) is a known nonlinear function.

Existence of a time-invariant change of coordinates for \( A \), stabilizability assumption by state feedback for considered system, restriction imposed on the unknown terms.
Recent developments: Continuous-discrete case

4. Constructed interval observer for a family of nonlinear discrete-time systems with input and output

\[ \dot{x}(t) = Ax(t) + Bu(t) + w_1(t) \]
\[ y(t) = Cx(t_i) + w_{2,i}, \forall t \in [t_i, t_{i+1}) \]

with \( x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^q \) and \( t_i : t_0 = 0, t_{i+1} - t_i = \nu, \forall i \in \mathbb{N} \), where \( \nu > 0 \) is a constant

Continuous-discrete interval observer consists of:

- two classical continuous-discrete observers (Andrieu et Nadri 2010, Deza et al. 1992......) · · · (i) when no measurement is available, the state estimate is computed by integrating the model; (ii) when a measurement occurs, the observer makes an impulsive correction on the state estimate
- a framer

as subsystems

An introduction to interval observers and some recent developments
Recent developments: Nonlinear systems

5. Constructed interval observer for a family of nonlinear continuous-time systems affine in the unmeasured part of state variables

\[ \dot{x} = \alpha(y)x + \beta(y, u) + \delta(t), \]

with both

- Continuous-time outputs: \( y = Cx \)
  
  Achieved an interesting result: we can even guarantee robustness of convergence with respect to the nonzero disturbances

- Discrete-time outputs: \( y(t) = Cx(t_i) + \delta_2(t_i) \) for \( t \in [t_i, t_{i+1}) \)
On flexibility in designing iiSS interval observers for nonlinear control systems
Interval observers are designed not only for monitoring, but also for robust controlling systems.
Notation

- Operators $\leq$ must be understood componentwise.
- $Q \in \mathbb{R}^{n \times n}$, $Q^+ = \max\{Q, 0\}$, $Q^- = Q^+ - Q$
- $Q = (q_{ij}) \in \mathbb{R}^{n \times n}$ Metzler if $\forall i \neq j : q_{ij} \geq 0$
- $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is positive definite ($\alpha \in \mathcal{P}$) if it is continuous and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$, $\forall s \in (0, \infty)$.
- $\alpha \in \mathcal{K}$ if $\alpha \in \mathcal{P}$ is strictly increasing.
- $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ is unbounded.
- $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is $\beta \in \mathcal{KL}$ if it is continuous and for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s > 0$, $\beta(s, \cdot)$ is strictly decreasing, $\lim_{t \to \infty} \beta(s, t) = 0$. 

An introduction to interval observers and some recent developments
Considered system

\[ \dot{x}(t) = A(y(t))x(t) + B(y(t))u(y(t), \hat{x}^+(t)) + \delta(t) \]
\[ y(t) = Cx(t) \]

Assumptions:
- \( x^-_0 \leq x_0 \leq x^+_0 \)
- \( \delta_-(t) \leq \delta(t) \leq \delta^+(t), \ \forall t \in \mathbb{R}_{\geq 0} \)
- \( A : \mathbb{R}^p \to \mathbb{R}^{n \times n} \) and \( B : \mathbb{R}^p \to \mathbb{R}^{n \times q} \) are locally Lipschitz
- \( u(y(t), \hat{x}^+(t)) \in \mathbb{R}^q \) is output feedback which is locally Lipschitz
- \( \hat{x}^+ : \) An estimate of \( x \)

Divide the control input: \( u(y, \hat{x}^+) = K(y)y + u_a(y, \hat{x}^+) \)

Locally Lipschitz \( K \) can be arbitrary since \( K(y)y \) can be absorbed by locally Lipschitz \( u_a \)
Observer candidates
Full-order interval observer (Dinh et al. 2014)

\[
\dot{\hat{x}}^+ = (A(y) + B(y)K(y)C)\hat{x}^+ + B(y)u_a + \
H(y)[C\hat{x}^+ - y] + S[R^+\delta^+ - R^-\delta^-] \\
\dot{\hat{x}}^- = (A(y) + B(y)K(y)C)\hat{x}^- + B(y)u_a + \
H(y)[C\hat{x}^- - y] + S[R^+\delta^- - R^-\delta^+] \\
\]

Initial condition

\[
\hat{x}^+(0) = \hat{x}_0^+ := S[R^+x_0^+ - R^-x_0^-] \\
\hat{x}^-(0) = \hat{x}_0^- := S[R^+x_0^- - R^-x_0^+] \\
\]

Output equation

\[
x^+ = S^+R\hat{x}^+ - S^-R\hat{x}^-, \quad x^- = S^+R\hat{x}^- - S^-R\hat{x}^+, \\
\quad \text{where } S = R^{-1}. \\
\]

Design parameters \( R, K, H \)
Discussions on observer design: Freedom of $K$

For $K = 0$

- Sufficient conditions for achieving
  \[ x^-(t) \leq x(t) \leq x^+(t) \]
  and the nominal convergence
  \[ x(t), x^+(t), x^-(t) \to 0 \text{ as } t \to \infty \text{ for } \delta(t) \equiv 0 \]
  are given in Dinh et al. 2014

- The convergence by the observer is made robust to allow $\delta(t) \neq 0$
  in Ito and Dinh 2016
Discussions on observer design: Freedom of $K$

For $K = 0$

- Sufficient conditions for achieving
  \[ x^{-}(t) \leq x(t) \leq x^{+}(t) \]
  and the nominal convergence
  \[ x(t), x^{+}(t), x^{-}(t) \to 0 \text{ as } t \to \infty \text{ for } \delta(t) \equiv 0 \]
  are given in Dinh et al. 2014
- The convergence by the observer is made robust to allow $\delta(t) \neq 0$
  in Ito and Dinh 2016

Here, the first goal is to allow $K \neq 0$

- Letting $K \neq 0$ is not technically hard, but it is useful in practice
Design Guidelines: Selecting $K$ and $H$

**Assumption (A1):** Securing the framer properties

$$\Gamma(y) = R[A(y) + B(y)K(y)C + H(y)C]R^{-1}$$ is Metzler for each $y$

**Assumption (A2):** Securing convergence for nonzero disturbances

$$\exists V, \nu, \bar{\nu} \in K_\infty, \omega \in P, \eta^+, \eta^- \in K \text{ s.t. } \nu(|\xi|) \leq V(\xi) \leq \bar{\nu}(|\xi|) \text{ and }$$

$$\frac{\partial V}{\partial \xi}(\xi) \left\{ [A(y) + B(y)K(y)C + H(y)C]\xi + S[R^+ \rho^+ ight. 
 \left. + R^- \rho^-] \right\} \leq -\omega(|\xi|) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|)$$

hold $\forall \xi, y, \rho^+$ and $\rho^-$

(A2) $\rightarrow$ iISS w.r.t $\rho^+ := \delta^+ - \delta$ and $\rho^- := \delta - \delta^-$
Design Guidelines: Selecting feedback input $u$

Assumption (A3): Separation of feedback gain design

$\exists U, \mu \in \mathcal{P}$ and $\gamma, \zeta \in \mathcal{K}$ s.t.

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + B(Cx)u(Cx, x + d) + \delta]$$

$$\leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|)$$

holds $\forall x, d, \delta$

(A3) $\rightarrow$ the fictitious state feedback system is iISS with respect to estimation error $d = \hat{x}^* - x$ and disturbance $\delta$
Guarantees under (A1), (A2), (A3): Framer + GAS

Recall (A3) \( \frac{\partial U}{\partial x}(x) \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \).
Guarantees under (A1), (A2), (A3): Framer + GAS

Recall (A3) \( \frac{\partial U}{\partial x}(x)[... \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|). \)

Assume \( \delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0. \)
Guarantees under (A1), (A2), (A3): Framer + GAS

Recall (A3) $\frac{\partial U}{\partial x}(x)[...] \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|)$.

Assume $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

If $\mu \in \mathcal{K}$, then

\begin{align*}
x^-(t) & \leq x(t) \leq x^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \quad \cdots \text{FP (framer property)} \\
\lim_{t \to \infty} |x^+(t) - x^-(t)| &= 0
\end{align*}

hold,
Guarantees under \((A1), (A2), (A3)\): Framer + GAS

Recall \((A3)\) \(\frac{\partial U}{\partial x}(x)\) \(\leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|)\).

Assume \(\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0\).

If \(\mu \in \mathcal{K}\), then

\[
x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \quad \cdots \quad \text{FP (framer property)}
\]

\[
\lim_{t \to \infty} |x^+(t) - x^-(t)| = 0
\]

hold, and moreover, \[
\begin{bmatrix}
x
\hat{x}^+
\hat{x}^-
\end{bmatrix} = 0
\]

is globally asymptotically stable.
Guarantees under \((A1), (A2), (A3)\): Framer + iISS

Recall \((A3)\) \(\frac{\partial U}{\partial x}(x)[...]\leq -\mu(|x|)+\gamma(|d|)+\zeta(|\delta|)\)

\((A2)\) \(\frac{\partial V}{\partial \xi}(\xi)\{...\}\leq -\omega(|\xi|)+\eta^+ (|\rho^+|)+\eta^- (|\rho^-|)\)
Guarantees under (A1), (A2), (A3): Framer + iISS

Recall (A3) \( \frac{\partial U}{\partial x}(x)[... \leq -\mu(|x|)+\gamma(|d|) + \zeta(|\delta|) \)
(A2) \( \frac{\partial V}{\partial \xi}(\xi) \{... \leq -\omega(|\xi|)+\eta^+(|\rho^+|)+\eta^-(|\rho^-|) \)

Define \( \eta = \eta^+ + \eta^- \), \( \delta^\pm = \delta^+ - \delta^- \).
Guarantees under (A1), (A2), (A3): Framer + iISS

Recall (A3) \( \frac{\partial U}{\partial x}(x)[...] \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \)

(A2) \( \frac{\partial V}{\partial \xi}(\xi) \{\ldots\} \leq -\omega(|\xi|) + \eta^+ (|\rho^+|) + \eta^- (|\rho^-|) \)

Define \( \eta = \eta^+ + \eta^-, \delta^\pm = \delta^+ - \delta^- \). If

\[ \omega \in \mathcal{K}_\infty \text{ or } \left[ \omega \in \mathcal{K} \text{ and } \left\{ \gamma \notin \mathcal{K}_\infty \text{ or } \lim_{s \to \infty} \omega(s) > \sup_{t \in \mathbb{R} \geq 0} \eta(\sqrt{2}|\delta^\pm(t)|) \right\} \right], \]

An introduction to interval observers and some recent developments
Guarantees under (A1), (A2), (A3): Framer + iISS

Recall (A3) \( \frac{\partial U}{\partial x}(x)[\ldots] \leq -\mu(|x|)+\gamma(|d|) + \zeta(|\delta|) \)

(A2) \( \frac{\partial V}{\partial \xi}(\xi) \{\ldots\} \leq -\omega(|\xi|)+\eta^+ (|\rho^+|)+\eta^- (|\rho^-|) \)

Define \( \eta = \eta^+ + \eta^- \), \( \delta^\pm = \delta^+ - \delta^- \). If

\[ \omega \in \mathcal{K}_\infty \text{ or } \left\{ \begin{array}{l} \omega \in \mathcal{K} \text{ and } \left\{ \gamma \not\in \mathcal{K}_\infty \text{ or } \lim_{s \to \infty} \omega(s) > \sup_{t \in \mathbb{R} \geq 0} \eta(\sqrt{2}|\delta^\pm(t)|) \right\} \end{array} \right\}, \]

then FP and \( \exists \hat{\chi} \in \mathcal{K}_\infty, \hat{\theta} \in \mathcal{K} \mathcal{L}, \hat{\psi} \in \mathcal{K} \) s.t.

\[ \hat{\chi}(|\hat{z}(t)|) \leq \hat{\theta}(|\hat{z}(0)|, t)+\int_0^t \hat{\psi}(|\hat{\rho}(\tau)|)d\tau, \forall t \in \mathbb{R} \geq 0 \]

\[ \int_0^\infty \hat{\psi}(|\hat{\rho}(\tau)|)d\tau < \infty \Rightarrow \lim_{t \to \infty} |\hat{z}(t)| = 0 \]

where \( \hat{z} = \begin{bmatrix} \hat{x}^+ - x \\ x^+ - x^- \end{bmatrix} \), \( \hat{\rho} = \begin{bmatrix} \delta^+ - \delta \\ \delta^- - \delta \end{bmatrix} \), \( X = \begin{bmatrix} x \\ \hat{x}^+ \end{bmatrix} \), \( \Delta = \begin{bmatrix} \delta \\ \delta^+ \\ \delta^- \end{bmatrix} \)

hold, and moreover, closed-loop system is iISS w.r.t \( \Delta \) and \( X \).
Guarantees under (A1), (A2), (A3): Framer + iISS

Recall (A3) \[ \frac{\partial U}{\partial x}(x) \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \]

(A2) \[ \frac{\partial V}{\partial \xi}(\xi) \leq -\omega(|\xi|) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|) \]

Define \[ \eta = \eta^+ + \eta^- \], \[ \delta^\pm = \delta^+ - \delta^- \]. If

\[ \omega \in C_\infty \] or \[ \omega \in \mathcal{K} \text{ and } \left\{ \gamma \notin C_\infty \text{ or } \lim_{s \to \infty} \omega(s) > \sup_{t \in \mathbb{R}_\geq 0} \eta(\sqrt{2}||\delta^\pm(t)||) \right\} \],

then FP and \[ \exists \hat{\chi} \in C_\infty, \hat{\theta} \in \mathcal{K}_\mathcal{L}, \hat{\psi} \in \mathcal{K} \text{ s.t.} \]

\[ \hat{\chi}(|\hat{z}(t)|) \leq \hat{\theta}(|\hat{z}(0)|, t) + \int_0^t \hat{\psi}(|\hat{\rho}(\tau)||)d\tau, \forall t \in \mathbb{R}_\geq 0 \]

\[ \int_0^\infty \hat{\psi}(|\hat{\rho}(\tau)||)d\tau < \infty \Rightarrow \lim_{t \to \infty} |\hat{z}(t)| = 0 \]

Appropriate convergence of disturbances implies convergence of the estimate and the closed-loop state. hold, and moreover, closed-loop system is iISS w.r.t \( \Delta \) and \( X \).
Guarantees under (A1), (A2), (A3): Framer + ISS

Recall (A3) \( \frac{\partial U}{\partial x}(x)[...] \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \)

(A2) \( \frac{\partial V}{\partial \xi}(\xi) \{...\} \leq -\omega(\xi) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|) \)

An introduction to interval observers and some recent developments
Guarantees under (A1), (A2), (A3): Framer + ISS

Recall (A3) $\frac{\partial U}{\partial x}(x)[...]\leq -\mu(|x|)+\gamma(|d|)+\zeta(|\delta|)$

(A2) $\frac{\partial V}{\partial \xi}(\xi)\{...\}\leq -\omega(|\xi|)+\eta^+(|\rho^+|)+\eta^-(|\rho^-|)$

If $\mu, \omega \in \mathcal{K}_\infty$, then FP and $\exists \hat{\theta} \in \mathcal{KL}$ and $\hat{\phi} \in \mathcal{K}$ s.t.

$$|\hat{z}(t)| \leq \hat{\theta}(|\hat{z}(0)|, t) + \hat{\phi}\left(\sup_{\tau \in [0,t]} |\hat{\rho}(\tau)|\right), \forall t \in \mathbb{R}_{\geq 0}$$

$$\lim_{t \to \infty} |\hat{\rho}(t)| = 0 \Rightarrow \lim_{t \to \infty} |\hat{z}(t)| = 0$$

where $\hat{z} = \begin{bmatrix} \hat{x}^+ - x \\ x^+ - x^- \end{bmatrix}$, $\hat{\rho} = \begin{bmatrix} \delta^+ - \delta \\ \delta^- - \delta \end{bmatrix}$, $X = \begin{bmatrix} x \\ \hat{x}^+ \\ \hat{x}^- \end{bmatrix}$, $\Delta = \begin{bmatrix} \delta \\ \delta^+ \\ \delta^- \end{bmatrix}$

hold, and moreover, the closed-loop system is ISS w.r.t $\Delta$ and $X$. An introduction to interval observers and some recent developments
Guarantees under (A1), (A2), (A3): Framer + ISS

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If \( \mu, \omega \in \mathcal{K}_\infty \), then FP and \( \exists \hat{\theta} \in \mathcal{KL} \) and \( \hat{\phi} \in \mathcal{K} \) s.t.

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hold, and moreover, the closed-loop system is ISS w.r.t \( \Delta \) and \( X \).

Convergence of disturbances implies convergence of the estimate and the closed-loop state.

An introduction to interval observers and some recent developments
Utility of $K$ compared with $H$

- Given and fixed feedback control law $u$, the choice of $H(y)$ does not influence the observer equation, while the choice of $K(y)$ does.
- This flexibility of $K(y)$ can be utilized to construct a bundle of interval observers for generating a tighter estimate.
- The standard observer aims at only closed-loop stability and convergence and it is not built for monitoring. In contrast, interval observers provide estimates in the transient phase and the freedom of $K(y)$ matters.

... to be illustrated by an example
Reduced-order interval observer: A natural idea

Aim: Possibly better estimate and convergence

\[ x = \begin{bmatrix} x_m \\ x_{\overline{m}} \end{bmatrix} \} \text{ } p \text{ components measured corresponding to } C = [I \ 0] \]
\[ n - p \text{ components} \]

\[ A(y) = \begin{bmatrix} A_{m,m}(y) & A_{m,\overline{m}}(y) \\ A_{\overline{m},m}(y) & A_{\overline{m},\overline{m}}(y) \end{bmatrix}, \ B(y) = \begin{bmatrix} B_m(y) \\ B_{\overline{m}}(y) \end{bmatrix} \]

\[ \delta = \begin{bmatrix} \delta_m \\ \delta_{\overline{m}} \end{bmatrix}, \ x_0 = \begin{bmatrix} x_{m,0} \\ x_{\overline{m},0} \end{bmatrix} \]

Output feedback control: \( u(y, \hat{x}_m) = u(y, \hat{x}_m^{+} - Gy) \)

\( \hat{x}_m \): An estimate of \( x_m \), \( G \): Constant matrix to be chosen

Assumptions:

\[ x_{m,0}^{-} \leq x_{m,0} \leq x_{m,0}^{+}, \]
\[ \delta_{m}(t) \leq G\delta_{m}(t) \leq \delta_{m}^{+}(t), \delta_{\overline{m}}^{-}(t) \leq \delta_{\overline{m}}(t) \leq \delta_{\overline{m}}^{+}(t) \]
Reduced-order interval observer: A natural idea

\[
\dot{\hat{x}}_m^+ = [A_{\bar{m},\bar{m}}(y) + GA_{\bar{m},\bar{m}}(y)] \hat{x}_m^+ + [B_{\bar{m}}(y) + GB_{\bar{m}}(y)] u \\
+ [A_{\bar{m},\bar{m}}(y) - A_{\bar{m},\bar{m}}(y) G - GA_{\bar{m},\bar{m}}(y) G + GA_{\bar{m},\bar{m}}(y)] y \\
+ S_{\bar{m}}[R_{m}^{+}\delta_{m}^{+} - R_{m}^{-}\delta_{m}^{-}] + S_{\bar{m}}[R_{m}^{+}\delta_{m}^{-} - R_{m}^{-}\delta_{m}^{+}]
\]

\[
\dot{\hat{x}}_m^- = [A_{\bar{m},\bar{m}}(y) + GA_{\bar{m},\bar{m}}(y)] \hat{x}_m^- + [B_{\bar{m}}(y) + GB_{\bar{m}}(y)] u \\
+ [A_{\bar{m},\bar{m}}(y) - A_{\bar{m},\bar{m}}(y) G - GA_{\bar{m},\bar{m}}(y) G + GA_{\bar{m},\bar{m}}(y)] y \\
+ S_{\bar{m}}[R_{m}^{+}\delta_{m}^{-} - R_{m}^{-}\delta_{m}^{-}] + S_{\bar{m}}[R_{m}^{+}\delta_{m}^{-} - R_{m}^{-}\delta_{m}^{+}]
\]

\[
x_m^+ = S_m^+ R_m \hat{x}_m^+ - S_m^- R_m \hat{x}_m^- - G y, \quad x_m^- = S_m^+ R_m \hat{x}_m^- - S_m^- R_m \hat{x}_m^+ - G y
\]

\[
\hat{x}_m^+(0) = \hat{x}_{m,0}^+ := S_{\bar{m}}[R_{m}^{+} x_{m,0}^+ - R_{m}^{-} x_{m,0}^-] + G y(0)
\]

\[
\hat{x}_m^-(0) = \hat{x}_{m,0}^- := S_{\bar{m}}[R_{m}^{+} x_{m,0}^- - R_{m}^{-} x_{m,0}^+] + G y(0)
\]

and \( x^+ = \begin{bmatrix} y \\ x_m^+ \end{bmatrix}, \quad x^- = \begin{bmatrix} y \\ x_m^- \end{bmatrix} \)
Design Guidelines: Selecting $G$ and $u$

Assumption (A4): Securing framer properties

$$\Gamma_m(y) = R_m [A_m, m(y) + GA_m, m(y)] R_m^{-1}$$ is Metzler for each $y$

Assumption (A5): Securing convergence for nonzero disturbances

$$\frac{\partial V}{\partial \xi}(\xi) \left\{ [A_m, m(y) + GA_m, m(y)] \xi + S_m [R_m^+ \rho_m^+ + R_m^- \rho_m^-] \right\} + S_m [R_m^+ \rho_m^+ + R_m^- \rho_m^-] \leq -\omega(|\xi|) + \eta^+ (|\rho^+|) + \eta^- (|\rho^-|)$$

Assumption (A6): Separation of feedback gain design

$$\frac{\partial U}{\partial x}(x)[A(x_m)x + B(x_m)u(x_m, x_m + d_m) + \delta] \leq -\mu(|x|) + \gamma(|d_m|) + \zeta(|\delta|)$$

An introduction to interval observers and some recent developments
A reduced-order observer exists almost always?

Recall $A(y) = \begin{bmatrix} A_{m,m}(y) & A_{m,m}(y) \\ A_{\bar{m},m}(y) & A_{\bar{m},m}(y) \end{bmatrix}, \ C = [I \ 0]$

**Proposition**

Suppose that $A_{\bar{m},m}(y)$ is independent of $y$. If Assumptions (A1)-(A2) (of the full-order design) are satisfied with a non-singular matrix $R = \begin{bmatrix} R_m & 0 \\ 0 & R_{\bar{m}} \end{bmatrix}$ and a quadratic function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$. Then Assumptions (A4)-(A5) (of the reduced-order design) hold with $G = 0$. 

Reduced-order design allows to get rid of unnecessarily "Metzlerization". For a meaningful full-order design, exploitation of a non-diagonal transformation matrix $R$ is crucial.
A reduced-order observer exists almost always?

Recall \( A(y) = \begin{bmatrix} A_{m,m}(y) & A_{m,m}(\overline{y}) \\ A_{\overline{m},m}(y) & A_{\overline{m},m}(\overline{y}) \end{bmatrix}, \quad C = [I \quad 0] \)

**Proposition**

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Reduced-order design allows to get rid of unnecessarily ”Metzlerization”.
For a meaningful full-order design, exploitation of a non-diagonal transformation matrix \( R \) is crucial.
Simulation Example

Plant

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & -x_1^2 - \frac{1}{2} \\
0 & -2x_2^2 - \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
\frac{x_2}{2} + u_1 + \delta_1 \\
-\frac{x_2}{2} + u_2 + \delta_2
\end{bmatrix}
\]

\[y = x_1\]

Feedback input \( u(y, \hat{x}^+) = \frac{1}{2} \begin{bmatrix} -4y^3 + \hat{x}_2^+ & -\hat{x}_2^+ \end{bmatrix} \]

Observer gain \( H(y) = \begin{bmatrix} -2y^2 - \frac{3}{4} & \frac{1}{2} \end{bmatrix} \)

Disturbance \( \delta(t) = \begin{bmatrix} \text{sgn}(\sin(t))\min \{|\sin(t)|, 5/t^2\} & \text{sgn}(\cos(t))\min \{|\cos(t)|, 5/t^2\} \end{bmatrix} \)

satisfying (A1)-(A3) (and (A4)-(A6))
Comparison of Three Designs

Full-order observer with $K(y) = 0$

Full-order observer with $K(y) = \begin{bmatrix} -2y^2 & -1 \end{bmatrix}$

Reduced-order observer with $G = 2$

An introduction to interval observers and some recent developments
Comparison of Three Designs

- Full-order observer with $K(y) = 0$
- Full-order observer with $K(y) = \begin{bmatrix} -2y^2 & 1 \\ 0 & 1 \end{bmatrix}$
- Reduced-order observer with $G = 2$

\[\text{better}\]
Comparison of Three Designs

Full-order observer with $K(y) = 0$

$\begin{bmatrix} -2y^2 & 0 \\ 0 & -1 \end{bmatrix}$

Full-order observer with $K(y) = \begin{bmatrix} -2y^2 & 0 \\ 0 & -1 \end{bmatrix}$

Reduced-order observer with $G = 2$

Result 1: Guarantees of framer property and convergence

Result 2: Flexibility for potentially better designs

An introduction to interval observers and some recent developments
Remarks, Conclusions and Perspectives

An introduction to interval observers and some recent developments
Remarks

Successful utilization of positivity

1. The Metzler property imposed for obtaining a framer

   The estimation errors with transformation $R$ are governed by positive systems.

2. Cascade of iISS systems utilized to secure the convergence of the frames and the closed-loop state

   The observer error system and the state feedback system are positive systems in terms of $V$ and $U$, respectively.
Conclusions

Reviewing the preceding result (Ito and Dinh 2016)

- The iISS approach to interval observer design for output feedback control of nonlinear systems
  Guarantees of the convergence of the estimated interval length to zero in the presence of converging disturbances

Incorporating feedback gain into the interval observer presented in the preceding study
- Flexibility to obtain better transient behavior of estimated intervals without altering the observer gain and the control law

A reduced-order observer allows us to avoid estimating measured variables for better estimation and convergence. Unless complex state transformation is employed, a reduced-order observer exists when a full-order one does.
Conclusions

Reviewing the preceding result (Ito and Dinh 2016)

• The iISS approach to interval observer design for output feedback control of nonlinear systems
  Guarantees of the convergence of the estimated interval length to zero in the presence of converging disturbances

New developments and messages

• Incorporating feedback gain into the interval observer presented in the preceding study → flexibility to obtain better transient behavior of estimated intervals without altering the observer gain and the control law

• A reduced-order observer allows us to avoid estimating measured variables for better estimation and convergence. Unless complex state transformation is employed, a reduced-order observer exists when a full-order one does.
Perspectives

- Research on monitoring, robust controlling systems based on interval observers: Theory and Applications. More systems can be considered:
  - Systems with delay (e.g. continuous-discrete systems with distributed delays in state......)
  - Continuous-discrete systems with non constant measurements
  - Hybrid systems
  - Applications to real-life systems such as in Robotics (drones as well as ground unmanned vehicles)

- Research on fault detection and diagnosis