

Constructions of Interval Observers for Discrete-time Systems of Luenberger type*

Frédéric Mazenc¹, Thach Ngoc Dinh² and Silviu-Iulian Niculescu³

Abstract—For a class of discrete-time systems with input, output and uncertain terms, a new interval observer is constructed. Its key feature is that it is composed of two copies of classical observers, whose corresponding error equations are in general not nonnegative. It is shown how this interval observer can be modified to cope with the presence of nonlinear terms and disturbances and how asymptotic stability of the resulting interval observer can be achieved through an appropriate choice of output feedback.

I. INTRODUCTION

The interval observer approach is a state estimation technique based on a guaranteed state estimator composed of a dynamic extension endowed with two outputs giving an upper and a lower bound for the solutions of the considered system. A method of this type makes it possible to cope with big disturbances and gives an information on the current value of the solutions at any time instant. To the best of the author's knowledge, the guaranteed state estimation technique can be traced back to the seminal work [18], but the notion of interval observer is more recent. It originates in [7] and has been developed in many directions because state estimation is essential for monitoring and control purposes. Some works on interval observers are devoted to various classes of linear systems [3], [10], [11], [14], [13] and others are devoted to some classes of nonlinear systems [16], [17], [15]. Most of these works are concerned with continuous-time systems only, although such a technique is appealing in the context of discrete-time systems: notice in particular that systems with sampled data often lead to discrete-time systems, as explained for instance in [1], which are frequently affected by disturbances. This motivates the development of robust state estimation techniques, like the one based on interval observers. This remark motivated the contributions [4] and [12] and motivates the present paper too.

In [4], interval observers are constructed for families of time-varying discrete-time systems without inputs and in [12], interval observers for two important families of discrete-time systems are proposed. The first is composed of time-invariant nonlinear systems which possess specific

stability and monotonicity properties. The second is the general family of the linear time-invariant exponentially stable systems. We established that these systems can be transformed into *nonnegative* and *exponentially stable time-invariant* systems through linear, possibly time-varying, changes of coordinates. Using this key result, interval observers for linear systems without input and an output have been constructed, under a detectability assumption. Such constructions are based on a dynamic extension, which is nonnegative when the output is identically equal to zero.

In the present paper, we complement [12] in several directions. In a first part, we consider the linear time-invariant discrete-time system with input and output:

$$\begin{cases} x_{k+1} &= \alpha x_k + \beta u_k, \\ y_k &= \mathcal{C}x_k, \end{cases} \quad (1)$$

with $k \in \mathbb{N}$, $x_k \in \mathbb{R}^n$, where $u_k \in \mathbb{R}^q$ is the input, $y_k \in \mathbb{R}^p$ is the output, where $\alpha \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^{n \times q}$, $\mathcal{C} \in \mathbb{R}^{p \times n}$ and show, under the classical detectability assumption, that two copies of Luenberger observers [9] endowed with appropriate outputs (the two bounds) and initial conditions compose an exponentially stable interval observer. This result may sound surprising because such observers or their associated error equations do not possess the property of being nonnegative systems (see, for instance, [8] for the definition of nonnegative system), although this property is usually used when constructing interval observers (see, for examples, [12], [4], [13], [6] and the discussions therein). In fact, we will use the notion of nonnegative system as well, but only indirectly to select appropriate initial conditions and upper and lower bounds for the interval observer. It turns out that these bounds may be time-varying. To the best of the authors' knowledge, there do not exist similar results in the literature even for continuous time systems. The main advantage of the new approach is that it makes it possible to let classical observers play simultaneously the role of observers and interval observers and therefore the introduction of extra dynamics with some nonnegativity property is not explicitly needed. Moreover, since the choice of initial conditions and bounding outputs for the interval observer is not unique, this technique may be used to construct a bundle of interval observers, as done for instance in [2], without having to introduce extra dynamics. Thus, better estimates can be obtained without having to consider interval observers of dimension larger than twice the dimension of the studied system.

In a second part, we consider a nonlinear system of the

*This work was supported by the Digiteo Project MOISYR - 2011-045D
¹Frédéric Mazenc is with EPI INRIA DISCO, Laboratoire des Signaux et Systèmes, CNRS-Supélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France Frederic.MAZENC at lss.supelec.fr

²Thach Ngoc Dinh is with EPI INRIA DISCO, Laboratoire des Signaux et Systèmes, CNRS-Supélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France Thach.Dinh at lss.supelec.fr

³Silviu-Iulian Niculescu is with EPI INRIA DISCO, Laboratoire des Signaux et Systèmes, CNRS-Supélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France Silviu.Niculescu@lss.supelec.fr

form

$$\begin{cases} x_{k+1} &= [\mathcal{A} + \mathcal{A}_s(x_k)]x_k + \beta u_k + \Phi(y_k), \\ y_k &= \mathcal{C}x_k, \end{cases} \quad (2)$$

with $k \in \mathbb{N}$, $\mathcal{A} \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^{n \times q}$, $\mathcal{C} \in \mathbb{R}^{p \times n}$, where $\mathcal{A}_s : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is an unknown bounded nonlinear function, and $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^n$ is a known nonlinear function. We give some conditions ensuring that the system admits an interval observer which is exponentially stable when the system is in closed-loop with a feedback which depends only on y_k and the values provided by the bounds of the interval observer. The proposed interval observer consists of the one designed for (1) with additional terms taking into account the presence of the uncertainties \mathcal{A}_s and of Φ . It is worth noticing that the idea of taking advantage of interval observers to design stabilizing control laws is not new: it is used in [5] to stabilize nonlinear systems.

The rest of this note is organized as follows. Notation, definitions and prerequisites are given in Section II. A construction of a time-varying interval observer for a linear system is proposed in Section III. In Section IV, we adapt the previous construction to the case where nonlinear unknown terms are present. An illustrative example is given in Section V. Concluding remarks are drawn in Section VI.

II. NOTATION, DEFINITIONS AND PREREQUISITES

The notation will be simplified whenever no confusion can arise from the context. Any matrix, whose entries are all 0 is simply denoted 0. The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimension are denoted $|\cdot|$. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. Then $|A|_p = (|a_{ij}|) \in \mathbb{R}^{n \times n}$. All the inequalities must be understood *componentwise* i.e. $v_a = (v_{a1}, \dots, v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, \dots, v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, \dots, r\}$, $v_{ai} \leq v_{bi}$. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (resp. negative semidefinite) if for all vectors $v \in \mathbb{R}^n$, $v^\top A v \geq 0$ (resp. $v^\top A v \leq 0$). Then we denote $A \preceq 0$ (resp. $A \succeq 0$). A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Schur stable if its spectral radius is smaller than 1. For two matrices $A = (a_{ij}) \in \mathbb{R}^{r \times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r \times s}$, $\max\{A, B\}$ is the matrix where each entry is $m_{ij} = \max\{a_{ij}, b_{ij}\}$. For a matrix $A \in \mathbb{R}^{r \times s}$, $A^+ = \max\{A, 0\}$, $A^- = \max\{-A, 0\}$. A matrix $A \in \mathbb{R}^{r \times s}$ is said to be *nonnegative* if every entry of A is nonnegative. A sequence (u_i) is *nonnegative* if for all $k \in \mathbb{N}$, u_k is nonnegative. A system $x_{k+1} = f(k, x_k)$ is *nonnegative* if for all integer k_0 and any initial condition $x_{k_0} \geq 0$, the solution x_k satisfies $x_k \geq 0$ for all integer $k \geq k_0$. Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then the inequality

$$|x + y|^2 \leq 2|x|^2 + 2|y|^2 \quad (3)$$

holds.

Due to the features of the systems we shall consider, we adopt a slightly less general definition of interval observer for discrete-time nonlinear systems than the one in [12]. We also define the framers. Those two notions have been introduced, with slightly different features, in several papers (see, for

instance, [14], [7] to cite only a few).

Definition 1: Consider a discrete-time system:

$$x_{k+1} = f_1(k, x_k), \quad (4)$$

with $x_k \in \mathbb{R}^n$, with an output $y_k = m(x_k) \in \mathbb{R}^p$, and where f_1 and m are two nonlinear functions. The initial condition at the instant $k_0 \in \mathbb{N}$, $x_{k_0} \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+. \quad (5)$$

Then, the dynamical system

$$z_{k+1} = f_2(k, z_k, y_k), \quad (6)$$

associated with the initial condition $z_{k_0} = g(k_0, x_{k_0}^+, x_{k_0}^-) \in \mathbb{R}^{n_z}$ and bounds for the solution x_k :

$$x_k^+ = h^+(k, z_k), \quad x_k^- = h^-(k, z_k) \quad (7)$$

where f_2, g, h^+ and h^- are nonlinear functions, is called (i) a framer for (4) if for any vectors $x_{k_0}, x_{k_0}^-$ and $x_{k_0}^+$ in \mathbb{R}^n satisfying (5), the solutions of (4)-(6) with respectively $x_{k_0}, z_{k_0} = g(k_0, x_{k_0}^+, x_{k_0}^-)$ as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k \geq k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+, \quad (8)$$

(ii) an interval observer for (4) if in addition any solution (x_k, z_k) of (4)-(6) is such that $\lim_{k \rightarrow +\infty} |h^+(k, z_k) - h^-(k, z_k)| = 0$.

The following result is a direct consequence of [8, Chapt. 5, Proposition 5.6].

Lemma 1: The system $z_{k+1} = \mathcal{A}z_k$ where $z_k \in \mathbb{R}^n$, $\mathcal{A} \in \mathbb{R}^{n \times n}$ is nonnegative if and only if the matrix \mathcal{A} is nonnegative.

Recall a result given in [12] because it is instrumental in establishing the results in Section III. A similar result, in the context of continuous-time systems, is presented in [11].

Lemma 2: Consider a constant Schur stable matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$. Then there exist a sequence of invertible matrices $\mathcal{R}_k \in \mathbb{R}^{n \times n}$, a constant Schur stable nonnegative matrix $\mathcal{E} \in \mathbb{R}^{n \times n}$ and a constant $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and $\mathcal{R}_{k+1} \mathcal{A} \mathcal{R}_k^{-1} = \mathcal{E}$.

III. TIME-VARYING INTERVAL OBSERVERS

In this section, we state and prove a result for the simple case of the linear time-invariant systems with input and output and no uncertainties. It applies to systems which do not possess any stability or stabilizability property, in contrast to the theorem of the next section, which applies only to stabilizable systems. The theorem below helps understanding the one proposed in the next section, which is more complicated due to the presence of nonlinearity and uncertainty.

Theorem 1: Consider the system:

$$\begin{cases} x_{k+1} &= \alpha x_k + \beta u_k, \\ y_k &= \mathcal{C}x_k, \end{cases} \quad (9)$$

with $x_k \in \mathbb{R}^n$, the output $y_k \in \mathbb{R}^p$, the input $u_k \in \mathbb{R}^q$, where $\alpha \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^{n \times q}$, $\mathcal{C} \in \mathbb{R}^{p \times n}$ are matrices such that there exists a matrix $\mathcal{K} \in \mathbb{R}^{n \times p}$ such that $\mathcal{A} = \alpha + \mathcal{K}\mathcal{C}$

is Schur stable.

Then, there exists a sequence of invertible real matrices (\mathcal{R}_k) and a real number $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and

$$\mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \mathcal{E}, \quad (10)$$

where $\mathcal{E} \in \mathbb{R}^{n \times n}$ is a nonnegative Schur stable matrix. Let $\mathcal{S}_k = \mathcal{R}_k^{-1}$ for all $k \in \mathbb{N}$. Then the system:

$$\begin{cases} \hat{x}_{k+1}^+ = \mathcal{A}\hat{x}_k^+ - \mathcal{K}y_k + \beta u_k, \\ \hat{x}_{k+1}^- = \mathcal{A}\hat{x}_k^- - \mathcal{K}y_k + \beta u_k, \end{cases} \quad (11)$$

associated with the initial conditions

$$\begin{aligned} \hat{x}_{k_0}^+ &= \mathcal{S}_{k_0} \left[\mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^- \right], \\ \hat{x}_{k_0}^- &= \mathcal{S}_{k_0} \left[\mathcal{R}_{k_0}^+ x_{k_0}^- - \mathcal{R}_{k_0}^- x_{k_0}^+ \right], \end{aligned} \quad (12)$$

and the bounds

$$\begin{aligned} x_k^+ &= \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^+ - \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^-, \\ x_k^- &= \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^- - \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^+, \end{aligned} \quad (13)$$

is an interval observer for system (9).

Remark. A remarkable feature of the two systems in (11) is that each of them is a standard Luenberger observer for the system (9).

Proof. Since the matrix \mathcal{A} is Schur stable, Lemma 2 provides with the required sequence of matrices \mathcal{R}_k .

Next, for an initial instant $k_0 \in \mathbb{N}$, we consider vectors $x_{k_0}^+, x_{k_0}^-, x_{k_0}^+, \hat{x}_{k_0}^+, \hat{x}_{k_0}^-$ in \mathbb{R}^n such that

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+ \quad (14)$$

and (12) is satisfied. Since the matrices \mathcal{R}_k^+ and \mathcal{R}_k^- are nonnegative, it follows from (14) that the inequalities:

$$\begin{aligned} \mathcal{R}_{k_0}^+ x_{k_0}^- &\leq \mathcal{R}_{k_0}^+ x_{k_0} \leq \mathcal{R}_{k_0}^+ x_{k_0}^+, \\ \mathcal{R}_{k_0}^- x_{k_0}^+ &\leq \mathcal{R}_{k_0}^- x_{k_0} \leq \mathcal{R}_{k_0}^- x_{k_0}^+ \end{aligned}$$

are satisfied. We deduce that:

$$-\mathcal{R}_{k_0}^- x_{k_0}^+ + \mathcal{R}_{k_0}^+ x_{k_0}^- \leq (\mathcal{R}_{k_0}^+ - \mathcal{R}_{k_0}^-) x_{k_0} \leq \mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^-.$$

According to (12), the inequalities

$$\mathcal{R}_{k_0} \hat{x}_{k_0}^- \leq \mathcal{R}_{k_0} x_{k_0} \leq \mathcal{R}_{k_0} \hat{x}_{k_0}^+ \quad (15)$$

hold. Next, let us consider the solutions $x_k, \hat{x}_k^+, \hat{x}_k^-$ of the systems (9) and (11) with the initial conditions $x_{k_0}, \hat{x}_{k_0}^+, \hat{x}_{k_0}^-$ selected above. Since, for all $k \geq k_0$,

$$\begin{cases} x_{k+1} = \mathcal{A}x_k + \beta u_k - \mathcal{K}y_k, \\ \hat{x}_{k+1}^+ = \mathcal{A}\hat{x}_k^+ + \beta u_k - \mathcal{K}y_k, \\ \hat{x}_{k+1}^- = \mathcal{A}\hat{x}_k^- + \beta u_k - \mathcal{K}y_k, \end{cases} \quad (16)$$

it follows from (10) that

$$\begin{aligned} \mathcal{R}_{k+1} x_{k+1} &= \mathcal{R}_{k+1} [\mathcal{A}x_k + \beta u_k - \mathcal{K}y_k] \\ &= \mathcal{E} \mathcal{R}_k x_k + \mathcal{R}_{k+1} \beta u_k - \mathcal{R}_{k+1} \mathcal{K}y_k \end{aligned} \quad (17)$$

and

$$\begin{aligned} \mathcal{R}_{k+1} \hat{x}_{k+1}^+ &= \mathcal{E} \mathcal{R}_k \hat{x}_k^+ + \mathcal{R}_{k+1} \beta u_k - \mathcal{R}_{k+1} \mathcal{K}y_k, \\ \mathcal{R}_{k+1} \hat{x}_{k+1}^- &= \mathcal{E} \mathcal{R}_k \hat{x}_k^- + \mathcal{R}_{k+1} \beta u_k - \mathcal{R}_{k+1} \mathcal{K}y_k. \end{aligned} \quad (18)$$

It follows that

$$g_{k+1}^+ = \mathcal{E} g_k^+, \quad g_{k+1}^- = \mathcal{E} g_k^-, \quad (19)$$

with $g_k^+ = \mathcal{R}_k \hat{x}_k^+ - \mathcal{R}_k x_k, g_k^- = \mathcal{R}_k x_k - \mathcal{R}_k \hat{x}_k^-$. From (15), the fact that \mathcal{E} is nonnegative and Lemma 1, we deduce that, for all $k \geq k_0, g_k^+ \geq 0, g_k^- \geq 0$, which implies that

$$\mathcal{R}_k \hat{x}_k^- \leq \mathcal{R}_k x_k \leq \mathcal{R}_k \hat{x}_k^+. \quad (20)$$

Since \mathcal{S}_k^+ and \mathcal{S}_k^- are nonnegative, it follows from the previous inequalities that, for all $k \geq k_0$,

$$\begin{aligned} \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^- &\leq \mathcal{S}_k^+ \mathcal{R}_k x_k \leq \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^+, \\ \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^+ &\leq \mathcal{S}_k^- \mathcal{R}_k x_k \leq \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^-, \end{aligned} \quad (21)$$

which implies that

$$\begin{aligned} \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^- - \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^+ &\leq (\mathcal{S}_k^+ - \mathcal{S}_k^-) \mathcal{R}_k x_k \\ &\leq \mathcal{S}_k^+ \mathcal{R}_k \hat{x}_k^+ - \mathcal{S}_k^- \mathcal{R}_k \hat{x}_k^-. \end{aligned} \quad (22)$$

From the definitions of x_k^+ and x_k^- in (13) and the fact that $\mathcal{S}_k^+ - \mathcal{S}_k^- = \mathcal{S}_k$ for all $k \in \mathbb{N}$, it follows that, for all $k \geq k_0, x_k^- \leq x_k \leq x_k^+$. In addition, for all $k \geq k_0, \hat{x}_{k+1}^+ - \hat{x}_{k+1}^- = \mathcal{A}(\hat{x}_k^+ - \hat{x}_k^-)$, which implies that $\lim_{k \rightarrow +\infty} |\hat{x}_k^+ - \hat{x}_k^-| = 0$, since \mathcal{A} is Schur stable. Finally, since $|\mathcal{R}_k| + |\mathcal{S}_k| \leq c$ for all $k \in \mathbb{N}$, we deduce that $\lim_{k \rightarrow +\infty} |x_k^+ - x_k^-| = 0$. This concludes the proof. \blacksquare

IV. ROBUST INTERVAL OBSERVER

In the sequel, we consider a family of nonlinear systems with uncertain terms. We will show that, despite the presence of these uncertainties, one can construct for them interval observers when they are in closed-loop with stabilizing output feedbacks which depend on the values of the bounds of the interval observer. More precisely, we consider:

$$\begin{cases} x_{k+1} = [\mathcal{A} + \mathcal{A}_s(x_k)]x_k + \beta u_k + \Phi(y_k), \\ y_k = \mathcal{C}x_k, \end{cases} \quad (23)$$

with $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^q$ is the input and $y_k \in \mathbb{R}^p$ is the output, where Φ is a nonlinear function of class C^1 , where $\mathcal{A} \in \mathbb{R}^{n \times n}, \beta \in \mathbb{R}^{n \times q}, \mathcal{C} \in \mathbb{R}^{p \times n}$ and the system (23) is affected by unknown disturbances $\mathcal{A}_s(x) \in \mathbb{R}^{n \times n}$, bounded in norm by a known constant.

We introduce some assumptions:

Assumption 1. *There exists an invertible matrix $\mathcal{R} \in \mathbb{R}^{n \times n}$ such that*

$$\mathcal{R}\mathcal{A}\mathcal{S} = \mathcal{E}, \quad (24)$$

where $\mathcal{E} \in \mathbb{R}^{n \times n}$ is a nonnegative Schur stable matrix and $\mathcal{S} = \mathcal{R}^{-1}$.

Notice for later use that Assumption 1 guarantees the existence of a symmetric positive definite matrix $Q_1 \in \mathbb{R}^{n \times n}$ such that

$$\mathcal{A}^\top Q_1 \mathcal{A} - Q_1 \preceq -I. \quad (25)$$

Assumption 2. *There exist matrices $\mathcal{M} \in \mathbb{R}^{q \times n}, \mathcal{L} \in \mathbb{R}^{n \times p}$ and a symmetric positive definite matrix $Q_2 \in \mathbb{R}^{n \times n}$ such that, for all $x \in \mathbb{R}^n$,*

$$[\mathcal{G} + \mathcal{A}_s(x)]^\top Q_2 [\mathcal{G} + \mathcal{A}_s(x)] - Q_2 \preceq -I, \quad (26)$$

with

$$\mathcal{G} = \mathcal{A} + \mathcal{L}\mathcal{C} + \beta\mathcal{M}. \quad (27)$$

Assumption 3. There exists a continuous function $\Omega : \mathbb{R}^p \rightarrow \mathbb{R}^q$ such that $\Omega(0) = 0$ and for all $y \in \mathbb{R}^p$,

$$\Phi(y) - \mathcal{L}y = \beta\Omega(y), \quad (28)$$

where β is the constant matrix given in (23) and \mathcal{L} is the matrix in Assumption 2.

Assumption 4. Let $c_{i,j}(x)$ denote the entries of the matrix $\mathcal{R}\mathcal{A}_s(x)\mathcal{S}$. Then the inequalities

$$|c_{i,j}(x)| \leq \epsilon, \quad \forall (i,j) \in \{1, \dots, n\}^2, \quad |\mathcal{A}_s(x)| \leq \epsilon \quad (29)$$

are satisfied with

$$\epsilon \leq \min\{b_1, b_2 b_3, |\mathcal{G}|\}, \quad (30)$$

$$b_1 = \frac{1}{4\sqrt{6sn}|\mathcal{S}||\mathcal{R}|}, \quad (31)$$

$$b_2 = \frac{1}{\sqrt{4(8|\mathcal{G}|^2|Q_2| + 1)|Q_2||\beta|^2|\mathcal{M}|^2 + 1}} \quad (32)$$

$$b_3 = \frac{1}{2\sqrt{2}\sqrt{s(1 + 6n^2|\mathcal{S}|^2|\mathcal{R}|^2)}} \quad (33)$$

and $s = 2|Q_1\mathcal{A}|^2 + |Q_1|$. We are ready to state the main result of the section.

Theorem 2: Let the system (23) satisfy Assumptions 1 to 4. Then this system with the dynamic extension

$$\begin{cases} \hat{x}_{k+1}^+ &= \mathcal{A}\hat{x}_k^+ + \beta u_k + \Phi(y_k) \\ &+ \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \\ \hat{x}_{k+1}^- &= \mathcal{A}\hat{x}_k^- + \beta u_k + \Phi(y_k) \\ &- \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \end{cases} \quad (34)$$

where ϵ is the positive real number in Assumption 4, $\mathcal{U} =$

$$\begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}, \text{ in closed-loop with the feedback}$$

$$u(\hat{x}_k^+, y_k) = \mathcal{M}\hat{x}_k^+ - \Omega(y_k) \quad (35)$$

is globally exponentially stable. Moreover, for this closed-loop system, the system (34) associated with the initial conditions

$$\begin{cases} \hat{x}_{k_0}^+ &= \mathcal{S}[\mathcal{R}^+ x_{k_0}^+ - \mathcal{R}^- x_{k_0}^-], \\ \hat{x}_{k_0}^- &= \mathcal{S}[\mathcal{R}^+ x_{k_0}^- - \mathcal{R}^- x_{k_0}^+], \end{cases} \quad (36)$$

the bounds for the solutions x_k

$$\begin{cases} x_k^+ &= \mathcal{S}^+ \mathcal{R} \hat{x}_k^+ - \mathcal{S}^- \mathcal{R} \hat{x}_k^-, \\ x_k^- &= \mathcal{S}^+ \mathcal{R} \hat{x}_k^- - \mathcal{S}^- \mathcal{R} \hat{x}_k^+, \end{cases} \quad (37)$$

is an exponentially stable interval observer for the system (23).

Discussion of Theorem 2.

• Assumption 1 ensures the existence of a time-invariant change of coordinates that transforms \mathcal{A} into a nonnegative Schur stable matrix. For the sake of simplicity, we have chosen to restrict ourselves to this case, although one can cope with the general case by taking advantage of the time-

varying change of coordinates provided by Lemma 2. The extension can be easily obtained by combining the proofs of Theorem 1 and Theorem 2.

• Assumption 1 implies that the matrix \mathcal{A} is Schur stable. But it does not imply that the system $x_{k+1} = \mathcal{A}x_k + \Phi(y_k)$ is globally stable and therefore the linear approximation at the origin of $x_{k+1} = \mathcal{A}x_k + \Phi(y_k)$ may be unstable. Indeed, for instance any system $x_{k+1} = \mathcal{A}_a x_k + \Phi_a(y_k)$ with $y_k = \mathcal{C}x_k$ such that the pair $(\mathcal{A}_a, \mathcal{C})$ is detectable satisfies Assumption 1 with $\mathcal{A} = \mathcal{A}_a + K_a \mathcal{C}$, where K_a is a constant matrix such that $\mathcal{A}_a + K_a \mathcal{C}$ is Schur stable. The associated function Φ is then $\Phi(y) = \Phi_a(y) - K_a y$.

• Assumptions 2 and 3 define a constraint on the function Φ . These assumptions imply that the system (23) is globally exponentially stabilized by the feedback $u(x) = \mathcal{M}x - \Omega(\mathcal{C}x)$. A way to check whether Assumption 2 is satisfied consists in determining a symmetric positive definite matrix Q_2 such that $\mathcal{G}^\top Q_2 \mathcal{G} - Q_2 \preceq -I$, and next in determining $\epsilon > 0$ such that Assumption 2 is satisfied when the inequality $|\mathcal{A}_s(x)| \leq \epsilon$ is satisfied for all $x \in \mathbb{R}^n$.

• Assumption 4 is a restriction imposed on the unknown terms. Such a restriction is used to establish the stability of (34) in closed-loop with (35). The system (34) is a framer for the system (23) for any input. But it is an interval observer only when the system (23) is in closed loop with suitably chosen stabilizing feedbacks. Of course feedbacks different from (35) can be used: for instance one can choose $u(\hat{x}_k^-, y_k) = \mathcal{M}\hat{x}_k^- - \Omega(y_k)$.

Proof. The proof splits up into two parts. The first establishes that (34)-(36)-(37) with suitable initial conditions and bounds is a framer for the system (23) for any sequence of inputs u_k . The second is devoted to the stability analysis of the systems (23)-(34) in closed-loop with the dynamic output feedback (35).

1. Property of framer.

For an initial instant $k_0 \in \mathbb{N}$, let us consider vectors $x_{k_0}, x_{k_0}^+, x_{k_0}^-$ in \mathbb{R}^n such that $x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+$. Then, arguing as we did at the beginning of the proof of Theorem 1, we obtain

$$\mathcal{R}\hat{x}_{k_0}^- \leq \mathcal{R}x_{k_0} \leq \mathcal{R}\hat{x}_{k_0}^+. \quad (38)$$

Next, let us consider the solutions $x_k, \hat{x}_k^+, \hat{x}_k^-$ of the systems (23) and (34) with the initial conditions $x_{k_0}, \hat{x}_{k_0}^+, \hat{x}_{k_0}^-$ selected above. Then:

$$\begin{cases} x_{k+1} &= [\mathcal{A} + \mathcal{A}_s(x_k)]x_k + \beta u_k + \Phi(y_k), \\ \hat{x}_{k+1}^+ &= \mathcal{A}\hat{x}_k^+ + \beta u_k + \Phi(y_k) \\ &+ \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \\ \hat{x}_{k+1}^- &= \mathcal{A}\hat{x}_k^- + \beta u_k + \Phi(y_k) \\ &- \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}. \end{cases} \quad (39)$$

Since Assumption 1 guarantees that $\mathcal{R}\mathcal{A} = \mathcal{E}\mathcal{R}$, one gets

$$\begin{cases} \mathcal{R}x_{k+1} &= \mathcal{E}\mathcal{R}x_k + \mathcal{R}\beta u_k + \mathcal{R}\Phi(y_k) \\ &+ \mathcal{R}\mathcal{A}_s(x_k)\mathcal{S}\mathcal{R}x_k, \\ \mathcal{R}\hat{x}_{k+1}^+ &= \mathcal{E}\mathcal{R}\hat{x}_k^+ + \mathcal{R}\beta u_k + \mathcal{R}\Phi(y_k) \\ &+ \epsilon \mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \\ \mathcal{R}\hat{x}_{k+1}^- &= \mathcal{E}\mathcal{R}\hat{x}_k^- + \mathcal{R}\beta u_k + \mathcal{R}\Phi(y_k) \\ &- \epsilon \mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}. \end{cases} \quad (40)$$

Now, we prove by induction that for all $k \geq k_0$,

$$\mathcal{R}\hat{x}_k^- \leq \mathcal{R}x_k \leq \mathcal{R}\hat{x}_k^+. \quad (41)$$

According to (38), the property is satisfied at the instant k_0 . Assume that there exists $j > k_0$ such that, for all $i \in \{k_0, \dots, j-1\}$, $\mathcal{R}\hat{x}_i^- \leq \mathcal{R}x_i \leq \mathcal{R}\hat{x}_i^+$. Then Assumption 4 implies that

$$\begin{aligned} -\epsilon \mathcal{U}(\max\{|\mathcal{R}\hat{x}_{j-1}^+|_p, |\mathcal{R}\hat{x}_{j-1}^-|_p\}) &\leq \mathcal{R}\mathcal{A}_s(x_{j-1})\mathcal{S}\mathcal{R}x_{j-1} \\ &\leq \epsilon \mathcal{U}(\max\{|\mathcal{R}\hat{x}_{j-1}^+|_p, |\mathcal{R}\hat{x}_{j-1}^-|_p\}). \end{aligned}$$

From (40), it follows immediately that $\mathcal{R}\hat{x}_j^- \leq \mathcal{R}x_j \leq \mathcal{R}\hat{x}_j^+$. Therefore the induction assumption is satisfied at the step $j+1$. We deduce that, for all $k \geq k_0$, the inequalities $\mathcal{R}\hat{x}_k^- \leq \mathcal{R}x_k \leq \mathcal{R}\hat{x}_k^+$ hold. Since the entries of \mathcal{S}^+ and \mathcal{S}^- are nonnegative, it follows that, for all $k \geq k_0$,

$$\mathcal{S}^+\mathcal{R}\hat{x}_k^- - \mathcal{S}^-\mathcal{R}\hat{x}_k^- \leq x_k \leq \mathcal{S}^+\mathcal{R}\hat{x}_k^+ - \mathcal{S}^-\mathcal{R}\hat{x}_k^+. \quad (42)$$

It follows that, for all integer $k \geq k_0$, $x_k^- \leq x_k \leq x_k^+$.

2. *Stability of the systems (23) - (34) - (35).*

In this part, we prove that the system

$$\begin{cases} x_{k+1} &= [\mathcal{A} + \mathcal{A}_s(x_k)]x_k + \beta(\mathcal{M}\hat{x}_k^+ - \Omega(y_k)) \\ &\quad + \Phi(y_k), \\ \hat{x}_{k+1}^+ &= \mathcal{A}\hat{x}_k^+ + \beta[\mathcal{M}\hat{x}_k^+ - \Omega(y_k)] + \Phi(y_k) \\ &\quad + \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \\ \hat{x}_{k+1}^- &= \mathcal{A}\hat{x}_k^- + \beta[\mathcal{M}\hat{x}_k^- - \Omega(y_k)] + \Phi(y_k) \\ &\quad - \epsilon \mathcal{S}\mathcal{U} \max\{|\mathcal{R}\hat{x}_k^+|_p, |\mathcal{R}\hat{x}_k^-|_p\}, \end{cases} \quad (43)$$

admits the origin as a globally exponentially stable equilibrium point. Using Assumption 3 and the change of coordinates $p_k = \hat{x}_k^+ - x_k$, $q_k = \hat{x}_k^- - x_k$ we obtain

$$\begin{cases} x_{k+1} &= [\mathcal{G} + \mathcal{A}_s(x_k)]x_k + \beta \mathcal{M}p_k, \\ p_{k+1} &= \mathcal{A}p_k + \epsilon \Gamma(x_k, p_k, q_k) - \mathcal{A}_s(x_k)x_k, \\ q_{k+1} &= \mathcal{A}q_k + 2\epsilon \Gamma(x_k, p_k, q_k), \end{cases} \quad (44)$$

with $\Gamma(x, p, q) = \mathcal{S}\mathcal{U} \max\{|\mathcal{R}(p+x)|_p, |\mathcal{R}(p+x-q)|_p\}$.

Let us establish that the system (44) admits the origin as a globally exponentially stable equilibrium point through a Lyapunov approach. Let $V_1(p, q) = p^\top Q_1 p + q^\top Q_1 q$, where Q_1 is the symmetric positive definite matrix satisfying (25) and let us introduce the notation $\Delta V_{1,k} = V_1(p_{k+1}, q_{k+1}) - V_1(p_k, q_k)$. Then, after some simple algebraic manipulations, we obtain

$$\begin{aligned} \Delta V_{1,k} &= p_k^\top (\mathcal{A}^\top Q_1 \mathcal{A} - Q_1) p_k + 2p_k^\top \mathcal{A}^\top Q_1 \mathcal{U}(x_k, p_k, q_k) \\ &\quad + \mathcal{U}(x_k, p_k, q_k)^\top Q_1 \mathcal{U}(x_k, p_k, q_k) \\ &\quad + q_k^\top (\mathcal{A}^\top Q_1 \mathcal{A} - Q_1) q_k + 4\epsilon q_k^\top \mathcal{A}^\top Q_1 \Gamma(x_k, p_k, q_k) \\ &\quad + 4\epsilon^2 \Gamma(x_k, p_k, q_k)^\top Q_1 \Gamma(x_k, p_k, q_k), \end{aligned} \quad (45)$$

with $\mathcal{U}(x_k, p_k, q_k) = \epsilon \Gamma(x_k, p_k, q_k) - \mathcal{A}_s(x_k)x_k$.

From (25), we deduce that

$$\begin{aligned} \Delta V_{1,k} &\leq -|p_k|^2 + 2p_k^\top \mathcal{A}^\top Q_1 \mathcal{U}(x_k, p_k, q_k) \\ &\quad + \mathcal{U}(x_k, p_k, q_k)^\top Q_1 \mathcal{U}(x_k, p_k, q_k) - |q_k|^2 \\ &\quad + 4\epsilon q_k^\top \mathcal{A}^\top Q_1 \Gamma(x_k, p_k, q_k) \\ &\quad + 4\epsilon^2 \Gamma(x_k, p_k, q_k)^\top Q_1 \Gamma(x_k, p_k, q_k). \end{aligned} \quad (46)$$

Using the triangle inequality, we obtain

$$\begin{aligned} \Delta V_{1,k} &\leq -\frac{1}{2}|p_k|^2 + s|\mathcal{U}(x_k, p_k, q_k)|^2 - \frac{1}{2}|q_k|^2 \\ &\quad + 4\epsilon^2 s |\Gamma(x_k, p_k, q_k)|^2. \end{aligned} \quad (47)$$

with s defined in Assumption 4. It follows from the inequality (3) that

$$\begin{aligned} \Delta V_{1,k} &\leq -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + 2s|\mathcal{A}_s(x_k)x_k|^2 \\ &\quad + 6\epsilon^2 s |\Gamma(x_k, p_k, q_k)|^2. \end{aligned} \quad (48)$$

Since, for all $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $|\Gamma(x, p, q)| \leq |\mathcal{S}\mathcal{U}|(|\mathcal{R}(p+x)|_p + |\mathcal{R}q|_p) \leq |\mathcal{S}\mathcal{U}||\mathcal{R}|(|p| + |q| + |x|)$, $|\mathcal{A}_s(x)| \leq \epsilon$ for all $x \in \mathbb{R}^n$ and $|\mathcal{U}| = n$, we deduce that

$$\Delta V_{1,k} \leq -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + 2s\alpha_k, \quad (49)$$

with $\alpha_k = \epsilon^2|x_k|^2 + 3\epsilon^2|\mathcal{S}\mathcal{U}|^2|\mathcal{R}|^2(|p_k| + |q_k| + |x_k|)^2$.

From the inequality (3), it follows:

$$\begin{aligned} \Delta V_{1,k} &\leq -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 \\ &\quad + 2s\epsilon^2 [\chi_k^2 + 12n^2|\mathcal{S}|^2|\mathcal{R}|^2(|p_k|^2 + |q_k|^2)], \end{aligned} \quad (50)$$

with $\chi_k = \sqrt{1 + 6|\mathcal{S}\mathcal{U}|^2|\mathcal{R}|^2}|x_k|$.

From the inequality (3) and (31), we deduce that

$$\Delta V_{1,k} \leq -\frac{1}{4}|p_k|^2 - \frac{1}{4}|q_k|^2 + \kappa_1 \epsilon^2 |x_k|^2, \quad (51)$$

with $\kappa_1 = 2s(1 + 6n^2|\mathcal{S}|^2|\mathcal{R}|^2)$. Let $V_2(x) = x^\top Q_2 x$, where Q_2 is the symmetric and positive definite matrix in Assumption 2 and let $\Delta V_{2,k} = V_2(x_{k+1}) - V_2(x_k)$. Then, from Assumption 2, we deduce that

$$\begin{aligned} \Delta V_{2,k} &\leq -|x_k|^2 + 2|\mathcal{G} + \mathcal{A}_s(x_k)||Q_2||\beta||\mathcal{M}||x_k||p_k| \\ &\quad + |Q_2||\beta|^2|\mathcal{M}|^2|p_k|^2. \end{aligned} \quad (52)$$

From the triangle inequality, we deduce that

$$\Delta V_{2,k} \leq -\frac{1}{2}|x_k|^2 + \kappa_2 |p_k|^2 \quad (53)$$

with $\kappa_2 = [4(|\mathcal{G}|^2 + |\epsilon|^2)|Q_2| + 1]|Q_2||\beta|^2|\mathcal{M}|^2$. Let $V_3(p, q, x) = (4\kappa_2 + 1)V_1(p, q) + V_2(x)$ and $\Delta V_{3,k} = V_3(p_{k+1}, q_{k+1}, x_{k+1}) - V_3(p_k, q_k, x_k)$.

Then (51) and (53) imply that

$$\begin{aligned} \Delta V_{3,k} &\leq -\frac{4\kappa_2+1}{4}|p_k|^2 - \frac{4\kappa_2+1}{4}|q_k|^2 \\ &\quad + (4\kappa_2 + 1)\kappa_1 \epsilon^2 |x_k|^2 - \frac{1}{2}|x_k|^2 + \kappa_2 |p_k|^2. \end{aligned} \quad (54)$$

Since (30) ensures that

$$\begin{aligned} \epsilon &\leq \frac{b_3}{\sqrt{4[4(|\mathcal{G}|^2 + |\epsilon|^2)|Q_2| + 1]|Q_2||\beta|^2|\mathcal{M}|^2 + 1}}} \\ &= \frac{1}{2\sqrt{4\kappa_2 + 1}\sqrt{\kappa_1}}, \end{aligned} \quad (55)$$

with b_3 defined in (33), it follows that $\Delta V_{3,k} \leq -W(p_k, q_k, x_k)$, with $W(p, q, x) = \frac{1}{4}|p|^2 + \frac{4\kappa_2+1}{4}|q|^2 + \frac{1}{4}|x|^2$. Since the functions V_3 and W are positive definite quadratic functions, we can conclude. ■

V. ILLUSTRATIVE EXAMPLE

We illustrate Theorem 2 with the following system:

$$\begin{cases} x_{k+1} = \begin{bmatrix} c_1 \sin(x_{2,k}) & -1 \\ c_2 + 9 & \frac{1}{4} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ \quad + \begin{pmatrix} 0 \\ -\frac{1}{4} \sin(x_{1,k}) \end{pmatrix}, \\ y_k = x_{1,k}, \end{cases} \quad (56)$$

with $x_k = (x_{1,k} \ x_{2,k}) \in \mathbb{R}^2$, where $u_k \in \mathbb{R}$ is the input, $y_k \in \mathbb{R}$ is the output and where c_1 and c_2 are unknown real constants such that

$$|c_i| \leq \frac{3}{16\sqrt{166}}, \quad i = 1, 2. \quad (57)$$

Notice that this system is unstable when $u_k = 0$ for all $k \in \mathbb{N}$. Throughout this section, we use the symbols of Section IV. Then we prove that Theorem 2 applies with the choices:

$$\mathcal{A} = \begin{bmatrix} 0 & -1 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad \mathcal{A}_s(x) = \begin{bmatrix} c_1 \sin(x_{2,k}) & 0 \\ c_2 & 0 \end{bmatrix},$$

$$\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad \Phi(y) = \begin{pmatrix} 9y - \frac{0}{4} \sin y \end{pmatrix}.$$

Let us check that Assumptions 1 to 4 are satisfied. We observe that \mathcal{A} is a Schur stable matrix and that the inequality

$$\mathcal{A}^\top Q_1 \mathcal{A} - Q_1 \preceq -I \quad \text{with} \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{32}{15} \end{bmatrix}$$

is satisfied. The matrix \mathcal{A} is not nonnegative and one can check readily that the equality $\mathcal{R}\mathcal{A}\mathcal{R}^{-1} = \mathcal{E}$ is satisfied with

$$\mathcal{R} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Now, we choose $\mathcal{L} = 0$, $\mathcal{M} = 0$ and then the corresponding matrix \mathcal{G} is $\mathcal{G} = \mathcal{A}$. By using (57), one can prove that Assumption 2 is satisfied with $Q_2 = 2Q_1$. We observe that $\Phi(y) = \Omega(y)\beta$, where $\Omega(y) = 9y - \frac{1}{4} \sin(y)$. Thus Assumption 3 is satisfied. Now, we observe that $|\mathcal{A}_s(x)| \leq \sqrt{c_1^2 + c_2^2} \leq \frac{3}{16\sqrt{83}}$ for all $x \in \mathbb{R}^2$ and $|\mathcal{G}| \geq 1$. Next, through simple calculations, one can prove that Assumption 4 is satisfied with $\varepsilon = \frac{1}{44}$ because $\frac{1}{8\sqrt{6(2|Q_1\mathcal{A}|^2 + |Q_1|)}} > \frac{1}{44}$. It follows that Theorem 2 applies to (56) with the dynamic extension

$$\begin{cases} \hat{x}_{k+1}^+ = \mathcal{A}\hat{x}_k^+ + \beta u_k + \Phi(y_k) \\ \quad + \epsilon \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \max\{|\hat{x}_k^+|_p, |\hat{x}_k^-|_p\}, \\ \hat{x}_{k+1}^- = \mathcal{A}\hat{x}_k^- + \beta u_k + \Phi(y_k) \\ \quad - \epsilon \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \max\{|\hat{x}_k^+|_p, |\hat{x}_k^-|_p\}, \end{cases} \quad (58)$$

with $\epsilon = \frac{1}{44}$. Theorem 2 leads us to consider the systems (56)-(58) in closed-loop with the feedback

$$u(\hat{x}^+, y) = \mathcal{M}\hat{x}^+ - \Omega(y) = \frac{1}{4} \sin(y) - 9y. \quad (59)$$

This closed loop system admits the origin as a globally exponentially stable equilibrium point. Moreover, the system (58) associated with the initial conditions

$$\hat{x}_{k_0}^+ = N_1 x_{k_0}^+ + N_2 x_{k_0}^-, \quad \hat{x}_{k_0}^- = N_1 x_{k_0}^- + N_2 x_{k_0}^+, \quad (60)$$

with $N_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the bounds for the solutions x_k

$$x_k^+ = N_1 \hat{x}_k^+ + N_2 \hat{x}_k^-, \quad x_k^- = N_1 \hat{x}_k^- + N_2 \hat{x}_k^+, \quad (61)$$

is an exponentially stable interval observer for the system (56) in closed loop with the output feedback (59).

VI. CONCLUSION

We have proposed a new technique of construction of time-varying exponentially stable interval observers for discrete-time nonlinear time-invariant systems with uncertainties. A key advantage of this approach is the simplicity of the dynamics of the proposed interval observer: basically, it is composed of two copies of a classical Luenberger observer with extra terms whose presence is due to the uncertain terms.

ACKNOWLEDGMENT

The authors acknowledge the financial support from the Digiteo Project MOISYR - 2011-045D.

REFERENCES

- [1] K.J. Astrom, B. Wittenmark, *Computer Controlled Systems. Theory and Design*. Printice-Hall, Englewood Cliffs, NJ, 3rd edition, 1997.
- [2] O. Bernard, J.-L. Gouzé, *Closed Loop Observers Bundle for Uncertain Biotechnological Models*. J. Process Control, vol. 14, pp. 765-774, 2004.
- [3] C. Combastel, S.A. Raka, *A Stable Interval Observer for LTI Systems with No Multiple Poles*. 18th IFAC World Congress, Milano, Italy, Aug. 28-Sept. 2, 2011.
- [4] D. Efimov, W. Perruquetti, T. Raissi, A. Zolghadri, *On Interval Observer Design for Discrete Systems*. Submitted, 2012.
- [5] D. Efimov, T. Raissi, A. Zolghadri, *Stabilization of nonlinear uncertain systems based on interval observer*. 50th IEEE Conference on Decision and Control and European Control Conference, Dec. 12-15, Orlando, USA, 2011.
- [6] G. Goffaux, A. Vande Wouwer, O. Bernard, *Improving continuous-discrete interval observers with application to microalgae-based bioprocess*. Journal of Process Control, 19 (7), pp. 1182-1190, 2009.
- [7] J.-L. Gouzé, A. Rapaport, Z. Hadj-Sadok, *Interval observers for uncertain biological systems*. Ecological Modelling, 133, pp. 45-56, 2000.
- [8] W.M. Haddad, V. Chellaboina, Q. Hui, *Nonnegative and Compartmental Dynamical Systems*. Princeton University Press, 2012.
- [9] D.G. Luenberger, *An Introduction to observers*, IEEE Trans. on Automatic Control, 16, pp. 596-602, 1971.
- [10] F. Mazenc, O. Bernard, *Asymptotically Stable Interval Observers for Planar Systems with Complex Poles*, IEEE Trans. on Automatic Control, Vol. 55, Issue 2, pp. 523-527, Feb. 2010.
- [11] F. Mazenc, O. Bernard, *Interval observers for linear time-invariant systems with disturbances*. Automatica, Vol. 47, No. 1, pp. 140-147, Jan. 2011.
- [12] F. Mazenc, T.N. Dinh, S.-I. Niculescu, *Interval observers for discrete-time systems*. Submitted to International Journal of Robust and Nonlinear Control.
- [13] F. Mazenc, M. Kieffer, E. Walter, *Interval observers for continuous-time linear systems with discrete-time outputs*. 2012 American Control Conference, Montreal, Canada, June 27-June 29, pp. 1889-1894, 2012.
- [14] F. Mazenc, S.-I. Niculescu, O. Bernard, *Exponentially Stable Interval Observers for Linear Systems with Delay*. SIAM J. Control Optim., Vol. 50, pp. 286-305, 2012.
- [15] M. Moisan, O. Bernard, J.-L. Gouzé, *Near optimal interval observers bundle for uncertain bioreactors*, Automatica, Vol. 45, pp. 291-295, 2009.
- [16] T. Raissi, D. Efimov, A. Zolghadri, *Interval State Estimation for a Class of Nonlinear Systems*. IEEE Trans. on Automatic Control, Vol. 57, No.1, pp. 260-265, Jan. 2012.
- [17] T. Raissi, N. Ramdani, Y. Candau, *Bounded error moving horizon state estimation for non-linear continuous time systems: application to a bioprocess system*. Journal of Process Control, 15, pp. 537-545, 2005.
- [18] F.C. Scheppe, *Recursive state estimation: unknown but bounded errors and system inputs*. IEEE Trans. on Automatic Control, vol. 13, pp. 22-28, Feb. 1968.