Interval Observers for Continuous-time Bilinear Systems with Discrete-time Outputs

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***Abstract*— For a family of continuous-time nonlinear systems with input, output and disturbances in the case where measurements are available only at discrete instants, a new interval observer design is proposed. The family covers bilinear (and linear) systems with static output feedback as special cases. Employing the structure of Luenberger-type observers, we construct framers that allow us to determine a lower and an upper bounds of state variables in the presence of disturbances, provided that intervals in which uncertain initial conditions and uncertain disturbances lie are known. It is proved that the lower and upper bounds of state variables converge to each other completely when the maximum time interval of measurements is sufficiently small in the absence of disturbances.**

I. Introduction

The paradigm of interval observers originates in the work [9] which is very different from classical observers which give an asymptotic estimation of the state vector of a given dynamical system. The theory of interval observers utilizes properties of positive dynamical systems and gives an auxiliary dynamical system that furnishes an upper and a lower bound of the state vector to be estimated component-wise in the presence of uncertainties. Although design of interval observers requires bounds of the uncertainties/disturbances and bounds of the initial conditions to be known *a priori*, this requirement is accepted in many applications, and practical usefulness of the state estimation coping with large uncertainties has been confirmed in industrial plants and biological systems (see e.g. [2], [1], [8], [18]).

Some works on interval observers have been devoted to families of continuous-time and discrete-time linear systems, e.g. [13], [12], [6], [16]. Several families of continuous-time and discrete-time nonlinear systems have been also addressed in [5], [15], [21], [20]. Recently, continuous-discrete-time linear systems have been studied in [17], [14] where states of continuous-time linear systems are estimated based on discrete-time measurements, i.e., measurements through the sample and hold mechanism. To the best of the authors' knowledge, the continuous-time estimation through discrete-time measurements has not yet been addressed satisfactorily, especially for nonlinear systems. The work reported in [8] with an application to microalgae-based bioprocesses has motivated the authors to deal with nonlinear systems in continuous-time estimation through discrete-time measurements.

In many practical situations, measurements are available only at discrete instants from sensors, although systems by themselves run in continuous-time. In fact, the problem of determining observers for continuous-time systems with discrete-time measurements has been acknowledged as an important topic of research (see e.g. [3], [4], [10], [19], [11] to name a few). The purpose of the present paper is to tackle this problem in the framework of interval observers for nonlinear systems. An interval observer has been proposed for nonlinear system in [5], but the study does not readily tell how to build an interval observer when measurements are available only at discrete-time instants. This paper pursue an extension of the approach [5] to discrete-time measurements by focusing on a family of nonlinear systems covering bilinear systems.

The rest of this paper is organized as follows: Notation and definitions for mathematically formulating the problem of the interval observer design are given in Section II. Section III is devoted to the development of a novel interval observer, i.e., the statement of the main result, which is followed by Section IV presenting its proof. A numerical example illustrates the main development in Section V. Concluding remarks are drawn in Section VI.

II. Preliminaries

A. Basic notation and definitions

- $\|\cdot\|$ denotes the Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions.
- Any $k \times n$ matrix, whose entries are all 0 is denoted 0.
- $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers, i.e., $[0, +\infty)$.
- I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.
- The inequalities must be understood *component-wise* (partial order of $\mathbb{R}^r$) *i.e.* $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$.
- $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$.
- For any square matrix[1], $M \in \mathbb{R}^{r\times r}$, we let $M^+ = \max(M, 0)$, $M^- = M^+ - M$.

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[1]The superscripts $+$ and $-$ in this meaning are used only for square constant matrices. The superscripts for other purposes are defined appropriately when they appear.

- A matrix $M \in \mathbb{R}^{r\times s}$ is said to be nonnegative if all its entries are nonnegative.
- A matrix $M \in \mathbb{R}^{r\times r}$ is said to be Metzler if each off-diagonal entry of this matrix is nonnegative. The system $\dot{v} = Mv$ with $v \in \mathbb{R}^r$ is said to be cooperative if and only if M is Metzler.
- A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive definite if for all non-zero vectors $v \in \mathbb{R}^n$, the inequality $v^\top Mv > 0$ is satisfied and we denote $M \succ 0$.

The notation will be simplified whenever no confusion can arise from the context.

B. Interval observer: Definition

In order to avoid ambiguity in defining interval observers, this paper slightly modify the formulation proposed in [17], [5] to deal with disturbances at discrete-time measurements[2].

Definition 1: Consider a continuous-time dynamical system

$$\dot{x}(t) = f\left(t, x(t), u(t), \delta_1(t)\right) \quad \text{for } t \in \mathbb{R}_{\geq 0} \tag{1a}$$
$$y(t) = h(x(t_i), \delta_2(t_i)) \quad \text{for } t \in [t_i, t_{i+1}) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^q$ is the input, $y(t) \in \mathbb{R}^p$ is the measurement output and $\delta_1(t) \in \mathbb{R}^{\ell_1}$ and $\delta_2(t) \in \mathbb{R}^{\ell_2}$ are disturbances[3]. The measuring instants t_i form an increasing sequence $\mathcal{T} := \{t_i \in \mathbb{R}_{\geq 0} : i = 0, 1, 2, ...\}$ with $t_0 = 0$ and are such that there are two constants $\epsilon > 0$, $\tau > \epsilon$ satisfying

$$t_{i+1} - t_i \in [\epsilon, \tau], \quad \forall i = 0, 1, 2, ... \tag{2}$$

Assume that for each initial condition[4] $x_0 := x(t_0) \in \mathbb{R}^n$, there exists a unique solution $x(t)$ to (1a) for all $t \in \mathbb{R}_{\geq 0}$, i.e., the system (1a) is forward complete. Let $x_0^-, x_0^+ \in \mathbb{R}^n$ and $\delta_{\max} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be such that

$$x_0^- \leq x_0 \leq x_0^+ \tag{3}$$
$$\|\delta_1(t)\| \leq \delta_{\max}(t), \ \|\delta_2(t)\| \leq \delta_{\max}(t), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{4}$$

Then the continuous-time dynamical system

$$\dot{\hat{x}}(t) = f_1(t, \hat{x}(t), \hat{x}(t_i), y(t), u(t)) \tag{5a}$$
$$\dot{\overline{m}}(t) = f_2^+\left(t, \overline{m}(t), \underline{m}(t), y(t), u(t), \delta_{\max}(t)\right) \tag{5b}$$
$$\dot{\underline{m}}(t) = f_2^-\left(t, \overline{m}(t), \underline{m}(t), y(t), u(t), \delta_{\max}(t)\right) \tag{5c}$$

defined for $t \in [t_i, t_{i+1})$ and nonnegative integers $i = 0, 1, 2, ...$ with

$$x^+(t) = h^+(t, \hat{x}(t), \overline{m}(t), \underline{m}(t)) \in \mathbb{R}^n \tag{6a}$$
$$x^-(t) = h^-(t, \hat{x}(t), \overline{m}(t), \underline{m}(t)) \in \mathbb{R}^n \tag{6b}$$

for the initial condition

$$\hat{x}(t_0) := \hat{x}_0 \in \mathbb{R}^n \tag{7a}$$
$$\overline{m}(t_0) := \overline{m}_0 = g^+(t_0, \hat{x}_0, x_0^+, x_0^-) \in \mathbb{R}^r \tag{7b}$$
$$\underline{m}(t_0) := \underline{m}_0 = g^-(t_0, \hat{x}_0, x_0^+, x_0^-) \in \mathbb{R}^r \tag{7c}$$

is called

(i) a framer for (1) if for any vectors $x_0, x_0^+, x_0^- \in \mathbb{R}^n$ satisfying (3), the unique solutions $\hat{x}(t)$, $\overline{m}(t)$ and $\underline{m}(t)$ to (5) with (7) exist and satisfy

$$x^-(t) \leq x(t) \leq x^+(t), \quad t \in \mathbb{R}_{\geq 0}. \tag{8}$$

(ii) an interval observer for (1) if, in addition, $\lim_{t\to+\infty} \|x^+(t) - x^-(t)\| = 0$ holds, provided that $\delta_{\max}(\cdot)$ is identically zero.

It is stressed that $x(t)$, $\delta_1(t)$ and $\delta_2(t)$ are not used in the interval observer (5)-(7), which means that they are unknown. Instead, we assume that x_0^-, x_0^+ and $\delta_{\max}(t)$ are known *a priori* and $y(t)$ and $u(t)$ are measured in building the interval observer.

As the first step to tackle interval observer design for general nonlinear systems with discrete-time measurements, this paper focuses on a special class of f in (1) by requiring the from $f(t, x(t), u(t), \delta_1(t)) = f_1(y(t))x(t) + f_2(t, y(t), u(t), \delta_1(t))$. Although the special class restricts the nonlinearities in (1a) to those of discrete-time measurement outputs, it covers bilinear (and linear) systems with static output feedback. To illustrate this, consider the continuous-time bilinear system

$$\dot{x}_1(t) = -2x_1(t) + x_2(t) + \delta_{1,1}(t) \tag{9a}$$
$$\dot{x}_2(t) = x_2(t) + x_2(t)u(t) + \delta_{1,2}(t) \tag{9b}$$
$$y(t) = x_1(t_i). \tag{9c}$$

Applying the (discrete-time, i.e., sample and hold) static output feedback

$$u(t) = -3 - k_1 \mathrm{sat}(k_2|y(t)|) \tag{10}$$

to (9) results in

$$\dot{x}_1(t) = -2x_1(t) + x_2(t) + \delta_{1,1}(t) \tag{11a}$$
$$\dot{x}_2(t) = -(2 + k_1 \mathrm{sat}(k_2|y(t)|))x_2(t) + \delta_{1,2}(t) \tag{11b}$$
$$y(t) = x_1(t_i), \tag{11c}$$

where $\mathrm{sat}(s) := \min\{1, \max\{-1, s\}\}$ for $s \in \mathbb{R}$. The second term in the output feedback (10) is practically useful in the sense that it can drive x_2 to the origin quickly if large $k_1, k_2 > 0$ are chosen appropriately taking account of actuator limitations and the interval of measurement instants t_i. This example of a bilinear system is not covered in [17], although it is simple.

III. Main Result

This paper focus on the system (1) in the following form:

$$\dot{x}(t) = \alpha(y(t))x(t) + \beta(y(t), u(y(t), t)) + \delta_1(t), \quad \text{for } t \in \mathbb{R}_{\geq 0} \tag{12a}$$
$$y(t) = Cx(t_i) + \delta_2(t_i), \quad \text{for } t \in [t_i, t_{i+1}) \tag{12b}$$

[2]By the discrete-time measurements, we mean that the measurement output defined by (1b) is piecewise constant.

[3]In (1b), the disturbance δ_2 which resides in the system physically in continuous-time affects the measurement in discrete-time.

[4]Definition 1 employs the initial time $t_0 = 0$ for the sake of notational simplicity throughout this paper. To let $t_0 \neq 0$, $t \in \mathbb{R}_{\geq 0}$ can be just replaced by with $t \in [t_0, \infty)$ throughout this paper.

with $x(t) \in \mathbb{R}^n$, $y(t) \in R^p$, $u(y(t),t) \in R^q$, $\delta_1(t) \in \mathbb{R}^n$, $\delta_2(t) \in \mathbb{R}^p$ and $C \in \mathbb{R}^{p\times n}$. Notice that (12) covers the motivating example (11). Indeed, the function $\alpha(y(t))$ in (12a) can be decomposed into

$$\alpha(y(t))x(t) = Ax(t) + \sum_{j=1}^{n} J_j(y(t))x_j(t),$$

where $\alpha(y) = A+[J_1, J_2, \ldots, J_n]$. The constant matrix $A \in \mathbb{R}^{n\times n}$ represents the linear autonomous part. The vector

$$J_i(y(t)) = [J_{1,j}(y(t)), J_{2,j}(y(t)), \cdots, J_{n,j}(y(t))]^\top \in \mathbb{R}^n$$

describes the bilinear input channel with respect to the state component $x_j(t) \in \mathbb{R}$, where each $J_{1,j}(t) \in \mathbb{R}$ represents an input coefficient multiplied by a static feedback law based on the measurement output $y \in \mathbb{R}^p$. It is stressed that $x(t)$ is the continuous (continuously updated) state, while $y(t)$ is the discrete-time (discontinuously updated, i.e., sample and hold) output. The function $\beta(y(t), u(y(t),t))$ is added to (12a) for describing the linear input channel whose input is supposed to be a static feedback law based on the measurement output $y \in \mathbb{R}^p$. Thus, a typical case is $\beta(y(t), u(y(t),t)) = Bu(y(t),t)$ with a constant matrix $B \in \mathbb{R}^{n\times q}$.

Assumption 1: Given a measurement sequence $\mathcal{T}$ and a feedback control law $u(w,\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ which is piecewise continuous for all $w \in \mathbb{R}^p$, for any piecewise continuous functions $\delta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\delta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, and any $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$ to (12a) for all $t \in \mathbb{R}_{\geq 0}$, i.e., system (12a) is forward complete. Furthermore, $\delta_{\max} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in (4) is piecewise continuous.

In contrast to the assumption made in [5] dealing with continuous-time measurements, this paper does not impose Lipschitzness on the functions on the right hand side of (12). This relaxation is possible since the measurement output y is piecewise constant in this paper. This paper makes use of the following assumption.

Assumption 2: There exist a positive constant η, a continuous function $\lambda : \mathbb{R}^p \to \mathbb{R}$, an invertible constant matrix $R \in \mathbb{R}^{n\times n}$ and a positive definite constant matrix $P \in \mathbb{R}^{n\times n}$ such that, for all $w \in \mathbb{R}^p$,

(i) $0 < \|\lambda(w)C\| \leq \eta$

(ii) The matrix

$$\Gamma(w) = R[\alpha(w) + \lambda(w)C]R^{-1} \tag{13}$$

is Metzler.

(iii) $[\alpha(w) + \lambda(w)C]^\top P + P[\alpha(w) + \lambda(w)C] \preceq -I$

It is emphasized that $\Gamma(w)$ is allowed to be a function of w. In (13), the matrix $\Gamma(w)$ defined with each fixed w is required to be Metzler[5]. Detectability of the system (12) is implied by Assumption 2 (iii). In addition, the detectability is required to be uniform in the sense that the Lyapunov inequality in (iii) is satisfied with a common P for all $w \in \mathbb{R}^p$. Item (ii) in Assumption 2 which is inspired by [5] provides the key to the structure of the observer candidate to be proposed in this paper. In order to guarantee the candidate to meet Definition 1 (i) and (ii), together with Items (i) and (iii) in Assumption 2, this paper introduces the following assumption on the maximum time interval between two consecutive measurement, which is inspired by [17].

Assumption 3: The function $\alpha : \mathbb{R}^p \to \mathbb{R}^n$ is bounded and there exists a real number $\alpha_* \in [a_m, +\infty)$ satisfying $\alpha_* > 0$ such that the constant τ introduced in (2) satisfies

$$0 < \tau \leq \frac{1}{\alpha_*} \ln\left(\frac{1}{2} + \Theta\right) \tag{14}$$

where

$$\Theta = \frac{1}{2}\min\left\{1 + \frac{\alpha_*}{\eta},\ \sqrt{1 + \frac{2\alpha_*}{q_m\|R\|^3\|R^{-1}\|\eta(2\eta+\alpha_*)}}\right\} \tag{15}$$

and $a_m = \sup_{w\in\mathbb{R}^p}\|\alpha(w)\|$ and $q_m = \|(R^{-1})^\top PR^{-1}\|$.

The smaller the norm of P achieving (iii) in Assumption 2 is, the bigger the maximum time interval τ allowed by (14) is. Define a matrix for $s \in \mathbb{R}_{\geq 0}$, $t \in [s, s+\tau]$ and $w \in \mathbb{R}^p$ by

$$\Xi(t,s,w) = \left[e^{\alpha(w)(t-s)} + \int_s^t e^{\alpha(w)(t-\ell)}\lambda(w)Cd\ell\right]. \tag{16}$$

With the help of Assumption 2 (i) and (iii), Assumption 3 allows one to invoke [17, Lemma 2] to prove the following.

Lemma 1: Let τ be a real number satisfying (14). Then for all $s \in \mathbb{R}_{\geq 0}$, $t \in [s, s+\tau]$ and $w \in \mathbb{R}^p$, the matrix $\Xi(t,s,w)$ is invertible and satisfies

$$\|\Xi^{-1}(t,s,w) - I\| \leq \left(2\frac{\|\lambda(w)C\|}{\alpha_*} + 1\right)(e^{\alpha_*\tau} - 1)\, e^{\alpha_*\tau} \tag{17}$$

$$\|\Xi^{-1}(t,s,w)\| \leq 2e^{\alpha_*\tau}\ . \tag{18}$$

It is stressed that the first term in (15) is sufficient for establishing Lemma 1. The remainder in (15) will be used later on. Due to the invertibility secured by Lemma 1, we can define

$$\Psi(t,s,w) = R\lambda(w)C(\Xi^{-1}(t,s,w) - I)R^{-1} \tag{19}$$

for $s \in \mathbb{R}_{\geq 0}$, $t \in [s, s+\tau]$ and $w \in \mathbb{R}^p$. Let $\gamma_{k,l}(w)$ (resp. $\psi_{k,l}(t,s,w)$) denote the (k,l)-entry of the matrix $\Gamma(w)$ (resp. $\Psi(t,s,w)$), which is expressed by $\Gamma(w) = (\gamma_{k,l}(w))_{k=1,l=1}^{n,n}$ (resp. $\Psi(t,s,w) = (\psi_{k,l}(t,s,w))_{k=1,l=1}^{n,n}$). Using

$$\overline{\psi}_{k,l}(t,s,w) = \begin{cases} \psi_{k,l}(t,s,w), & \text{if } k = l \text{ or} \\ & \quad \gamma_{k,l} + \psi_{k,l}(t,s,w) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

for $k,l = 1,2,...,n$, we define matrices $\overline{\Psi}(t,s,w)$, $\underline{\Psi}(t,s,w) \in \mathbb{R}^{n\times n}$ by

$$\overline{\Psi}(t,s,w) = \left(\overline{\psi}_{k,l}(t,s,w)\right)_{k=1,l=1}^{n,n} \tag{21a}$$

$$\underline{\Psi}(t,s,w) = \overline{\Psi}(t,s,w) - \Psi(t,s,w) \tag{21b}$$

[5]Some conditions of the existence of R as in (13) are discussed in [7].

for $s \in \mathbb{R}_{\geq 0}$, $t \in [s, s+\tau]$ and $w \in \mathbb{R}^p$. For $t \in [t_i, t_{i+1})$, define $\delta_*(t) \in \mathbb{R}^n$ as

$$\delta_*(t) = \|R\|(1 + \|\lambda(y(t_i))\|)\Delta(t, y(t_i))[1, ..., 1]^\top \tag{22}$$

$$\Delta(t) = \delta_{\max}(t) + 2\|\lambda(y(t_i))C\|e^{2\alpha_*\tau}\int_{\max\{t_0, t-\tau\}}^{t} \delta_{\max}(\ell)d\ell,$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}_p$ is the unique solution to (12) given with x_0 and δ_1, δ_2. Note that in each time interval $t \in [t_i, t_{i+1})$, the function $\delta_*(t)$ can be computed, i.e., updated at the information available at $t = t_i$ since $y(t_i)$ is measured and the entire function $\delta_{\max} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is given *a priori*. It is stressed that x_0 and δ_1 and δ_2 are not needed to obtain δ_*. In each interval of $t \in [t_i, t_{i+1})$, the function δ_* is piecewise continuous since $\delta_{\max} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is piecewise continuous and $\lambda : \mathbb{R}^p \to \mathbb{R}$ is continuous.

Now we are ready to state the main result.

Theorem 1: Suppose that Assumptions 1, 2 and 3 hold. Consider the system given by

$$\begin{aligned}\dot{\hat{x}}(t) =& \alpha(y(t))\hat{x}(t) + \beta(y(t), u(y(t), t)) \\ &+ \lambda(y(t))[C\hat{x}(t_i) - y(t)] \end{aligned}\tag{23a}$$

$$\begin{aligned}\dot{\overline{m}}(t) =& [\Gamma(y(t)) + \overline{\Psi}(t, t_i, y(t))]\overline{m}(t) - \underline{\Psi}(t, t_i, y(t))\underline{m}(t) \\ &+ S\delta_*(t) \end{aligned}\tag{23b}$$

$$\begin{aligned}\dot{\underline{m}}(t) =& [\Gamma(y(t)) + \overline{\Psi}(t, t_i, y(t))]\underline{m}(t) - \underline{\Psi}(t, t_i, y(t))\overline{m}(t) \\ &- S\delta_*(t) \end{aligned}\tag{23c}$$

in the interval $t \in [t_i, t_{i+1})$ for each $i = 1, 2, ...$ with the initial conditions

$$\begin{bmatrix} \hat{x}(0) \\ \overline{m}(0) \\ \underline{m}(0) \end{bmatrix} = \begin{bmatrix} \hat{x}_0 \\ \overline{m}_0 \\ \underline{m}_0 \end{bmatrix} := \begin{bmatrix} \hat{x}_0 \\ S\left[R^+\mathcal{E}_0^+ - R^-\mathcal{E}_0^-\right] \\ S\left[R^+\mathcal{E}_0^- - R^-\mathcal{E}_0^+\right] \end{bmatrix}, \tag{23d}$$

where $\mathcal{E}_0^+ = x_0^+ - \hat{x}_0$, $\mathcal{E}_0^- = x_0^- - \hat{x}_0$, and $S = R^{-1}$. Then the above equations (23) with

$$x^+(t) = \hat{x}(t) + S\left[S^+R\overline{m}(t) - S^-R\underline{m}(t)\right] \tag{24a}$$

$$x^-(t) = \hat{x}(t) + S\left[S^+R\underline{m}(t) - S^-R\overline{m}(t)\right] \tag{24b}$$

is an interval observer for the system (12).

IV. Proof

This section presents the proof of Theorem 1.

First, note that endogenous signals in (23a)-(23c) are $\hat{x}$, $\overline{m}$ and $\underline{m}$. The differential equation given by (23a)-(23c) is linear in these endogenous variables $\hat{x}$, $\overline{m}$ and $\underline{m}$. Recall that signal y is generated by (12), but it is constant in each time interval $[t_i, t_{i+1})$. By definition, the coefficient and the other functions in (23a)-(23c) are exogenous and piecewise continuous (including constant) in t during the interval $[t_i, t_{i+1})$. Thus, for any x_0, x_0^+, $x_0^- \in \mathbb{R}^n$, there exists a unique solution $(\hat{x}(t), \overline{m}(t), \overline{m}(t))$ to (23a)-(23c) for all $t \in \mathbb{R}_{\geq 0}$.

Next, define $\mathcal{E}(t) = x(t) - \hat{x}(t)$ which denotes the error between trajectories of (12a) and (23a). Then its time-derivative satisfies

$$\begin{aligned}\dot{\mathcal{E}}(t) =& \alpha(y(t))x(t) - \alpha(y(t))\hat{x}(t) \\ &- \lambda(y(t))[C\hat{x}(t_i) - y(t)] + \delta_1(t) \\ =& \alpha(y(t))\mathcal{E}(t) + \lambda(y(t))C\mathcal{E}(t_i) + \delta_1(t) + \lambda(y(t))\delta_2(t_i)\end{aligned}\tag{25}$$

for all $t \in [t_i, t_{i+1})$. Since $\alpha(y(t))$ is constant over $[t_i, t_{i+1})$, integrating both sides of (25) yields , for all $t \in [t_i, t_{i+1})$,

$$\begin{aligned}\mathcal{E}(t) =& e^{\alpha(y(t))(t-t_i)}\mathcal{E}(t_i) \\ &+ \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\lambda(y(t))C\mathcal{E}(t_i) + \delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell \\ =& e^{\alpha(y(t))(t-t_i)}\mathcal{E}(t_i) \\ &+ \left(\int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\lambda(y(t))Cd\ell\right)\mathcal{E}(t_i) \\ &+ \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell \\ =& \left[e^{\alpha(y(t))(t-t_i)} + \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\lambda(y(t))Cd\ell\right]\mathcal{E}(t_i) \\ &+ \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell \\ =& \Xi(t, t_i, y(t))\mathcal{E}(t_i) \\ &+ \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell\ ,\end{aligned}$$

where $\Xi(t, s, y(t))$ is given by (16). Since $\Xi(t, s, y(t))$ is invertible, due to Lemma 1, we have

$$\begin{aligned}\mathcal{E}(t_i) =& \Xi^{-1}(t, t_i, y(t_i)) \\ &\times \left(\mathcal{E}(t) - \int_{t_i}^{t} e^{\alpha(y(t_i))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t_i))\delta_2(t_i)\right)d\ell\right).\end{aligned}\tag{26}$$

By substituting (26) in (25), the error equation is rewritten for all $t \in [t_i, t_{i+1})$ as

$$\begin{aligned}\dot{\mathcal{E}}(t) =& \left[\alpha(y(t)) + \lambda(y(t))C\Xi^{-1}(t, t_i, y(t))\right]\mathcal{E}(t) \\ &+ \delta_1(t) + \lambda(y(t))\delta_2(t_i) - \lambda(y(t))C\Xi^{-1}(t, t_i, y(t)) \\ &\times \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell \\ =& \left[\alpha(y(t)) + \lambda(y(t))C\Xi^{-1}(t, t_i, y(t))\right]\mathcal{E}(t) \\ &+ \delta_3(t, t_i, y(t))\end{aligned}\tag{27}$$

where

$$\begin{aligned}&\delta_3(t, t_i, y(t)) = \\ &\quad \delta_1(t) + \lambda(y(t))\delta_2(t_i) - \lambda(y(t))C\Xi^{-1}(t, t_i, y(t)) \\ &\quad \times \int_{t_i}^{t} e^{\alpha(y(t))(t-\ell)}\left(\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\right)d\ell\end{aligned}\tag{28}$$

is defined for $t \in [t_i, t_{i+1})$. Let

$$m(t) = R\mathcal{E}(t). \tag{29}$$

Through elementary calculations, we obtain

$$\begin{aligned}\dot{m}(t) =& R\dot{\mathcal{E}}(t)\\ =& R\left[\alpha(y(t)) + \lambda(y(t))C\Xi^{-1}(t,t_i,y(t))\right]R^{-1}m(t)\\ &+ R\delta_3(t,t_i,y(t))\\ =& R\left[\alpha(y(t)) + \lambda(y(t))C\right.\\ &\left.+\lambda(y(t))C(\Xi^{-1}(t,t_i,y(t)) - I)\right]R^{-1}m(t)\\ &+ R\delta_3(t,t_i,y(t))\\ =& \left[\Gamma(y(t)) + \Psi(t,t_i,y(t))\right]m(t) + R\delta_3(t,t_i,y(t))\\ =& \left[\Gamma(y(t)) + \overline{\Psi}(t,t_i,y(t))\right]m(t) - \underline{\Psi}(t,t_i,y(t))m(t)\\ &+ R\delta_3(t,t_i,y(t)) \end{aligned} \quad (30)$$

for $t \in [t_i, t_{i+1})$, where $\Psi(t,t_i,y(t))$, $\overline{\Psi}(t,t_i,y(t))$ and $\underline{\Psi}(t,t_i,y(t))$ are given by (19), (21a) and (21b), respectively. By the definitions (20), (21a) and Assumption 2 (ii), $\Gamma(y)+\overline{\Psi}$ is Metzler. By the definitions (21a) and (21b), all functions in $\overline{\Psi}$ and $\underline{\Psi}$ are nonnegative.

On the other hand, from (28) and (18) in Lemma 1 one can check readily that for all $t \in [t_i, t_{i+1})$,

$$\begin{aligned}\|R\delta_3(t,t_i,y(t))\| \le& \|R\|\|\delta_1(t) + \lambda(y(t))\delta_2(t_i)\|\\ &+ 2\|R\|\|\lambda(y(t))C\|e^{2\alpha_*\tau}\\ &\times \int_{t_i}^{t} \|\delta_1(\ell) + \lambda(y(t))\delta_2(t_i)\| d\ell\ .\end{aligned}$$

From (4) and (22) it is verified that

$$-\delta_*(t) \le R\delta_3(t,t_i,y(t)) \le \delta_*(t) \quad (31)$$

holds for all $t \in [t_i, t_{i+1})$ at all $i = 1, 2, ...$

Now, consider vectors $x_0, x_0^+, x_0^-, \hat{x}_0, \overline{m}_0, \underline{m}_0 \in \mathbb{R}^n$ satisfying (3) and (23d). It can be verified that

$$R^+x_0^- - R^-x_0^+ \le Rx_0 \le R^+x_0^+ - R^-x_0^-.$$

Then from defining $\mathcal{E}_0 = x_0 - \hat{x}_0$ and $m_0 = SR\mathcal{E}_0 = \mathcal{E}_0$ it follows that

$$R\underline{m}_0 \le Rm_0 \le R\overline{m}_0. \quad (32)$$

From (23b), (23c) and (30) it follows that

$$\begin{aligned}R\dot{\overline{m}}(t) - R\dot{m}(t) =& [\Gamma(y(t)) + \overline{\Psi}(t,t_i,y(t))](R\overline{m}(t) - Rm(t))\\ &+ \underline{\Psi}(t,t_i,y(t))(Rm(t) - R\underline{m}(t))\\ &+ \delta_*(t) - R\delta_3(t,t_i,y(t)) \end{aligned} \quad (33a)$$

$$\begin{aligned}R\dot{m}(t) - R\dot{\underline{m}}(t) =& [\Gamma(y(t)) + \overline{\Psi}(t,t_i,y(t))](Rm(t) - R\underline{m}(t))\\ &+ \underline{\Psi}(t,t_i,y(t))(R\overline{m}(t) - Rm(t))\\ &+ R\delta_3(t,t_i,y(t)) + \delta_*(t) \end{aligned} \quad (33b)$$

in the interval $t \in [t_i, t_{i+1})$ for all $i = 1, 2, ...$ Recall that $\Gamma(y(t)) + \overline{\Psi}(t,t_i,y(t))$ is Metzler and that $\underline{\Psi}(t,t_i,y(t))$ is nonnegative. From (31) and (32) it follows that $R\underline{m}(t) \le Rm(t) \le R\overline{m}(t)$ for all $t \in [t_0, t_1)$. By induction, we obtain

$$R\underline{m}(t) \le Rm(t) \le R\overline{m}(t), \quad \forall t \in [t_i, t_{i+1})$$

for any nonnegative integer i. Consequently, we arrive at

$$R\underline{m}(t) \le Rm(t) \le R\overline{m}(t), \quad \forall t \in \mathbb{R}_{\ge 0}.$$

Then It can be verified that

$$S^+R\underline{m}(t) - S^-R\overline{m}(t) \le m(t) \le S^+R\overline{m}(t) - S^-R\underline{m}(t),\quad \forall t \in \mathbb{R}_{\ge 0}. \quad (34)$$

By virtue of (24) and (29), multiplying both sides of (34) by S from left, we have, for all $t \in \mathbb{R}_{\ge 0}$,

$$x^- - \hat{x} \le \mathcal{E} \le x^+ - \hat{x} \quad (35)$$

which is equivalent to (8). Therefore, Definition 1 (i) is proved.

Finally, to prove Definition 1 (ii), let $\delta_{\max}(t) \equiv 0$ which implies $\delta_*(t) \equiv 0$ in (22). Define $\widetilde{m} = \overline{m} - \underline{m}$. Then, from (23b) and (23c) we have

$$\dot{\widetilde{m}}(t) = \left[\Gamma(y(t)) + \overline{\Psi}(t,t_i,y(t)) + \underline{\Psi}(t,t_i,y(t))\right]\widetilde{m}(t),\quad \forall t \in [t_i, t_{i+1}) \quad (36)$$

for any nonnegative integer i. Define a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ by $Q := (R^{-1})^\top PR^{-1}$, where P is given in Assumption 2 (iii). Let

$$V(\zeta) = \zeta^\top Q\zeta \quad (37)$$

for $\zeta \in \mathbb{R}^n$. The right derivative of $V(\widetilde{m}(t))$ with respect to time t along the trajectories of (36) satisfies

$$\begin{aligned}\dot{V}(t) = 2\widetilde{m}(t)^\top Q&\left[\Gamma(y(t))\right.\\ &\left.+\overline{\Psi}(t,t_i,y(t)) + \underline{\Psi}(t,t_i,y(t))\right]\widetilde{m}(t)\end{aligned}$$

in the interval $t \in [t_i, t_{i+1})$ for any nonnegative integer i. Since Assumption 2 (iii) implies that $\Gamma(w)^\top Q + Q\Gamma(w) \preceq -(R^{-1})^\top R^{-1}$ holds for all $w \in \mathbb{R}^p$, from (19) we obtain

$$\begin{aligned}\dot{V}(t) \le& -\|R\|^{-2}\|\widetilde{m}(t)\|^2\\ &+ 2\|Q\|\|\overline{\Psi}(t,t_i,y(t)) + \underline{\Psi}(t,t_i,y(t))\|\|\widetilde{m}(t)\|^2\\ \le& -\|R\|^{-2}\|\widetilde{m}(t)\|^2 + 2\|Q\|\|\Psi(t,t_i,y(t))\|\|\widetilde{m}(t)\|^2\\ \le& -\|R\|^{-2}\|\widetilde{m}(t)\|^2\\ &+ 2\|Q\|\|R\lambda(y(t))C(\Xi^{-1}(t,t_i,y(t)) - I)R^{-1}\|\\ &\times \|\widetilde{m}(t)\|^2\\ \le& -\|R\|^{-2}\|\widetilde{m}(t)\|^2 + 2\|Q\|\|R\|\|R^{-1}\|\|\lambda(y(t))C\|\\ &\times \|(\Xi^{-1}(t,t_i,y(t)) - I)\|\|\widetilde{m}(t)\|^2 \end{aligned} \quad (38)$$

for $t \in [t_i, t_{i+1})$. Recall that Assumptions 2 and 3, i.e., the second term in (15), imply

$$\begin{aligned}\tau \le& \frac{1}{\alpha_*}\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2\alpha_*}{q_m\|R\|^3\|R^{-1}\|\eta(2\eta + \alpha_*)}}\right)\\ \le& \frac{1}{\alpha_*}\ln\left(\tfrac{1}{2}\right.\\ &\left.+\tfrac{1}{2}\sqrt{1+\tfrac{2\alpha_*}{\|Q\|\|R\|^3\|R^{-1}\|\|\lambda(y(t))C\|(2\|\lambda(y(t))C\|+\alpha_*)}}\right) \end{aligned} \quad (39)$$

for all $t \in [t_i, t_{i+1})$. From (17) in Lemma 1 and (39) it follows that

$$\begin{aligned}\dot{V}(t) \leq & -\|R\|^{-2}\|\widetilde{m}(t)\|^2 + 2\|Q\|\|R\|\|R^{-1}\|\|\lambda(y(t))C\| \\ & \times \left(2\frac{\|\lambda(y(t))C\|}{\alpha_*}+1\right)(e^{\alpha_*\tau}-1)\, e^{\alpha_*\tau}\|\widetilde{m}(t)\|^2 \\ \leq & -\|R\|^{-2}\|\widetilde{m}(t)\|^2 + \frac{1}{2}\|R\|^{-2}\|\widetilde{m}(t)\|^2 \\ \leq & -\frac{1}{2}\|R\|^{-2}\|\widetilde{m}(t)\|^2 \qquad (40)\end{aligned}$$

holds for all $t \in [t_i, t_{i+1})$. Since (40) is satisfied for any non-negative integer i, we have $\dot{V}(t) \leq -\frac{1}{2}\|R\|^{-2}\|\widetilde{m}(t)\|^2$ for all $t \in \mathbb{R}_{\geq 0}$. Therefore, $\lim_{t\to+\infty}\|\widetilde{m}\| = \lim_{t\to+\infty}\|\overline{m}(t) - \underline{m}(t)\| = 0$. Hence, using (24) we arrive at Definition 1 (ii). This allows us to conclude the proof.

V. An Illustrative Example

This section illustrates Theorem 1 with the two dimensional system :

$$\dot{x}_1(t) = x_2(t) + c\sin(t), \qquad t \in \mathbb{R}_{\geq 0} \quad (41a)$$

$$\dot{x}_2(t) = -x_1(t)+k(t)\sin(y(t))+c\sin(t), \quad t \in \mathbb{R}_{\geq 0} \quad (41b)$$

$$y(t) = x_1(t_i) + c\sin(t_i), \qquad t \in [t_i, t_{i+1}) \quad (41c)$$

with $x(t) = [x_1(t), x_2(t)]^\top \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, where $\delta_1(t) = \begin{bmatrix} c\sin(t) \\ c\sin(t) \end{bmatrix} \in \mathbb{R}^2$ and $\delta_2(t) = c\sin(t) \in \mathbb{R}$ with a non-negative constant c. The term $k(t)\sin(y(t))$ represents a discrete-time signal of output static feedback, where k is a piecewise continuous function of a time-varying gain in discrete-time. With $\alpha(y) = [0, 1; -1, 0]$ and $u = k$, the equations (41) are represented in the form of (12). Due to the Lipschitzness of the right hand sides of (41a) and (41b) and the definition of δ_1 and δ_2, Assumption 1 and (4) are satisfied with $\delta_{\max}(t) = c$ for all $t \in \mathbb{R}_{\geq 0}$. Choosing $\lambda(w) = [-3\ 0]^\top$ in Assumption 2, we obtain

$$\alpha(w) + \lambda(w)C = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Although the matrix $\alpha(w)+\lambda(w)C$ is not Metzler, Assumption 2 (ii) is satisfied with $R = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ since

$$\Gamma(w) = R[\alpha(w) + \lambda(w)C]R^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix},$$

which is Metzler. Moreover, the positive definite matrix $P = \begin{bmatrix} 7 & -\frac{2}{3} \\ -\frac{2}{3} & 7 \end{bmatrix}$ satisfies Assumption 2 (iii). We also obtain Assumption 2 (i) with $\eta := 3 = \|\lambda(w)C\|$. Through simple calculations, it is verified from $q_m = \|(R^{-1})^\top PR^{-1}\| = 16.772$ that Assumption 3 holds with $\alpha_* := 1 = a_m = \|\alpha(w)\|$ and $\tau = 5.4 \times 10^{-4}$. On the other hand, using

$$e^{\alpha(w)(t-t_i)} = \begin{bmatrix} \cos(t-t_i) & \sin(t-t_i) \\ -\sin(t-t_i) & \cos(t-t_i) \end{bmatrix},$$

one can verify that

$$\begin{aligned}\Psi(t, t_i, y) = & \frac{1}{1 - 3\sin(t - t_i)} \\ & \times \begin{bmatrix} -3\cos(t-t_i) - 6\sin(t-t_i) + 3 & 3\sin(t-t_i) \\ 3\cos(t-t_i) + 6\sin(t-t_i) - 3 & -3\sin(t-t_i) \end{bmatrix}.\end{aligned}$$

Since $0 \leq t - t_i \leq \tau = 5.4 \times 10^{-4}$, the matrix $\Psi(t, t_i)$ is Metzler. From (20) and (21) it follows that

$$\overline{\Psi}(t, t_i, y) = \Psi(t, t_i, y)\ ,\ \underline{\Psi}(t, t_i, y) = 0\ .$$

From (22) one can obtain

$$\delta_*(t) = \begin{bmatrix} 8c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \\ 8c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \end{bmatrix}.$$

Since Assumptions 1, 2 and 3 are satisfied, Theorem 1 applies and provides us with the following interval observer :

$$\begin{cases} \dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} 0 \\ k(t)\sin(y(t)) \end{bmatrix} \\ \qquad + \begin{bmatrix} -3 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \end{bmatrix}\hat{x}(t_i) - y(t)\right) \\ \dot{\overline{m}}(t) = \frac{1}{1-3\sin(t-t_i)} \\ \qquad \times \begin{bmatrix} 1 - 3\cos(t-t_i) & 1 \\ 3\left(\cos(t-t_i) + \sin(t-t_i)\right) - 2 & -1 \end{bmatrix}\overline{m}(t) \\ \qquad + \begin{bmatrix} 8c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \\ 16c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \end{bmatrix} \\ \dot{\underline{m}}(t) = \frac{1}{1-3\sin(t-t_i)} \\ \qquad \times \begin{bmatrix} 1 - 3\cos(t-t_i) & 1 \\ 3\left(\cos(t-t_i) + \sin(t-t_i)\right) - 2 & -1 \end{bmatrix}\underline{m}(t) \\ \qquad - \begin{bmatrix} 8c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \\ 16c\left(1 + 6e^{2\tau}(t - \max\{0, t-\tau\})\right) \end{bmatrix} \end{cases} \quad (42)$$

in the interval $t \in [t_i, t_{i+1})$ for each $i = 1, 2, \ldots$ with the initial conditions

$$\begin{bmatrix} \hat{x}_0 \\ \overline{m}_0 \\ \underline{m}_0 \end{bmatrix} = \begin{bmatrix} \hat{x}_0 \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathcal{E}_0^+ - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\mathcal{E}_0^-\right] \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathcal{E}_0^- - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\mathcal{E}_0^+\right] \end{bmatrix} \quad (43)$$

with $\mathcal{E}_0^+ = x_0^+ - \hat{x}_0$, $\mathcal{E}_0^- = x_0^- - \hat{x}_0$ and $\tau = 5.4 \times 10^{-4}$. Then the equations (42) and (43) with

$$\begin{aligned} x^+(t) &= \hat{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\overline{m}(t) \\ x^-(t) &= \hat{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\underline{m}(t) \end{aligned} \quad (44)$$

is an interval observer for the system (41).

Figure 1 illustrates the true state $x(t)$ of (41) and the end points $x^+(t)$ and $x^-(t)$ of the estimated interval generated by (42)-(44) in the case where there is no disturbances ($c = 0$). They are numerically computed with $k(t) \equiv 50$, $x_0 = [5, 10]^\top$, $\hat{x}_0 = [10, 7]^\top$ $x_0^+ = [7, 13]^\top$, $x_0^- = [3, 7]^\top$ for the discretization $\epsilon = \tau = 5.4 \times 10^{-4}$. It is seen clearly in Fig.1 that the estimated bounds $x^+(t)$ and $x^-(t)$ converge to each other.

Next we consider (41) with the disturbances amplitude $c = 1/9$. The simulation result is shown in Fig.2. The interval observer still provides $x^+(t)$ and $x^-(t)$ giving an upper and a lower bounds of $x(t)$ for all times, although the bounds do not converge to each other in the presence of the persistent disturbances. The initial conditions and parameters used for Fig.2 are the same as those for Fig.1.

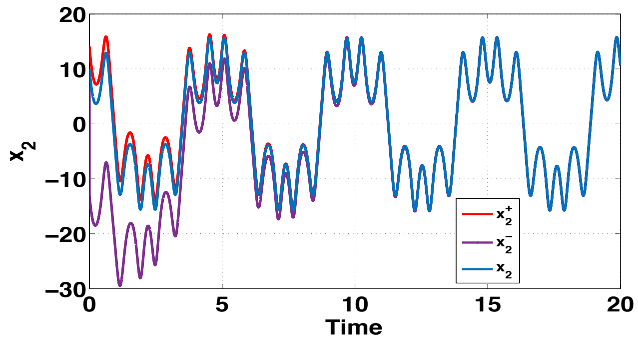

Fig. 1. Evolution of x_2, x_2^+ and x_2^- without uncertainties $\delta_1(t)$ and $\delta_2(t)$

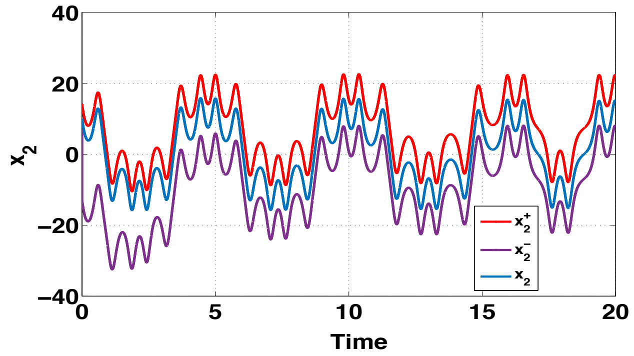

Fig. 2. Evolution of x_2, x_2^+ and x_2^- with uncertainties $\delta_1(t)$ and $\delta_2(t)$

VI. Conclusion

For a family of nonlinear systems covering continuous-time bilinear (and linear) systems with discrete-time output static feedback, this paper has proposed an interval observer design coping with disturbances. The development complements the previous result on nonlinear systems [5] in case where the output is discretized. It also complements the previous results on linear systems [17] in the presence of nonlinearities. Under a detectability-type assumption, we have presented a new technique that allows us to prove the properties required for the synthesis of interval observers. The unique feature in this paper is that the proposed observer design does not impose global Lipschitzness on the system as long as the system by itself is forward complete. Making use of the discretized measurements and the binonlinearity has led to such a unique development.

Covering general nonlinearities beyond the family of bilinear systems is one of important topics of future research. Extensions to systems with delays are also practically useful.

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