Interval Observers for Nonlinear Systems with Appropriate Output Feedback

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Abstract: The problem of designing interval observers for nonlinear systems exposed to disturbances is addressed in this paper. Intervals in which components of the state vector reside in are estimated by the interval observer consisting of two Luenberger-type observers making use of the information of the range of the initial state and the disturbances. The main feature of the proposed method is to encompass a larger class of nonlinearities than previously existing approaches. This paper demonstrates that an integral input-to-state stability (iISS) framework not only accomplishes this goal, but also provides us with robustness of convergence with respect to the disturbances.

Keywords: Interval observer, Nonlinear systems, Lyapunov functions, Guaranteed state estimation.

1. INTRODUCTION

One of important roles of observers is to reproduce information of dynamical systems for feedback control. For the purpose of control such as stabilization and tracking, precise information of the state vector in transient periods is not necessary. However, practically there is a great demand for estimation of the state of a system with guarantees at all times. The Luenberger observer in the popular formulation does not provide component-wise information of the state vector. Although mathematical models are never accurate, the Luenberger observer by itself is not equipped with explicit guarantees in the presence of perturbations and disturbances. The notion of interval observer has been one of useful approaches to meeting the practical demand [9]. The interval observer generates component-wise bounds of the state vector of a system to be estimated, and methods of designing such mechanism have been proposed and applied in many studies, (see e.g., [1, 4, 7, 8, 11-18] and references therein) .

An interval observer for a nonlinear model of bioreactors has been proposed in [16]. It is notable that nonlinear systems have also been approached by linearization and Taylor expansions with discretization in [17, 18]. In [6], an interval observer is proposed for a class of of nonlinear systems which are affine in the unmeasured part of the state vector. Interestingly, in contrast to the aforementioned approaches the proposed observer is in a simple structure consisting of two modified Luenberger-type observers. A sufficient condition under which the constructed structure is guaranteed to work as an interval observer is presented in [6]. Recently, the study of constructing interval observers with discrete-time measurements has been extended to nonlinear systems in [5]. The technique is, however, basically effective only for bilinearlities, and generalizing the result to nonlinearities covered by the continuous measurement case [6] is not trivial at all. This paper focuses on enlargement of classes of systems for which interval observers are guaranteed to work. Even in the continuous measurement case, the class of nonlinear systems satisfying the sufficient condition proposed in [6] is not satisfactorily broad. To guarantee proper estimation by the interval observer in the presence of disturbances, one needs to rely on global Lipschitzness of the system in the framework of [6]. The sufficient condition proposed there requires the feedback control input to be globally mild in accord with the observer, which restricts the use of nonlinear damping. This paper deals with nonlinear systems with continuous-time measurements and provides an integral input-to-state stability (iISS) based approach to the construction of interval observers in the presence of disturbances under milder assumptions.

<u>Notation</u>
The set of real numbers is denoted by $\mathbb{R}$. The set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$, i.e., $\mathbb{R}_{\geq 0} := [0, \infty)$. The symbol I denotes the identity matrix in $\mathbb{R}^{n\times n}$ of any dimension n. The symbol $|\cdot|$ denotes Euclidean norm of vectors of any dimension. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, ..., x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, ..., x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \{1, ..., n\}$, $x_{a,i} \leq x_{b,i}$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, let the matrix $Q^+ \in \mathbb{R}^{n\times n}$ denote $Q^+ = (\max\{q_{i,j}, 0\})_{i,j=1,1}^{n,n}$, where the notation $Q = (q_{i,j})_{i,j=1,1}^{n,n}$ is used. Let $Q^- \in \mathbb{R}^{n\times n}$ be defined by $Q^- = Q^+ - Q$. This notation is limited to square matrices, and the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be Metzler if each off-diagonal entry of this matrix is nonnegative. For functions $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, by $\alpha(s) \equiv \beta(s)$ we mean $\alpha(s) = \beta(s)$ for all $s \in \mathbb{R}_{\geq 0}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be positive definite and written as $\alpha \in \mathcal{P}$ if α is continuous and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s \in (0, \infty)$. A function $\alpha \in \mathcal{P}$ is said to be of class $\mathcal{K}$ if α is strictly increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively.

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2. PROBLEM SETUP: INTERVAL OBSERVER

Consider the system[1]

$$\dot{x}(t) = A(y(t))x(t) + \beta(y(t), u(t)) + \delta(t) \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

with time $t \in \mathbb{R}_{\geq 0}$, the state $x(t) \in \mathbb{R}^n$, the output $y(t) \in \mathbb{R}^p$, the input $u(t) \in \mathbb{R}^q$ and the initial condition $x(0) = x_0$, where the function $A : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ is continuous and $C \in \mathbb{R}^{p\times n}$ is a constant matrix. The functions $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ and $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ are supposed to be locally Lipschitz, where $\alpha(x) := A(Cx)x$. The disturbance vector $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is supposed to be a piecewise continuous function. This paper tackles the problem of designing an interval observer for the given system (1), which is defined as follows:

Definition 1: Let the vectors $x_0^-, x_0^+ \in \mathbb{R}^n$ and the piecewise continuous functions $\delta^+, \delta^- : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be such that

$$x_0^- \leq x_0 \leq x_0^+ \tag{2}$$
$$\delta^-(t) \leq \delta(t) \leq \delta^+(t), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{3}$$

Then the system

$$\dot{z}(t) = f(z(t), y(t), u(t), \delta^+(t), \delta^-(t)) \tag{4}$$

defined with

$$x^+(t) = h^+(z(t)) \in \mathbb{R}^n, \quad x^-(t) = h^-(z(t)) \in \mathbb{R}^n \tag{5}$$

and the initial condition

$$z(0) := z_0 = g(x_0^+, x_0^-) \in \mathbb{R}^r \tag{6}$$

for some functions $f : \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^n \times \to \mathbb{R}^r$, $h^+, h^- : \mathbb{R}^r \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^r$ is called

(i) a framer for (1) with a given input u if unique solutions $x(t)$ and $z(t)$ to (1) and (4) exist for all time and satisfy

$$x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \tag{7}$$

for any x_0 and δ satisfying (2) and (3).

(ii) an interval observer for (1) with a given input u if, in addition to (i),

$$\lim_{t\to\infty} |x^+(t) - x^-(t)| = 0 \tag{8}$$

is guaranteed for any x_0 satisfying (2) in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

(iii) a primitive interval observer for (1) with a given input u if, in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$, unique solutions $x(t)$ and $z(t)$ to (1) and (4) exist for all time and satisfy (7) and (8) for any x_0 satisfying (2).

Finally, the entire system consisting of (1) and (4) with (5), (6) and a given input u is said to be 0-GAS if $[x^\top, z^\top]^\top$=0 is globally asymptotically stable in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

Note that primitive interval observers guarantee neither the global existence nor (7) for non-zero disturbances. The observer candidate (4)-(6) is not allowed to use $x(t)$ and $\delta(t)$ which are unknown. Instead, x_0^-, x_0^+, $\delta^-(t)$ and $\delta^+(t)$ are supposed to be known and allowed to be used. The above definition of interval observers is basically the same as the one employed in [6]. This paper, however does not require 0-GAS as a property of an interval observer. Instead, the 0-GAS property which is useful and reasonable in choosing the control input u is stated separately in the above definition. For the sake of following the popular definition in the literature (see e.g., [6] and references therein), being an interval observer requires the convergence (8) only for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$, although the frame property (7) needs to be guaranteed for non-zero disturbances. Note that an interval observer is always a primitive interval observer. It is emphasized here that Subsection 5.2 in this paper addresses properties going beyond Definition 1 of interval observers

[1]Nonlinearities in (1) are restricted to functions of the measured output y. This class of systems has been studied extensively in studies of (non-interval) global observer designs. The class can be considered to be reasonably broad when one resort to neither global Lipschitzness nor global slope restrictions of nonlinearities in allowing the state to be arbitrarily large (see [3] and references therein).

3. MAIN RESULTS

This section proposes an iISS-based approach to designing interval observers. Its capability is demonstrated in the standard framework of Definition 1. Demonstration of an additional desirable property is postponed until Subsection 5.2. To be ready for presenting main theorems, we first state two assumptions borrowed from [6].

Assumption 1: Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exists an invertible matrix $R \in \mathbb{R}^{n\times n}$ such that, for all $y \in \mathbb{R}^p$, the matrix

$$\Gamma(y) = R[A(y) + \Lambda(y)C]R^{-1} \tag{9}$$

is Metzler.

Assumption 2: Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}, \overline{\nu} \in \mathcal{K}_\infty$ and $\omega \in \mathcal{P}$ such that

$$\underline{\nu}(|\xi|) \leq V(\xi) \leq \overline{\nu}(|\xi|) \tag{10}$$
$$\frac{\partial V}{\partial \xi}(\xi)[A(y) + \Lambda(y)C]\xi \leq -\omega(|\xi|) \tag{11}$$

hold for all $\xi \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$.

This paper introduces iISS-type assumptions. The next one gets rid of an unnecessary requirement the study in [6] relies on.

Assumption 3: Given a locally Lipschitz function $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$, there exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathcal{P}$ and $\gamma \in \mathcal{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + \beta(Cx, u_s(Cx, x+d))] \leq -\mu(|x|) + \gamma(|d|) \quad (12)$$

holds for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$.

The following two assumptions will be employed to broaden capabilities of the observer design.

Assumption 4: Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}, \overline{\nu} \in \mathcal{K}_\infty$, $\omega \in \mathcal{P}$ and $\eta^+, \eta^- \in \mathcal{K}$ such that (10) and

$$\frac{\partial V}{\partial \xi}(\xi)\left\{[A(y) + \Lambda(y)C]\xi + S[R^+\rho^+ + R^-\rho^-]\right\} \leq -\omega(|\xi|) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|) \quad (13)$$

hold for all $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $\rho^+ \in \mathbb{R}^n$ and $\rho^- \in \mathbb{R}^n$.

Assumption 5: Given a locally Lipschitz function $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$, there exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathcal{P}$ and $\gamma, \zeta \in \mathcal{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + \beta(Cx, u_s(Cx, x+d)) + \delta] \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \quad (14)$$

holds for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^n$.

Note that by definition, the fulfillment of Assumption 4 (resp. Assumption 5) implies that Assumption 2 (resp. Assumption 3) is satisfied.

Let $S = R^{-1}$. Consider the differential equations

$$\dot{\hat{x}}^+ = A(y)\hat{x}^+ + \beta(y, u) + \Lambda(y)[C\hat{x}^+ - y] + S[R^+\delta^+ - R^-\delta^-] \quad (15a)$$

$$\dot{\hat{x}}^- = A(y)\hat{x}^- + \beta(y, u) + \Lambda(y)[C\hat{x}^- - y] + S[R^+\delta^- - R^-\delta^+] \quad (15b)$$

defined with the initial condition

$$\hat{x}^+(0) = \hat{x}_0^+ := S[R^+x_0^+ - R^-x_0^-] \quad (16a)$$

$$\hat{x}^-(0) = \hat{x}_0^- := S[R^+x_0^- - R^-x_0^+] \quad (16b)$$

and the output equation

$$x^+ = S^+R\hat{x}^+ - S^-R\hat{x}^- \quad (17a)$$

$$x^- = S^+R\hat{x}^- - S^-R\hat{x}^+. \quad (17b)$$

Equations (15)-(17) are in the form of (4)-(6) by considering z as the vector consisting of $\hat{x}^+$ and $\hat{x}^-$. The equations (15)-(17) are proposed in [6] as an interval observer candidate. By employing the same structure of the interval observer, this paper develops improved design guidelines which are less restrictive and more capable than those developed in [6]. We are now in position to state the first main result.

Theorem 1: Suppose that locally Lipschitz functions $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ and $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ satisfy Assumptions 1, 2, 3. If

$$\mu \in \mathcal{K} \quad (18)$$

holds, then the system consisting of (15)-(17) is a primitive interval observer for the system (1) with $u = u_s(y, \hat{x}^+)$, and the entire system is 0-GAS. Moreover, if any of

(i) $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ and $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ are globally Lipschitz.

(ii) Assumptions 4 and 5 and

$$\mu \in \mathcal{K}_\infty \wedge \omega \in \mathcal{K}_\infty \quad (19)$$

are satisfied.

(iii) Assumptions 4 and 5, $\omega \in \mathcal{K}$ and

$$\omega \in \mathcal{K}_\infty \vee \gamma \notin \mathcal{K}_\infty \vee \lim_{s\to\infty} \omega(s) > \sup_{t\in\mathbb{R}_{\geq 0}} \eta(\sqrt{2}|\delta^{\pm}(t)|) \quad (20)$$

are satisfied, where $\eta := \eta^+ + \eta^-$ and $\delta^{\pm} := \delta^+ - \delta^-$.

holds true in addition, then the system consisting of (15)-(17) is an interval observer for the system (1) with $u = u_s(y, \hat{x}^+)$.

The second main result to be presented as the next theorem removes the requirement (18) and allows the convergence rate μ to be radially vanishing at the cost of introducing a growth order constraint on the coupling between the observer (15) and the plant (1).

Theorem 2: Suppose that locally Lipschitz functions $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ and $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ satisfy Assumptions 1, 2, 3. If

$$\int_0^1 \frac{\gamma \circ \underline{\nu}^{-1}(s)}{\omega \circ \overline{\nu}^{-1}(s)} ds < \infty \quad (21)$$

holds, then the system consisting of (15)-(17) is a primitive interval observer for the system (1) with $u = u_s(y, \hat{x}^+)$, and the entire system is 0-GAS. Moreover, if any of

(i) $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ and $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ are globally Lipschitz.

(ii) Assumptions 4 and 5, $\omega \in \mathcal{K}$ and

$$\exists c > 0, k \geq 1 \text{ s.t. } c\gamma \circ \underline{\nu}^{-1}(s) \leq [\omega \circ \overline{\nu}^{-1}(s)]^k, \forall s \in \mathbb{R}_{\geq 0} \quad (22)$$

are satisfied.

holds true in addition, then the system consisting of (15)-(17) is an interval observer for the system (1) with $u = u_s(y, \hat{x}^+)$.

4. SCKETCH OF PROOFS

4.1. Scketch of the Proof of Theorem 1

By the change of coordinates

$$p = \hat{x}^+ - x, \quad q = \hat{x}^- - x \quad (23)$$

from (1) and (15) with $u = u_s(y, \hat{x}^+)$ we obtain

$$\dot{x} = A(y)x + \beta(y, u_s(y, x+p)) + \delta(t) \quad (24)$$

$$\dot{p} = [A(y) + \Lambda(y)C]p + S[R^+\delta^+ - R^-\delta^-] - \delta$$

$$\dot{q} = [A(y) + \Lambda(y)C]q + S[R^+\delta^- - R^-\delta^+] - \delta.$$

Since $SR = I$ and $R = R^+ - R^-$ yield

$$S[R^+\delta^+ - R^-\delta^-] - \delta = S[R^+(\delta^+ - \delta) - R^-(\delta^- - \delta)]$$
$$S[R^+\delta^- - R^-\delta^+] - \delta = S[R^+(\delta^- - \delta) - R^-(\delta^+ - \delta)],$$

it holds that

$$\dot{p} = [A(y) + \Lambda(y)C]p + S[R^+\hat{\rho}^+ + R^-\hat{\rho}^-] \quad (25a)$$
$$\dot{q} = [A(y) + \Lambda(y)C]q - S[R^+\hat{\rho}^- + R^-\hat{\rho}^+], \quad (25b)$$

where $\hat{\rho}^+ = \delta^+ - \delta$ and $\hat{\rho}^- = \delta - \delta^-$. The entire system consists of (24), (25a) and (25b). The two subsystems (24) and (25a) form a cascade in which (24) is driven by (25a) through p. The local Lipschitzness imposed on the functions in (1) and stated in Assumptions 2, 3, 4 and 5 ensures the existence of a unique maximal solution i.e., local in time, to (24), (25a) and (25b). By virtue of Assumption 1, it can be verified as in [6] that equations (15)-(17) with the restrictions (2) and (3) give a framer for the system (1) up to the maximum time of the existence of the solution $(x(t), p(t), q(t))$ to (24), (25a) and (25b).

Next, suppose that (18) holds. Since Assumptions 2 and 3 are satisfied, application of [10, Corollary 1 (ii)] to the cascade of (24) and (25a) with $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$ yields the existence of $\alpha_{xp} \in \mathcal{P}$ and continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\lambda_i(s) > 0, \forall s \in (0, \infty), \qquad \int_1^\infty \lambda_i(\tau)d\tau = \infty \quad (26)$$

hold and

$$\dot{W} \leq -\alpha_{xp}(V_{xp}) - \omega(|q|) \quad (27)$$

is satisfied along the trajectories of (24), (25a) and (25b) for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$ with

$$W(x, p, q) = V_{xp}(x, p) + V(q) \quad (28)$$
$$V_{xp}(x, p) = \int_0^{U(x)} \lambda_1(s)ds + \int_0^{V(p)} \lambda_2(s)ds. \quad (29)$$

Since W is a positive definite and radially unbounded function of (x, p, q) due to (26), inequality (27) with α_{xp}, $\omega \in \mathcal{P}$ implies that the maximum time of the existence $(x(t), p(t), q(t))$ is unbounded in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. Property (27) with α_{xp}, $\omega \in \mathcal{P}$ also implies that $(x, p, q) = (0, 0, 0)$ of the entire system consisting of (24), (25a) and (25b) is globally asymptotically stable for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. Consequently, taking (23) and (17) into account, property (8) holds for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

In the case of the global Lipschitzness assumption (i) it is clear that the maximum time of the existence $(x(t), p(t), q(t))$ is unbounded for any piecewise continuous δ, δ^+ and δ^-.

Next, without assuming (i), suppose that Assumptions 4 and 5 hold. Assume that (19) holds. Then application of [10, Theorem 1 (ii)] with the help of the argument in [10, Theorem 2] to the cascade of (24) and (25a) proves the existence of $\alpha_{xp} \in \mathcal{K}_\infty$, $\sigma, \sigma^\pm \in \mathcal{K}$ and continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that (26) holds and

$$\dot{W} \leq -\alpha_{xp}(V_{xp}) - \omega(|q|) + \sigma(|\delta|) + \sigma^\pm(|\delta^\pm|) \quad (30)$$

is satisfied along the trajectories of (24), (25a) and (25b) with (28) and (29). Again, the function W is a positive definite and radially unbounded function of (x, p, q) due to (26). Inequality (30) with α_{xp}, $\omega \in \mathcal{K}_\infty$ implies that the maximum time of the existence $(x(t), p(t), q(t))$ is unbounded for any piecewise continuous δ, δ^+ and δ^-. Property (30) with α_{xp}, $\omega \in \mathcal{K}_\infty$ also implies that $(x, p, q) = (0, 0, 0)$ of the entire system consisting of (24), (25a) and (25b) is globally asymptotically stable in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. Hence, due to (23) and (17), $\lim_{t\to\infty} |\hat{x}^+(t) - \hat{x}^-(t)| = 0$ implies that (8) is satisfied in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

Finally, to prove the claim with (iii), the argument in [10, Theorem 1 (i)] can be invoked.

4.2. Scketch of the Proof of Theorem 2

The clams can be proved by applying [10, Corollary 1 (i)] and [10, Remark 1] to the cascade of (24) and (25a).

5. NEW FEATURES

5.1. Broader Nonlinearities

Under Assumptions 1, 2, 3 and (i) in Theorem 2, the result in [6] proposes

$$\exists \kappa \in \mathcal{K},\ l \in \mathbb{R}_{\geq 0} \text{ s.t.}$$
$$\gamma(\underline{\nu}^{-1}(s)) \leq (l + \kappa(s))\omega(\overline{\nu}^{-1}(s)),\ \forall s \in \mathbb{R}_{\geq 0} \quad (31)$$

as a sufficient condition[2] to establish that the system consisting of (15)-(17) is an interval observer for the system (1) with $u = u_s(y, \hat{x}^+)$. It can be verified that (31) is a special case of (21). In fact, the assumption (31) yields

$$\int_0^1 \frac{\gamma \circ \underline{\nu}^{-1}(s)}{\omega \circ \overline{\nu}^{-1}(s)} ds \leq \int_0^1 (l + \kappa(s))ds.$$

By virtue of $\kappa \in \mathcal{K}$, the above guarantees (21). Thus, property (31) assumed by [6] is a sufficient condition of (21) which is one of several options provided by this paper. It is worth stressing that (21) is only a local property around the origin of functions γ and ω. Moreover, this paper does not require (21) when (18) is satisfied. In other words, the restrictions on the coupling can be removed completely if the convergence rate of the plant is not vanishing as the magnitude of the state tends to infinity.

5.2. Convergence in the presence of disturbances

The iISS-based design proposed in this paper furnishes an observer with a more desirable property than being interval observers in the standard notion defined in Section 2. To see this, let the state and the input of the entire system be defined as

$$X(t) = \begin{bmatrix} x(t) \\ \hat{x}^+(t) \\ \hat{x}^-(t) \end{bmatrix}, \quad \Delta(t) = \begin{bmatrix} \delta(t) \\ \delta^+(t) \\ \delta^-(t) \end{bmatrix}. \quad (32)$$

[2] The expression (31) is not exactly the same as the one in [6], but they are qualitatively equivalent to each other.

The proofs of Theorems 1 and 2 establish iISS of the entire system with respect to the state X and the input Δ. More precisely, assuming iISS of the system (1) with the feedback control $u = u_s(y, \hat{x}^+)$ in the form of Assumption 5, the iISS of the entire system is established by the existence of $\alpha_{xp} \in \mathcal{P}$, σ, $\sigma^{\pm} \in \mathcal{K}$ satisfying (30). The iISS property guarantees the existence of $\overline{\beta} \in \mathcal{KL}$, $\overline{\mu} \in \mathcal{K}$ and $\overline{\chi} \in \mathcal{K}_\infty$ such that

$$\overline{\chi}(|X(t)|) \leq \overline{\beta}(|X(0)|, t) + \int_0^t \overline{\mu}(|\Delta(\tau)|)d\tau, \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{33}$$

holds for all $X(0) \in \mathbb{R}^{3n}$ and all δ satisfying (3)[3]. Due to this iISS property [2, 19], the entire state $X(t)$ is guaranteed to converge to the origin even in the presence of $\Delta(t)$ if $\Delta(t)$ converges to zero appropriately, i.e., if $\int_0^\infty \overline{\mu}(\Delta(t))dt < \infty$ holds.

It is also confirmed from the existence of $\alpha_{xp} \in \mathcal{K}_\infty$, σ, $\sigma^{\pm} \in \mathcal{K}$ satisfying (30) demonstrated in the proof of Theorems 1 that if $\mu, \omega \in \mathcal{K}_\infty$ holds, the entire system is input-to-state stable (ISS) with respect to the state X and the input Δ, i.e., there exist $\overline{\beta} \in \mathcal{KL}$ and $\overline{\gamma} \in \mathcal{K}$ such that

$$|X(t)| \leq \overline{\beta}(|X(0)|, t) + \overline{\gamma}\left(\sup_{\tau \in [0,t]} |\Delta(\tau)|\right), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{34}$$

Due to this ISS property [20], the entire state $X(t)$ converges to the origin for any input $\Delta(t)$ converging to zero.

Recall that the definition of interval observers in Section 2 requires convergence of $x^+(t) - x^-(t)$ to zero only when disturbances are absent. In contrast, the aforementioned converging properties of the entire state $X(t)$ implies the convergence of $x^+(t) - x^-(t)$ to zero in the presence of disturbances, which is one of useful and novel features of the results in this paper.

6. AN EXAMPLE

Consider

$$\dot{x}_1 = -x_1^2 x_2 - \frac{x_2}{2} + u_1 + \delta_1 \tag{35a}$$

$$\dot{x}_2 = -2x_1^2 x_2 - \frac{x_2}{2} + u_2 + \delta_2 \tag{35b}$$

$$y = x_1 \tag{35c}$$

with the feedback control input $u = u_s$ and the observer gain Λ given by

$$u_s(y, \hat{x}^+) = \frac{1}{2}\begin{bmatrix} -4y^3 + \hat{x}_2^+ \\ -\hat{x}_2^+ \end{bmatrix}, \ \Lambda(y) = \begin{bmatrix} -2y^2 - \frac{1}{2} \\ 0 \end{bmatrix}. \tag{36}$$

For these equations, the functions α, β, u_s and Λ are locally Lipschitz. Notice that α, u_s and Λ are not globally Lipschitz. Define $A(y)$ and choose R as follows:

$$A(y) = \begin{bmatrix} 0 & -y^2 - 1/2 \\ 0 & -2y^2 - 1/2 \end{bmatrix}, \ R = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}.$$

[3]Unless we invoke $\lim_{s \to \infty} \omega(s) > \sup_{t \in \mathbb{R}_{\geq 0}} \eta(\sqrt{2}|\delta^{\pm}(t)|)$ in (20), this iISS property can be established without the restriction (3).

Then we obtain

$$\Gamma(y) = \begin{bmatrix} -2y^2 - 1/2 & 2y^2 + 1 \\ 0 & -2y^2 - 1/2 \end{bmatrix}$$

which is Metzler for all $y \in \mathbb{R}$. Let $V(\xi) = \xi^\top \xi$. Then (10) is satisfied with $\underline{\nu}(s) = s^2$ and $\overline{\nu}(s) = s^2$. It can be verified that (13) is satisfied with

$$\omega(s) = \frac{1}{10}s^2, \ \eta^+(s) = 5s^2, \ \eta^-(s) = \frac{13}{2}s^2. \tag{37}$$

The choice $U(x) = x^\top x$ yields (14) with

$$\mu(s) = \frac{1}{4}\min\{s^4, s^2\}, \ \gamma(s) = \max\left\{\frac{3}{2}s^{\frac{4}{3}}, s^2\right\} \tag{38}$$

$$\zeta(s) = \max\left\{3s^{\frac{4}{3}}, 2s^2\right\}. \tag{39}$$

Therefore, Assumptions 1, 4, 5 are satisfied. It is stressed that the pair of ω and μ in (37) and (38) does not satisfy (31) since the order of γ is smaller than that of ω around the origin [4]. However, due to $\mu, \omega \in \mathcal{K}_\infty$, Theorem 1 (ii) is applicable. Therefore, an interval observer is constructed as (15)-(17). First, the persistent disturbances

$$\delta(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \tag{40}$$

the simulation result is plotted in Fig. 1 with the initial conditions $x_0 = [5, -5]^\top$, $x_0^+ = [10, 0]^\top$ and $x_0^- = [0, -10]^\top$. The component $x_2(t)$ which is not measured remains in the estimated interval $[x_2^-(t), x_2^+(t)]$ all the time. Next, for the convergent disturbances

$$\delta(t) = \begin{bmatrix} \mathrm{sgn}(\sin(t)) \min\left\{|\sin(t)|, 5/t^2\right\} \\ \mathrm{sgn}(\cos(t)) \min\left\{|\cos(t)|, 5/t^2\right\} \end{bmatrix}, \tag{41}$$

Figure 2 shows the interval computed with the initial conditions $x_0 = [5, -5]^\top$, $x_0^+ = [10, 0]^\top$ and $x_0^- = [0, -10]^\top$. The component $x_2(t)$ again stays in the estimated interval all the time. Moreover, the length of the interval converges to zero as well as the state variables $x_1(t)$ an $x_2(t)$ converge to the origin. The figure for $x_(t)$ is omitted. This is consistent with Theorem 1 (ii) and the ISS property demonstrated in Subsection 5.2.

7. CONCLUSIONS

The problem of designing interval observers for nonlinear systems has been addressed and a new approach has been proposed in this paper. Basically, this proposed design method has extended the earlier result in [6] in view of three points. Firstly, this paper has shown how to guarantee forward completeness of the entire system including the interval observer in the presence of disturbances by making use of an iISS framework, in addition to estimating intervals of the state variables. The iISS Lyapunov characterizations replace the global Lipschitz conditions on which the approach in [6] relies on in the

[4]The exponent in $\gamma(s)$ fulfilling (14) cannot be larger than $4/3$ around the origin, due to u_s containing y^3.

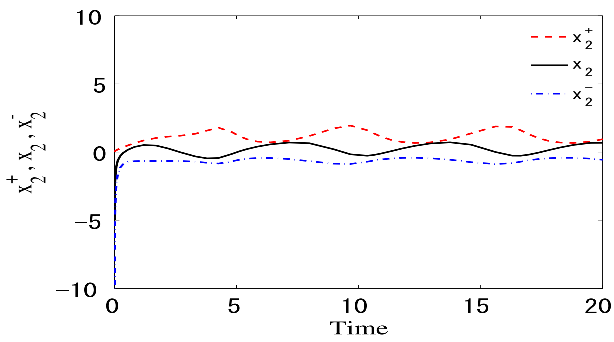

Fig. 1 Evolution of 2nd component of x, x^+, x^- in the presence of persistent disturbances (40).

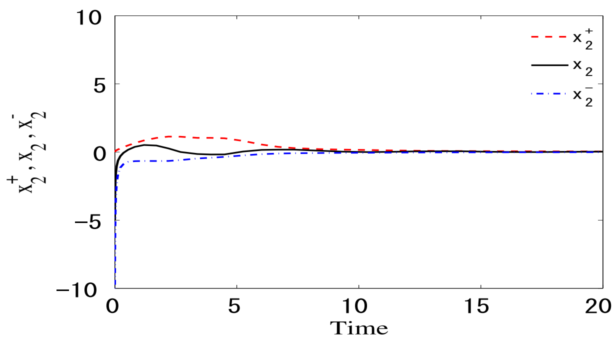

Fig. 2 Evolution of 2nd component of x, x^+, x^- in the presence of converging disturbances (41).

presence of disturbances. In contrast to the global Lipschitzness which has strong limitations, the iISS framework covers a broad class of nonlinearities including saturations which are not covered by ISS. Secondly, in the absence of disturbances, restrictions on the coupling between the interval observer and the controlled plant have been relaxed in several ways. For instance, a growth order condition is imposed only around the origin. This paper has also demonstrated that the growth order condition can be completely removed when the convergence rate of the system is not radially vanishing. Thirdly, the iISS approach to the interval observer design allows us to ensure the convergence to zero of the estimated intervals even in the presence of disturbances if the disturbances are converging to zero.

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