Immersion and Invariance in delayed input sampled-data stabilization

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Abstract—In this paper, a nonlinear continuous-time dynamics with input-delay is considered. Assuming the existence of a continuous-time stabilizing strategy in the delay free case, it is shown how the problem can be reformulated in the sampled-data context as an Immersion and Invariance I&I stabilizing one.

Index Terms—Systems with delays, Nonlinear stabilization, Nonlinear hybrid dynamics.

I. INTRODUCTION

The purpose of this paper is to show how the Immersion and Invariance I&I technique proposed in [1], [2] provides a natural context for the design of sampled-data controllers for delayed input dynamics. Nonlinear stabilization is hereinafter addressed with reference to a single-input-affine continuous-time dynamics with input-delay $\tau$. The existence of a feedback law ensuring Global Asymptotic Stability - GAS of the equilibrium in the delay free case is assumed.

Choosing a sampling period $\delta$ satisfying $\tau = N\delta$ for a positive integer $N \in N^+$, the problem is set in the sampled context with state measures available at the sampling instants $t = k\delta, k \geq 0$ and control constant over each sampling period. The representation of the input-delayed dynamics as an equivalent finite dimensional hybrid dynamics makes it possible to recast the problem in the discrete-time I&I stabilizing context [1], [2]. At first, a sampled-data stabilizing feedback as described in [15], [16], [11] is designed on the delay free dynamics so defining the discrete-time target dynamics and its associated invariant manifold. Then, stabilization under sampled-data feedback is reduced to drive the off-the-manifold component to zero so recovering the sampled-data stable delay free dynamics on the manifold. With respect to sampled-data predictor-based techniques (see [5], [12], [8], [9], [10], [6], [17], [19]), the I&I stabilizing controller achieves asymptotic convergency to the target delay free dynamics so recovering on the manifold the predictor-based one. Such an approach prevents from big control effort and improves robustness to model uncertainties. An example concludes the paper.

The paper is organized as follows. Instrumental tools and preliminary results are given in section II. The main result is discussed in section III. An example is in section IV.

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II. PROBLEM SETTLEMENT AND PRELIMINARIES

A. Problem settlement

Consider the single input-affine dynamics with delayed input

$$\dot{x}(t) = f(x) + u(t-\tau)g(x)$$

(1)

where $f$ and $g$ are smooth (i.e. $C^\infty$) vector fields on $\mathbb{R}^n$; $x_e$ denotes the equilibrium point $f(x_e) = 0$, supposed without loss of generality equal to zero; the delay $\tau$ is known. It is assumed in the following that:

- maps and vector fields are smooth (i.e. infinitely differentiable $- C^\infty$), vector fields are forward complete. The set $\mathcal{W}$ (resp. $\mathcal{W}_d$) of admissible inputs consists of all $U$-valued piecewise continuous (resp. piecewise constant) functions on $\mathbb{R}$;
- in the sampled context here considered, the sampling period is assumed regular and equal to $\delta \in [0,T^+[$, a finite time interval chosen so that $\tau = N\delta$ for a suitable integer $N$;
- assumption A - The delay free system (1) is smoothly stabilizable; i.e. there exists a feedback $u(t) = \gamma(x)$ with $\gamma(0) = 0$ and a Lyapunov function $V > 0$ with $V(0) = 0$ such that $\dot{V} = (L_f + V_{\gamma})V < 0$.

B. Sampled-data stabilization in the delay free case - recalls

To define the discrete-time target dynamics let us first recall that assumption A is sufficient to prove the existence of a piecewise constant control preserving GAS of the equilibrium at the sampling instants. Let (1) with $\tau = 0$ and assume the input constant over successive intervals of length $\delta > 0$, $u(t) = u_k$ for $t \in [k\delta,(k+1)\delta[); i.e.

$$\dot{x}(t) = f(x) + u_k g(x).$$

(2)

Through integration over the same time-interval with initial condition $x_k = x_{k-\delta}$, one describes the equivalent sampled-data dynamics in the form of a map as

$$x_{k+1} = F^\delta(x_k,u_k) = e^{\delta(f+u_k g)}x_k.$$ (3)

Following [15], the existence of a sampled-data feedback $u_k = \gamma^\delta(x_k)$ which stabilizes (3) with the same performances as the continuous-time controller at the sampling instants directly follows from assumption A. The following result is recalled from [16].

Theorem 2.1: Let the input-affine dynamics (1) assumed forward complete and let $u = \gamma(x)$ be a smooth feedback
satisfying $A$ with Lyapunov function $V$. Then, there exists $T^*$ and for all $\delta \in [0, T^*)$ a feedback law of the form
\[ \gamma^\delta (x) \]
with $\gamma^\delta (0) = 0$ which assures one step Lyapunov-matching at the sampling instants; i.e. $\forall k \geq 0$ and initial condition $x_k = x|_{t=k \delta}$ satisfying $L_\delta V(x_k) \neq 0$, the equality
\[ V(x_{k+1}) - V(x_k) = \int_{k \delta}^{(k+1) \delta} (L_\delta V + \gamma^\delta(t)) (x(t)) dt \] (5)
is satisfied by an appropriate choice of $u_k = \gamma^\delta (x_k)$.

The proof is constructive and works out by equating terms of the same power in $\delta$ in the respective expansions with respect to $\delta$ of both members of the equality (5) (see [16]). A solution exists because $L_\delta V(x) \neq 0$ when $x \neq 0$. For the first terms one computes
\[ \gamma_0 (x_k) = \gamma(x(t))|_{t=k \delta} \] (6)
\[ \gamma_1 (x_k) = \gamma(x(t))|_{t=k \delta} \] (7)
\[ \gamma_2 (x_k) = \gamma(x(t))|_{t=k \delta} + \frac{\gamma_0 (x_k)}{2L_\delta V(x(t))} \ad_{f^\delta} V(x(t))|_{t=k \delta} \delta \] (8)
recovering respectively the continuous-time solution (6) and its time derivative (7) computed at time $t = k \delta$. The higher order terms can be computed from the previous ones through an executable algorithm.

Remark. $\gamma_0 (x_k)$ constitutes the emulated solution which satisfies (5) with an error in $O(\delta)$. Denoting by $\gamma^{\hat{\delta}} (x_k)$ with $p \geq 0$, the $d^p$-order approximate controller (truncation in $O(\delta^{d+1})$ of the exact solution (4)), it follows that by construction $\gamma^{\hat{\delta}} (x_k) := \gamma_0 (x_k) + \frac{\hat{\delta}}{d} \gamma_1 (x_k)$ satisfies (5) with an error in $O(\delta^2)$. $\gamma^{\hat{\delta}} (x_k) := \gamma_0 (x_k) + \frac{\hat{\delta}}{d} \gamma_1 (x_k) + \frac{\hat{\delta}^2}{2d^2} \gamma_2 (x_k)$ with an error in $O(\delta^3)$ and $\gamma^{\hat{\delta}} (x_k)$ with an error in $O(\delta^{d+1})$.

Remark. $\gamma^{[0]} (x)$ and $\gamma^{[1]} (x)$ do not depend on the control Lyapunov function $V(x)$ while the computation of the additional terms in $\gamma^{[p]} (x)$ with $p \geq 2$ requires the explicit knowledge of the function $V(x)$.

C. Hybrid representation and target dynamics

Setting $\tau = N \delta$ in (1) and assuming the input constant over time intervals of length $\delta$, one can represent continuous-time dynamics with input-delay as hybrid dynamics over $\mathbb{R}^{n+N}$

\[ \dot{x}(t) = f(x) + v^1 g(x) \quad \forall t \in [k \delta, (k+1) \delta] \]
\[ v^1_{k+1} = v^2_k \]
\[ ... = ... \]
\[ v^N_{k+1} = u_k. \] (9)

On these bases, the stabilizing problem can be set on an extended but finite dimensional dynamics with dynamic extension of order $N$ strictly related to the delay length. Accordingly, the discrete-time target dynamics is defined in the delay free case as the sampled-data equivalent (3) of the continuous-time dynamics (2) when $u_k = \gamma^\delta (x_k)$; i.e. for $t \in [k \delta, (k+1) \delta]$

\[ \dot{x}(t) = f(x) + \gamma^\delta (x_k) g(x). \]

Through integration with initial condition $x_k := x|_{t=k \delta}$, one gets for each $\delta \in [0, T^*)$, the discrete-time target dynamics over $\mathbb{R}^n$.

\[ x_{k+1} = F^\delta (x_k, \gamma^\delta (x_k)) = e^{\delta (f + \gamma^\delta (x_k))} x_k \] (10)
which is parameterized by $\delta$, with GAS equilibrium by construction of $\gamma^\delta (.)$.

D. I&I stabilization in discrete time

The Immersion and Invariance methodology firstly introduced in [1] has been completely formalized and applied to several physical domains in [2]. Discrete-time versions are introduced in [7] and [20] regarding especially adaptive stabilizing strategies. Accordingly, I&I stabilizability can be formulated for generally nonlinear discrete-time dynamics following the lines of the continuous time.

**Theorem 2.2:** Consider any nonlinear discrete-time dynamics in the form of a map
\[ x_{k+1} = F(x_k, u_k) \] (11)
with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}_d$, with an equilibrium point $x_e$ to be stabilized. Let $p < n$ and assume that we can find mappings
\[ \alpha (\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p; \quad \pi (\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^n \]
\[ \phi (\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n-p}; \quad \psi (\cdot, \cdot) : \mathbb{R}^{n \times (n-p)} \rightarrow \mathbb{R} \]
such that the following four conditions hold:

H1 (Target dynamics) - The dynamics with state $\xi \in \mathbb{R}^p$
\[ \xi_{k+1} = \alpha (\xi_k) \] (12)
has a globally asymptotically stable equilibrium at $\xi_s \in \mathbb{R}^p$ and $x_e = \pi (\xi_s)$.

H2 (Immersion and invariance condition) - For all $\xi \in \mathbb{R}^p$, there exists $c (\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}$ such that
\[ F(\pi (\xi_k), c (\xi_k)) = \pi (\alpha (\xi_k)) \] (13)

H3 (Implicit manifold) - The following identity between sets holds
\[ \{ x \in \mathbb{R}^p | \phi (x) = 0 \} = \{ x \in \mathbb{R}^n | x = \pi (\xi) \text{ for } \xi \in \mathbb{R}^p \} \] (14)

H4 (Manifold attractivity and trajectory boundedness) - All the trajectories of the system
\[ z_{k+1} = \phi (F(x_k, \psi (x_k, z_k))) \]
\[ x_{k+1} = F(x_k, \psi (x_k, z_k)) \] (15a)
with $z_0 = \phi (x_0)$ are bounded for all $k \geq 0$ and satisfy
\[ \lim_{k \rightarrow \infty} z_k = 0 \quad \text{and} \quad \psi (\cdot, 0) \big|_{\pi (\cdot)} = c (\cdot) \] (16)
Then $x^*$ is a globally asymptotically stable equilibrium of the closed loop dynamics
\[ x_{k+1} = F(x_k, \psi(x_k, \phi(x_k))). \]  
(17)

**Definition.** A system described by equations (11) is said to be I&I stabilizable with target dynamics $\bar{x}_{k+1} = \alpha(\bar{x}_k)$ if the hypotheses H1) to H4) of Theorem 2.2 are satisfied.

**Remark.** Rewriting (11) as $F(x, u) = F(x) + G(x, u)$ with $G(x, 0) = 0$, condition (28) can be relaxed. To prove asymptotic convergence of $x_k$ to $x^*$, it is sufficient to require
\[ \lim_{k \to \infty} (G(x_k, \psi(x_k, z_k)) - G(x_k, \psi(x_k, 0))) = 0. \]  
(18)

### III. Main Result

Let the discrete-time equivalent of the hybrid dynamics (9) rewritten in compact form (11) as
\[ x^e_{k+1} = \Phi^\delta(x_k, v_k, u_k) \]  
(20)
with $(x^e)' = (x', v) \in R^{n+1}; v = (v^1, ..., v^N)' \in R^N$. The main result can now be proved.

**Theorem 3.1:** Consider the input-affine continuous-time dynamics (11) satisfying Assumption A with input delay $\tau = N\delta$. Let the extended dynamics (20) with equilibrium $x^e = (x', 0)'$. Set $\bar{z} = (z^1, ..., z^N)'$, $\Phi^\delta = (\phi^1, ..., \phi^N)'$ with
\[ z^1 \equiv \phi^1(x_k, v_k) = v_k^1 - \gamma^\delta(x_k) \]
\[ z^2 \equiv \phi^2(x_k, v_k) = v_k^2 - \gamma^\delta(x_{k+1}) \]
\[ \vdots \]
\[ z^N \equiv \phi^N(x_k, v_k) = v_k^N - \gamma^\delta(x_{k+N-1}) \]  
(21)

Set
\[ \pi^\delta = (\pi^1, ..., \pi^N, \pi^\delta \circ (\alpha^\delta)^{N-1}(\cdot)) = F(\cdot, \gamma^\delta), \]
then the feedback $\psi^\delta(x', \bar{z}) : R^{n+1} \times R^N \to R$ designed to satisfy
\[ \lim_{k \to \infty} \bar{z}_k = 0 \quad \text{with} \quad \psi^\delta(0, 0) = \pi^\delta(\cdot) = \alpha^\delta(\cdot) \]  
and boundedness of the trajectories of the dynamics
\[ z^1_{k+1} = \phi^1_{\delta}(F(x_k, \psi^\delta(x_k, z_k))) \]  
(22a)
\[ x^e_{k+1} = F(x_k, \psi^\delta(x_k, z_k)) \]  
(22b)
with $z^i = \phi^i(x_k); i = 1, ..., N$, achieves globally asymptotic stability of the equilibrium of the closed loop dynamics
\[ x^e_{k+1} = F^\delta(x_k, \psi^\delta(x_k, \phi^\delta(x_k))). \]  
(23)

Proof: For one has to show that the four conditions in Theorem 2.2 are satisfied. Given $\gamma^\delta(\cdot) : R^n \to R$ defined according to (5) with control Lyapunov function $V(.) : R^n \to R$, let $\bar{z} \in R^n$ and define $\pi^\delta(\cdot) : R^n \to R^{n+N}$ as
\[ \bar{x}_k = (\bar{x}_k^1, ..., \bar{x}_k^N)' \]
with target dynamics on $R^n$ and $x^e_k = \alpha^\delta(\bar{x}_k)$.

**H^\delta 1 (Target dynamics)** - The dynamics with state $\bar{x} \in R^n$
\[ \bar{x}_{k+1} = F^\delta(\bar{x}_k, \gamma^\delta(\bar{x}_k)) = \alpha^\delta(\bar{x}_k) \]  
(24)
has a globally asymptotically stable equilibrium at $\bar{x}_0 \in R^n$.

**H^\delta 2 (Immersion and invariance condition)** - There exists $c^\delta(\cdot) : R^n \to R$ so that the so defined $\pi^\delta(\cdot) : R^n \to R^{n+N}$ satisfies
\[ F^\delta(\pi^\delta(\bar{x}_k), c^\delta(\bar{x}_k)) = \pi^\delta(F^\delta(\bar{x}_k, \gamma^\delta(\bar{x}_k))). \]  
(25)
This is satisfied by setting
\[ c^\delta(\bar{x}_k) = \gamma^\delta(\bar{x}_k) - \gamma^\delta((\alpha^\delta)^N(\bar{x}_k)). \]  
(26)

**H^\delta 3 (Implicit manifold)** - The following identity between sets holds by construction
\[ \{x' \in R^{n+N} | \bar{z}(x') = 0\} \subset \{x' \in R^{n+N} | x' = \pi^\delta(\bar{x}); \bar{x} \in R^n\}. \]

On these bases, it follows directly from Theorem 2.2 that the feedback $u = \psi^\delta(x, v, \bar{z})$ designed to drive $\bar{z}$ to zero (21) with boundedness of the extended state trajectories (22a-22b) and $\psi^\delta(\cdot, 0) = c^\delta(\cdot)$ achieves global asymptotic stability of the equilibrium of the closed loop dynamics (23).

**Remark** - The feedback $c^\delta(\cdot)$ in (26) corresponds to the $N$-steps ahead predicted feedback $\gamma^\delta(\cdot)$ with target dynamics (24) as predictor. In the delay free case $c^\delta(\cdot) = \gamma^\delta(\cdot)$.

**A. Input delayed sampled-data I&I stabilization when $\delta = \tau$**

Let us discuss more in detail the choice of the feedback $u = \psi^\delta(x, v, z)$ when $\delta = \tau$, equivalently $N = 1$. In such a case, Theorem 3.1 specifies as follows.

**Corollary 3.1:** Consider the input-affine continuous-time dynamics (1) satisfying Assumption A with input delay $\tau = \delta$. Let the extended dynamics on $R^{n+1}$
\[ x_{k+1} = F(x_k, v_k) \]  
(27)
\[ v_{k+1} = u_k \]
and or equivalently $x^e_{k+1} = F^\delta(x^e_k, u_k)$ with $x^e = (x', v)'$ and equilibrium $x^e = (x', 0)'$. Setting
\[ z = \phi^\delta(x', v) \quad \text{the feedback} \quad u = \psi^\delta(x, v, z) : R^{n+1} \times R \to R \]
achieves $z_k = 0$ with $\psi^\delta(\cdot, 0) = \gamma^\delta \circ \alpha^\delta(\cdot) = c^\delta(\cdot)$ and satisfying boundedness of the trajectories of
\[ z_{k+1} = \phi^\delta(F^\delta(x_k, v_k)) \]  
(29a)
\[ x_{k+1} = F^\delta(x_k, v_k) \]  
(29b)
\[ v_{k+1} = \psi^\delta(x_k, v_k, z_k) \]  
(29c)
with $z_0 = \phi^\delta(x_0)$ achieves globally asymptotic stability of the equilibrium $(x^*_e)$ of the closed loop dynamics

\[ x_{k+1} = F^\delta(x_k, v_k) \]
\[ v_{k+1} = \psi^\delta(x_k, v_k, \phi^\delta(x_k, v_k)) \].

In that particular case, one sets:

- $\pi^\delta : R^n \rightarrow R^{n+1}$ as $\pi^\delta(\xi) = (\xi^+, \psi(\xi))$,
- $\phi^\delta(x, v) : R^{n+1} \rightarrow R$ as $\phi^\delta(x, v) = v - \gamma^\delta(x) = z_k$,
- $c^\delta : R^n \rightarrow R$ as $c^\delta(\xi) = \phi^\delta(F^\delta(\xi, \psi(\xi))) = \phi^\delta(\alpha^\delta(\xi))$.

The four conditions $H^1, H^2, H^3, H^4$ for $N = 1$ as:

**Target dynamics** - The dynamics over $R^n$

\[ \xi_{k+1} = F^\delta(\xi_k, \psi^\delta(\xi_k)) = \alpha^\delta(\xi_k) \] (31)

has a GAS equilibrium $\xi^*_e \in R^n$ and $x^*_e = \pi^\delta(\xi^*_e)$;

**Immersion condition** - $\pi^\delta$ and $\phi^\delta$ satisfy

\[ F^\delta(\pi^\delta(\xi_k), c^\delta(\xi_k)) = \pi^\delta(F^\delta(\pi^\delta(\xi_k))) \] (32)

**Implicit manifold** - $\phi^\delta$ and $c^\delta$ satisfy the identity

\[ \{x' \in R^{n+1} | \phi^\delta(x') = 0\} = \{x' | (x' = \pi^\delta(\xi), \xi \in R^n)\} \]

**Manifold attractivity and trajectory boundedness** - all the trajectories of the system

\[ z_{k+1} = \phi^\delta(F^\delta(x_k, \psi^\delta(x_k, \xi_k))) \]
\[ x_{k+1}^* = F^\delta(x_k, \psi^\delta(x_k, v_k)) \] (33a)
\[ v_{k+1}^* = \psi^\delta(x_k, v_k, \xi_k) \].

with $z_0 = \phi^\delta(x_0)$ are bounded and $\lim_{k \rightarrow \infty} z_k = 0$ with

$\psi^\delta(\cdot, 0)|_{\pi^\delta(\cdot)} = c^\delta(\cdot)$.

**B. Some constructive aspects**

In practice, approximated solutions only are implemented. Some constructive aspects are sketched for the computation of the solution around the continuous-time delay free control law. More in detail, when $N = 1$ dynamics (33) rewrites as

\[ z_{k+1} = \psi^\delta(x_k, v_k, z_k) - \gamma^\delta(x_{k+1}) \]
\[ x_{k+1} = F^\delta(x_k, \gamma^\delta(x_k) + z_k) \] (34a)
\[ v_{k+1} = \psi^\delta(x_k, v_k, z_k) \].

(34b) can be expanded around the target dynamics $\alpha^\delta(\cdot)$ as

\[ x_{k+1} = \alpha^\delta(x_k) + \frac{\partial F^\delta(x_k, v_k)}{\partial v} \Big|_{\alpha^\delta(x_k)} z_k + O(z^2) \]

and in (34a), $\gamma^\delta(x_{k+1})$ as

\[ \gamma^\delta(x_{k+1}) = \gamma^\delta(\alpha^\delta(x_k)) + \frac{\partial \gamma^\delta(x_k)}{\partial x} \frac{\partial F^\delta(x_k, v_k)}{\partial v} |_{\alpha^\delta(x_k)} z_k + O(z^2) \]

Choosing

\[ \psi^\delta(x, v, z) = \gamma^\delta(\alpha^\delta(x)) + \Gamma_1(x)z \] (35)

with $\Gamma_1(x)$ bounded such that

\[ |\Gamma_1(x)| \frac{\partial \gamma^\delta(x)}{\partial x} |_{\alpha^\delta(x)} \frac{\partial F^\delta(x, v)}{\partial v} |_{\alpha^\delta(x)} | \gamma^\delta(x) | < 1 \]

one gets in $O(z^2)$

\[ z_{k+1} = (\Gamma_1(x_k) - \frac{\partial \gamma^\delta(x_k)}{\partial x} |_{\alpha^\delta(x_k)} \frac{\partial F^\delta(x_k, v_k)}{\partial v} |_{\alpha^\delta(x_k)} z_k) + \gamma^\delta(\alpha^\delta(x_k)) \]
\[ x_{k+1} = \alpha^\delta(x_k) + \frac{\partial F^\delta(x_k, v_k)}{\partial v} |_{\alpha^\delta(x_k)} \gamma^\delta(\alpha^\delta(x_k)) \]
\[ v_{k+1} = \gamma^\delta(\alpha^\delta(x_k)) + \Gamma_1(x_k)z_k \]

where $x_k = \alpha^\delta(x_k)$ is the target dynamics assumed GAS and $\lim_{k \rightarrow \infty} z_k = 0$. Moreover, $\gamma^\delta(\alpha^\delta(x_k)) = \epsilon^\delta(x_k)$, the feedback computed one step ahead over the target dynamics which ensures the immersion condition (32). More in detail, because in $O(z^2)$

\[ F_1^\delta(x, v) = x + \delta(f(x) + g(x)v) + O(z^2) \]
\[ \gamma^\delta(x) = \gamma(x) + \frac{\delta}{2} \gamma(x) + O(z^2) \]
\[ \alpha^\delta(x) = x + \delta(f(x) + g(x)\gamma(x)) + O(z^2) \]

one gets in $O(z^2)$ and $O(z^3)$

\[ z_{k+1} = (\Gamma_1(x_k) - \frac{\partial \gamma^\delta(x_k)}{\partial x} |_{\alpha^\delta(x_k)} \frac{\partial F^\delta(x_k, v_k)}{\partial v} |_{\alpha^\delta(x_k)} z_k) + \gamma^\delta(\alpha^\delta(x_k)) \]
\[ x_{k+1} = \alpha^\delta(x_k) + \frac{\partial F^\delta(x_k, v_k)}{\partial v} |_{\alpha^\delta(x_k)} \gamma^\delta(\alpha^\delta(x_k)) \]
\[ v_{k+1} = \gamma^\delta(\alpha^\delta(x_k)) + \Gamma_1(x_k)z_k \]

From the previous equalities, one verifies that $\lim_{k \rightarrow \infty} z_k = 0$ with boundedness of the evolutions of $x$ and $v$. This implies the asymptotic stability of the approximate $x^*$ dynamics when $z = v - \gamma(x) - \frac{\delta}{2} \gamma(x)$ and consequently the asymptotic stability in first approximation of

\[ x_{k+1} = F^\delta(x_k, v_k) \]
\[ v_{k+1} = \Gamma_1(x_k)v_k + \psi^\delta(\alpha^\delta(x_k)) - \Gamma_1(x_k)\gamma^\delta(x_k) \]

i.e. the local asymptotic stabilization of the sampled-data feedback $u_k = \Gamma_1(x_k)v_k + \psi^\delta(\alpha^\delta(x_k)) - \Gamma_1(x_k)\gamma^\delta(x_k)$ over the input-delay continuous time dynamics (1).

**IV. THE LINEAR CASE AS AN EXAMPLE**

Consider the linear time invariant - LTI - single input continuous time dynamics with input-delay $\tau = N\delta$

\[ \dot{x}(t) = Ax(t) + Bu(t - \tau) \] (37)

and assume the pair (A, B) stabilizable - AL; i.e. there exist a row gain matrix $K \in R^{n\times r}$ and a symmetric positive definite matrix $Q \in R^{n\times n}$ such that $H^TQ + QH < 0$ is satisfied with $H = A + BK$.

**A. The sampled-data predictor solution**

As well known, a sampled-data stabilizing controller can be computed through a $N$-steps ahead predictor based controller designed on the delay free dynamics. A sampled-data stabilizing controller of the delay free dynamics exists under
and can be computed starting from the sampled-data equivalent model
\[ x_d(k + 1) = A^\delta x_d(k) + B^\delta u_d(k) \] (38)
with \( A^\delta := e^{\delta A}, B^\delta = \int_0^\delta e^{\tau A} B d\tau. \)

Following Section II, the solution takes the form \( u_d(k) = K^\delta x_d(k) \) where the gain \( K^\delta \) satisfies the equality
\[ x^T_k (A^\delta + B^\delta K^\delta)^T Q (A^\delta + B^\delta K^\delta) x_k - x^T_k Q x_k = \int_{k\delta}^{(k+1)\delta} x^T(s)(H^T Q + QH) x(s) ds. \]

When such an equality is satisfied we say that the digital feedback \( u_d(t) = K^\delta x_d(k) \) achieves at the sampling instants \( k \delta \) the same stabilizing performances as the continuous-time feedback \( u(t) = Kx(t) \). It is a matter of computation to verify that \( K^\delta \) admits an asymptotic expansion in \( \delta \)
\[ K^\delta := K_0 + \sum_{i=1}^{\infty} \frac{\delta^i}{(i+1)!} K_i = K + \frac{\delta}{2} KH + O(\delta^2) \]
which can be computed because \( B^\tau Q \neq 0 \) by equating terms of the same power in \( \delta \) in the equality below
\[ (A^\delta + B^\delta K^\delta)^T Q (A^\delta + B^\delta K^\delta) - Q = \int_{k\delta}^{(k+1)\delta} (e^{(A+BK)}\delta)^T (H^T Q + QH) e^{(A+BK)\delta} ds. \]

As recalled before, starting from \( K^\delta \), the predictor-based sampled-data state feedback takes the form \( u_d(k) = K^\delta z_d(k) \) with
\[ z_d(k) = e^{N\delta A} x_d(k) + \sum_{i=1}^{N} e^{(i-1)\delta A} B^\delta u_d(k-i) \] (39)
and \( \forall k \in [-N,\ldots,-1] \) initial conditions \( (x_d(0), z_d(0), u_d(i)) \) satisfying \( z_d(0) = e^{N\delta A} x_d(0) + \sum_{i=1}^{N} e^{(i-1)\delta A} B^\delta u_d(-i) \).

**Remark** - Such a solution corresponds to assign to the extended discrete-time equivalent model the eigenvalues of the matrix \( A^\delta + B^\delta K^\delta \) and the other \( N \) to 0.

**Remark** - In this linear context, several different approaches can be pursued to stabilize the discrete-time delay free dynamics independently on the continuous-time stabilizing controller.

### B. The sampled-data predictor based I&I stabilization

Let us now recast the problem in the I&I context assuming for simplicity of notations that \( N = 1 \) (\( \tau = \delta \)) so that the equivalent extended discrete-time dynamics takes the form
\[ x_{k+1} = A^\delta x_k + B^\delta v_k \]
\[ v_{k+1} = u_k. \]

According to the definitions in section III, we set \( \pi^\delta(\xi) = (\xi, K^\delta \xi), z^\delta(x,v) = v - K^\delta x \) and the target dynamics as
\[ x_{k+1} = (A^\delta + B^\delta K^\delta) x_k = H^\delta x_k. \]

The design results in finding \( u_k \) such that \( z_k \) goes to zero with boundedness of
\[ z_{k+1} = u_k - K^\delta x_{k+1} \]
\[ x_{k+1} = H^\delta x_k + B^\delta z_k \]
\[ v_{k+1} = u_k. \]

Assuming the following structure to the feedback \( u_k \):
\[ u = \Gamma_0 x + \Gamma_1^0 z + \Gamma_2^0 v \]
a possible solution is obtained by choosing:
1. \( \Gamma_0^0 = K^\delta H^\delta \)
2. \( \Gamma_1^0 \) so that \( \Gamma_1^0 - K^\delta B^\delta < 1 \)
3. \( \Gamma_2^0 = 0 \)

which gives:
\[ z_{k+1} = (\Gamma_1^0 - K^\delta B^\delta) z_k \]
\[ x_{k+1} = H^\delta x_k + B^\delta z_k \]
\[ v_{k+1} = \Gamma_1^0 v_k + (K^\delta H^\delta - \Gamma_1^0 K^\delta) x_k. \]

**Remark** - It is easily verified that the eigenvalues are equal to the ones of \( A^\delta + B^\delta K^\delta \) and the one of \( \Gamma_1^0 - K^\delta B^\delta \). It results that setting \( \Gamma_1^0 = K^\delta B^\delta \), the predictor based solution is recovered.

**Remark** - A different solution can be set by choosing \( \Gamma_0^0 = K^\delta A^\delta, \Gamma_1^0 < 1, \Gamma_2^0 = K^\delta B^\delta \).

### V. The van der Pol example

#### A. Example 2-predicted based controller

Let us consider the system in strict feedforward form studied in [4], which represents the Van der Pol oscillator
\[ x_1(t) = x_2(t) - x_2^3(t)u(t) \]
\[ x_2(t) = u(t). \]

with stabilizing continuous-time controller
\[ \gamma(x) = -x_1 - 2x_2 - \frac{x_3^2}{3} \]

and equivalent sampled-data delay free controller \( \gamma^\delta(x) \) computed according to section II. Assuming \( u(t) = u_d(k) \) for \( t \in [k\delta, (k+1)\delta) \), one gets the sampled data equivalent dynamics of order 3 in \( \delta \) (finite sampling)
\[ x_{1}(k+1) = x_{1}(k) + \delta(x_2(k) - x_3^2(k)u_d(k)) \]
\[ + \frac{\delta^2}{2!}(u_d(k) - 2x_2(k)u_d(k)) - \frac{\delta^3}{3!} u_d^3(k) \]
\[ x_{2}(k+1) = x_{2}(k) + \delta u_d(k) \]

which specifies the predictor dynamics. When the delay is equal to \( \delta \) (\( N = 1 \)), one computes at time \( t = k\delta \) the
state predictor $z(k)$ and the sampled-data stabilizing feedback takes the form $u_d(k) = \gamma \delta (z(k))$ with

$$
\begin{align*}
    z_1(k) &= x_1(k) + \delta (x_2(k) - x_2^2(k) u_d(k-1)) \\
    &+ \frac{\delta^2}{2!} (u_d(k-1) - 2x_2(k)u_d^2(k-1)) - \frac{\delta^3}{3} u_d^3(k-1) \\
    z_2(k) &= x_2(k) + \delta u_d(k-1).
\end{align*}
$$

(45)

(46)

B. Example 2-I &I stabilizing data controller

According to section III, one sets:

- Extended dynamics

$$
\begin{align*}
    x_1(k+1) &= x_1(k) + \delta (x_2(k) - x_2^2(k) v(k)) \\
    &+ \frac{\delta^2}{2!} (v(k) - 2x_2(k)v^2(k)) - \frac{\delta^3}{3} v^3(k) \\
    x_2(k+1) &= x_2(k) + \delta v(k).
\end{align*}
$$

- Target dynamics: given $\gamma(x)$, there exists $\gamma^\delta(x)$ which stabilizes the dynamics $x(k+1) = \alpha^\delta(x(k))$ below

$$
\begin{align*}
    x_1(k+1) &= x_1(k) + \delta (x_2(k) - x_2^2(k) \gamma^\delta(x_1(k))) \\
    &+ \frac{\delta^2}{2!} (\gamma^\delta(x(k)) - 2x_2(k)\gamma^\delta(x(k))^2) - \frac{\delta^3}{3} \gamma^\delta(x(k))^3 \\
    x_2(k+1) &= x_2(k) + \delta \gamma^\delta(x(k)).
\end{align*}
$$

- Manifold: $z = \theta^\delta(x) = v - \gamma^\delta(x)$.

The problem results in finding $\psi^\delta(x,v,z)$ such that $\lim_{k \to \infty} = 0$ with boundedness of the trajectories of the system below

$$
\begin{align*}
    z(k+1) &= \psi^\delta(x(k),v(k),z(k)) - \gamma^\delta(x(k+1)) \\
    x_1(k+1) &= x_1(k) + \delta (x_2(k) - x_2^2(k)v(k)) \\
    &+ \frac{\delta^2}{2!} (v(k) - 2x_2(k)v^2(k)) - \frac{\delta^3}{3} v^3(k) \\
    x_2(k+1) &= x_2(k) + \delta v(k) \\
    v(k+1) &= \Gamma^\delta_1(x(k),v(k),z(k),z(k+1)).
\end{align*}
$$

Setting: $\psi^\delta(x,v,z) = \Gamma^\delta_1(x(k),z(k)) + \gamma^\delta(x(k+1))$, one has to assure $\lim_{k \to \infty} z(k)$ and boundedness of

$$
\begin{align*}
    z(k+1) &= \Gamma^\delta_1(x(k),z(k)) \\
    x_1(k+1) &= x_1(k) + \delta (x_2(k) - x_2^2(k)v(k)) \\
    &+ \frac{\delta^2}{2!} (v(k) - 2x_2(k)v^2(k)) - \frac{\delta^3}{3} v^3(k) \\
    x_2(k+1) &= x_2(k) + \delta v(k) \\
    v(k+1) &= \Gamma^\delta_1(x(k),v(k) - \Gamma^\delta_1(x(k),z(k)) + \gamma^\delta(x(k+1)).
\end{align*}
$$

to ensure stability of the closed loop sample-data input-delayed dynamics

$$
\begin{align*}
    x_1(k+1) &= x_1(k) + \delta (x_2(k) - x_2^2(k)v(k)) \\
    &+ \frac{\delta^2}{2!} (v(k) - 2x_2(k)v^2(k)) - \frac{\delta^3}{3} v^3(k) \\
    x_2(k+1) &= x_2(k) + \delta v(k) \\
    v(k+1) &= \Gamma^\delta_1(x(k),v(k) - \Gamma^\delta_1(x(k),z(k)) + \gamma^\delta(x(k+1)).
\end{align*}
$$

(18)

When $z = 0$, the manifold is reached and one recovers $\psi^\delta(x(k),v(k),0) = \gamma^\delta(\alpha^\delta(x(k)))$, the closed loop stable dynamics on the manifold.

VI. CONCLUSIONS

In this paper the I&I stabilizing approach has been used to investigate sampled-data feedback stabilization of input-affine-delayed continuous-time dynamics. The sampled-data predictor based controller is recovered as a particular case in the I&I approach (dead beat strategy). Work is progressing to properly address the computational aspects and robustness properties.

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