The Price of Anarchy in Routing Games

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Eliciting Coordination with Rebates

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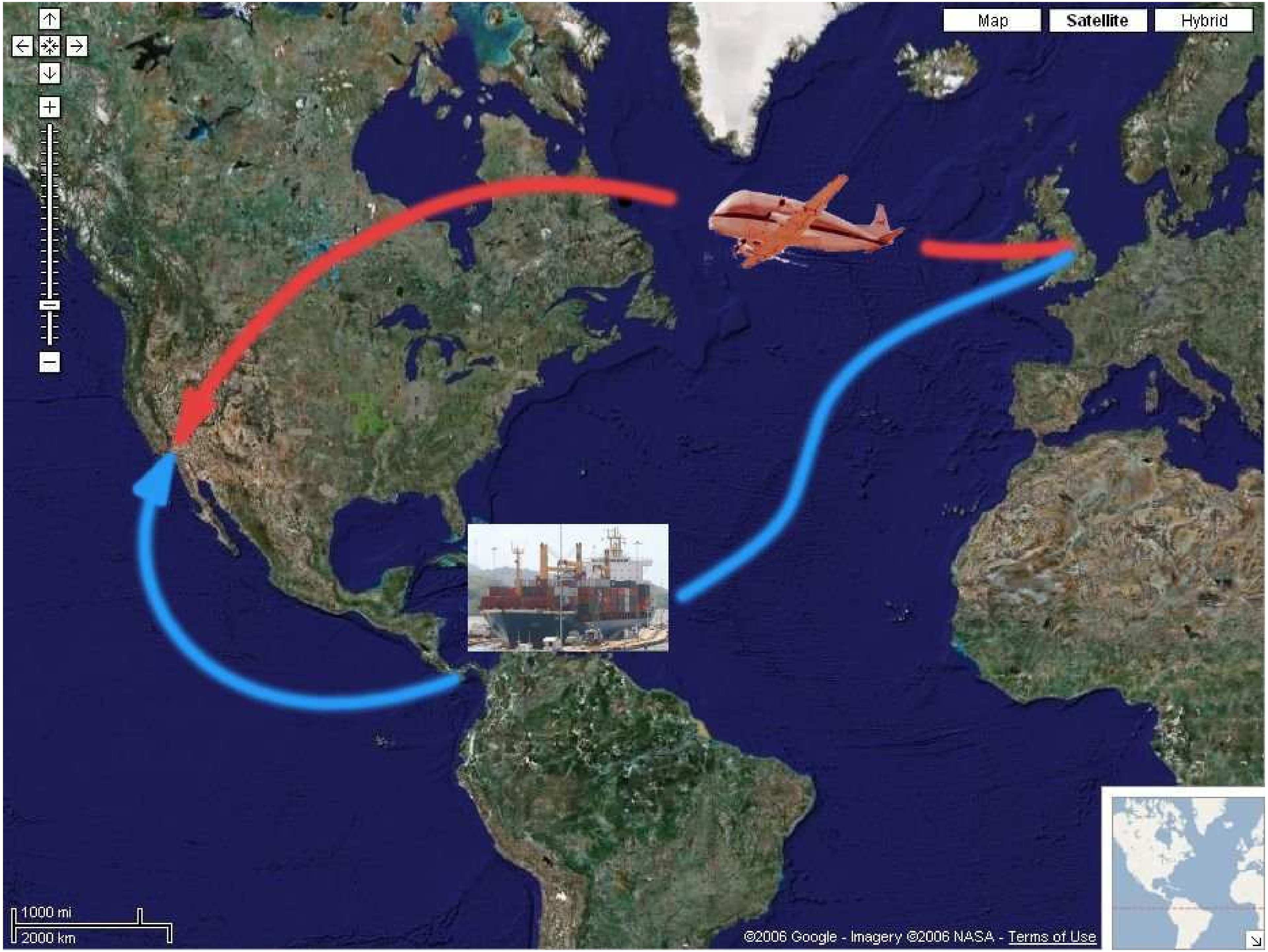
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1

Competitors make selfish decisions

How efficient is the system?

centralized	**competition**
global perspective	selfish
social optimum =	**Nash equilibrium** =
min total cost s.t. solution is feasible	$\forall$ user: min individual cost s.t. solution is feasible
efficient, unfair	not efficient, no regret


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HSBC

Price of Anarchy

- Ideas and techniques: in the interface of Management Science, Economics and Computer Science

- Use worst-case analysis to measure efficiency of equilibria

- Compare equilibria to good upper bounds on social surplus

$\longrightarrow$ Usually social optimum

POA=worst-case performance ratio of equilibrium to optimum

POA measures the "price" of not having central coordination in system

Price of Anarchy: Consequences

If **POA** small, owner may want to let participants choose:
there is not much to gain from dictating what people should do!

If **POA** large, owner may want to re-design system or to give incentives to achieve more efficient results:

- Mechanism Design
  e.g., if certain design guidelines are used, then equil. $\approx$ opt.

- Pricing and Stackelberg games
  e.g., system's owner prices resources s.t. equil. $=$ opt.

This talk: system's owner offers rebates

Outline

- Competition among Price-Taking Participants
  Nash Equilibrium & Social Optimum

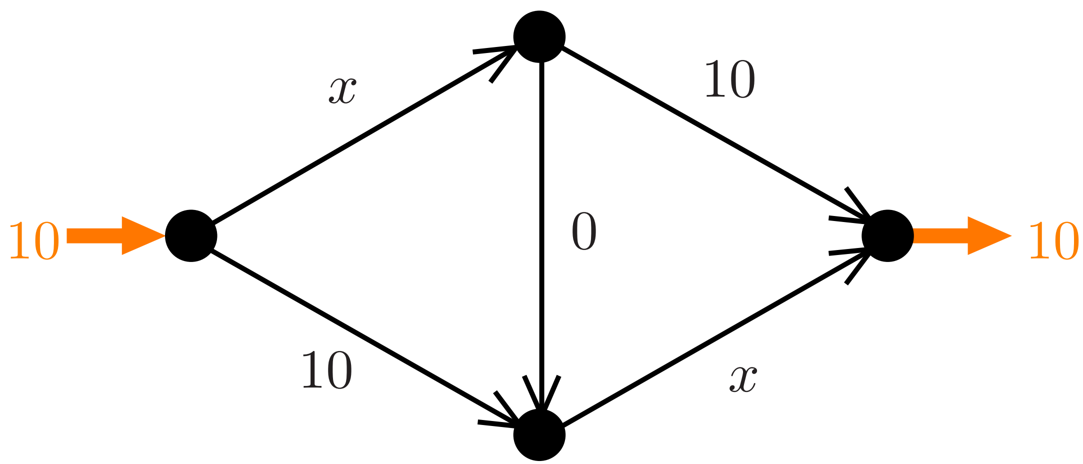
- The Price of Anarchy

- Some coordination mechanisms

- Eliciting Coordination with Rebates

The Competitive Model

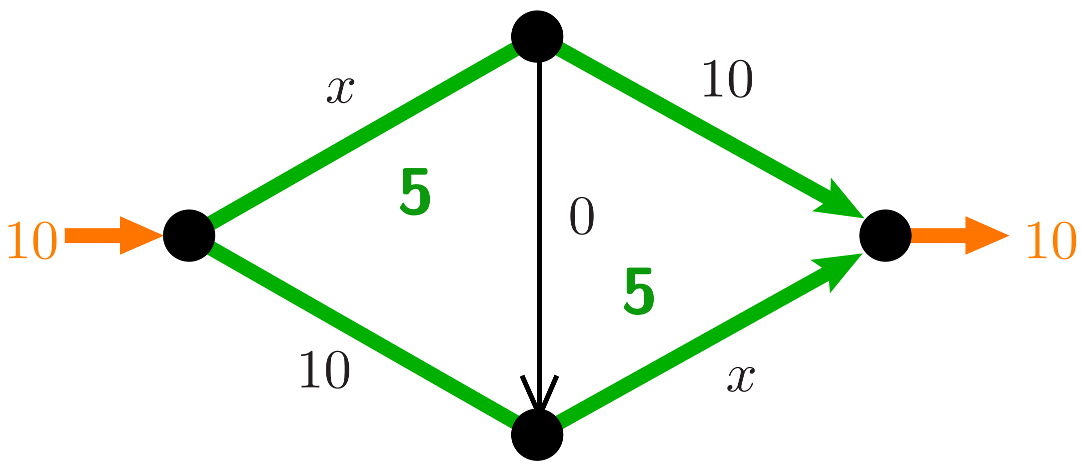
- Network $G = (N, A)$ with OD pairs of rate r_k, $k \in K$
- Resources have cost functions c_a depending on total demand x_a
- Participants are price-takers
- Example: Braess' Instance



Social Optimum

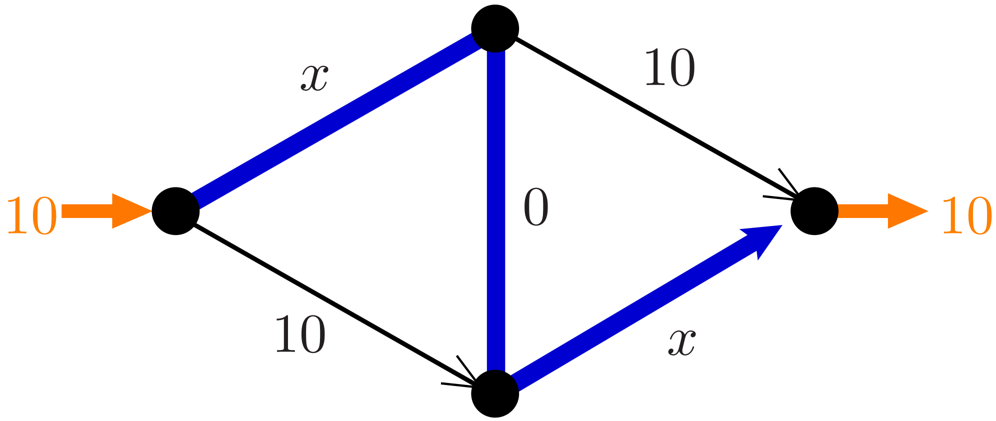
- Participants' Cost $C(x) \ := \ \sum_{a \in A} c_a(x_a)x_a$

Definition: A **SO** is a feasible flow x^{SO} that minimizes $C(\cdot)$



Nash (Wardrop) Equilibrium

Definition: A **NE** is a flow x^{NE} such that nobody can switch to a path with smaller travel time Wardrop'52



- **NE** characterized by a Variational Inequality: Smith'79

  $$\sum_{a \in A} c_a(x_a^{\text{NE}}) x_a^{\text{NE}} \leq \sum_{a \in A} c_a(x_a^{\text{NE}}) x_a \text{ for all } x$$ Dafermos'80

- **NE** exists & essentially unique Beckmann, McGuire & Winsten'56

Price of Anarchy

Price of Anarchy measures impact of lack of central coordination

Papadimitriou STOC'01

$$\textbf{POA} := \max_{\text{instances}} \frac{C(\textbf{NE})}{C(\textbf{SO})}$$

- For unrestricted cost functions, **POA** is unbounded
- We will assume a fixed set of cost functions $\mathcal{C}$, e.g., affine

Price of Anarchy — Affine Costs

Theorem.
In networks with *affine* costs,

Roughgarden & Tardos JACM'02
Correa, Schulz & Stier-Moses MOR'04
Correa, Schulz & Stier-Moses IPCO'05

$$C(\textbf{NE}) \leq \frac{4}{3} C(\textbf{SO})$$

Selfishness drives the system close to optimality

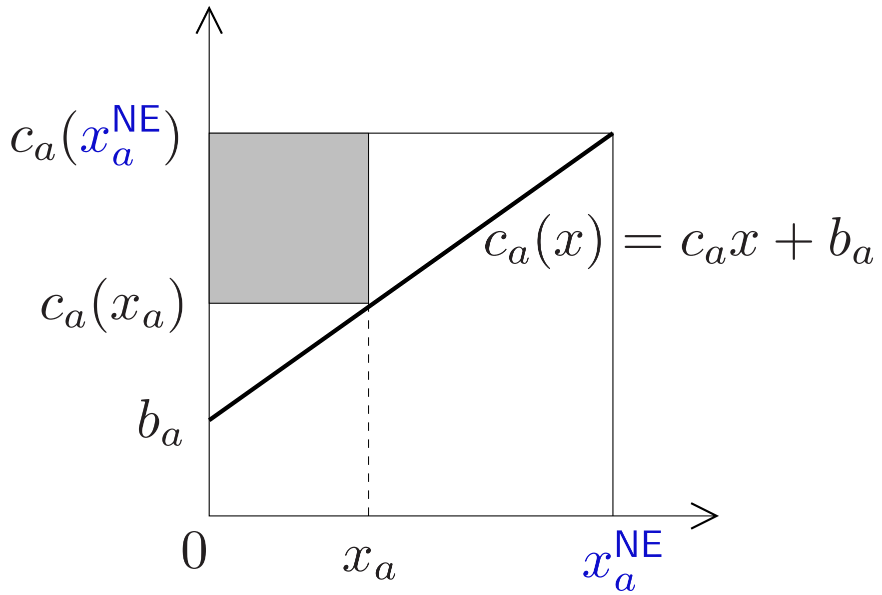
Corollary. Braess' Instance is worst possible

Proof of 4/3 Result

Correa, Schulz & Stier-Moses IPCO'05

$$C(x^{\text{NE}}) = \sum c_a(x_a^{\text{NE}})x_a^{\text{NE}} \leq \sum c_a(x_a^{\text{NE}})x_a \qquad (\leftarrow \textbf{VI})$$

$$= \sum c_a(x_a)x_a + \sum (c_a(x_a^{\text{NE}}) - c_a(x_a))x_a$$



$$\leq C(x) + \tfrac{1}{4} \sum c_a(x_a^{\text{NE}})x_a^{\text{NE}} = C(x) + \tfrac{1}{4} C(x^{\text{NE}})$$

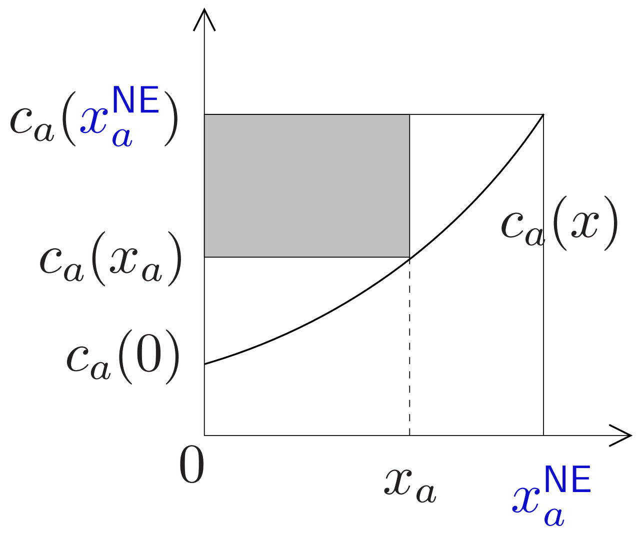
Price of Anarchy — General Costs

Roughgarden JCSS'03

Correa, Schulz & Stier-Moses MOR'04

Correa, Schulz & Stier-Moses IPCO'05

Let $\beta(\mathcal{C}) = \max_{c \in \mathcal{C}} \left\{ \frac{\text{shaded area}}{\text{big rectangle}} \right\}$



Theorem. If costs are drawn from a family of *continuous* costs $\mathcal{C}$,

$$C(\textbf{NE}) \leq (1 - \beta(\mathcal{C}))^{-1} \; C(\textbf{SO})$$

Bounds on the Price of Anarchy

Roughgarden & Tardos '02, Roughgarden '03

Correa, Schulz & Stier-Moses '04,'05

Theorem. For polynomials of maximum degree p,
$C(\textbf{NE})/C(\textbf{SO})$ is bounded by

degree	1	2	3	4	...	p
POA	$4/3$	1.626	1.896	2.151	...	$\Omega(p/\ln p)$

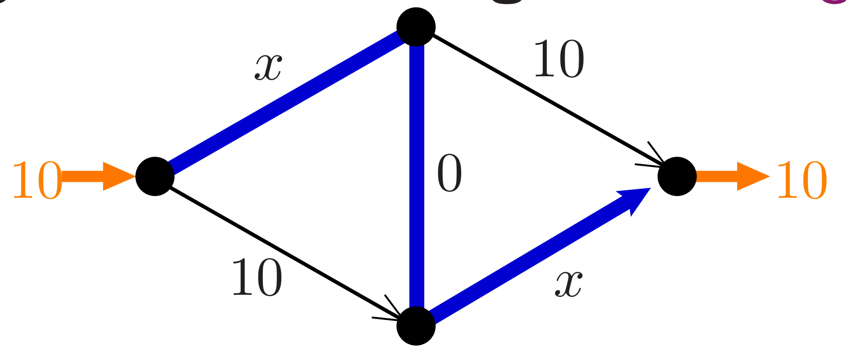
Corollary. Braess' Instance is worst possible

Game and system objectives are partially 'aligned'.
Then, selfishness drives the system close to optimality

How to enforce coordination among users?

Stackelberg games: the *leader* determines his actions, predicting that users *(followers)* will react selfishly

- Removing some edges: **network design** Roughgarden FOCS'02



$\Rightarrow$ finding the best set of edges to remove is NP-hard...

- Controlling a given **proportion of the demand**

Roughgarden STOC'01
Kaporis, Spirakis SPAA'06

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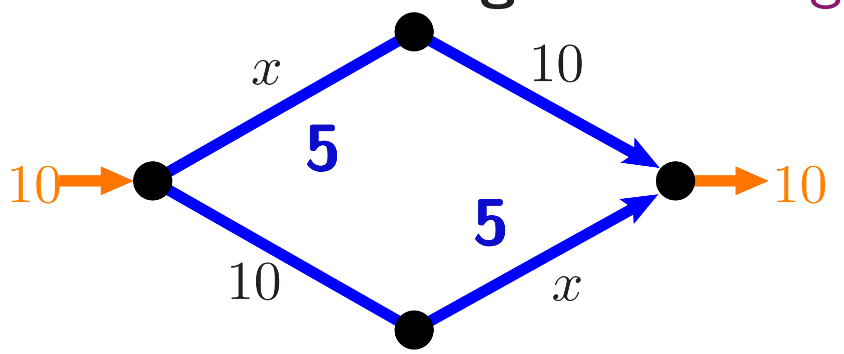

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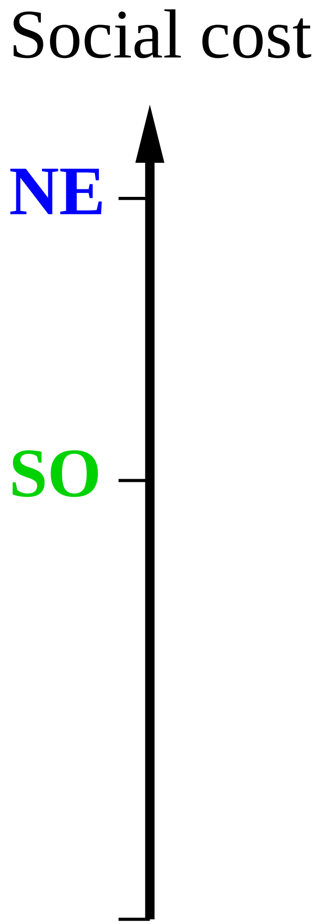
Network Pricing

The following articles use Stackelberg games and pricing to

- min. participants' cost — Dupuit 1849, Pigou'20, Knight'24, . . .
  - Some articles did not put constraints on the sign of prices (hence, rebates allowed)
  - When users pay for externalities they create, system is optimal
- max. owner's profit — Acemoglu & Ozdaglar'03, Correa & Stier-Moses'07
- min. social cost (including prices paid) — Cole, Dodis & Roughgarden'06

We focus on rebates and minimize **social cost**

The Owner Could Save by Eliciting Coordination



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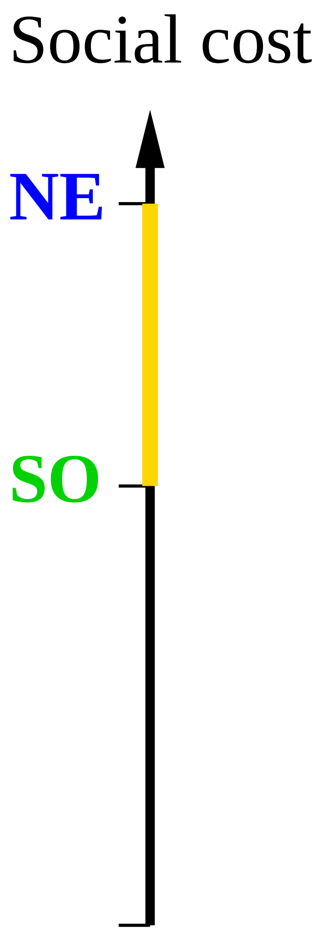

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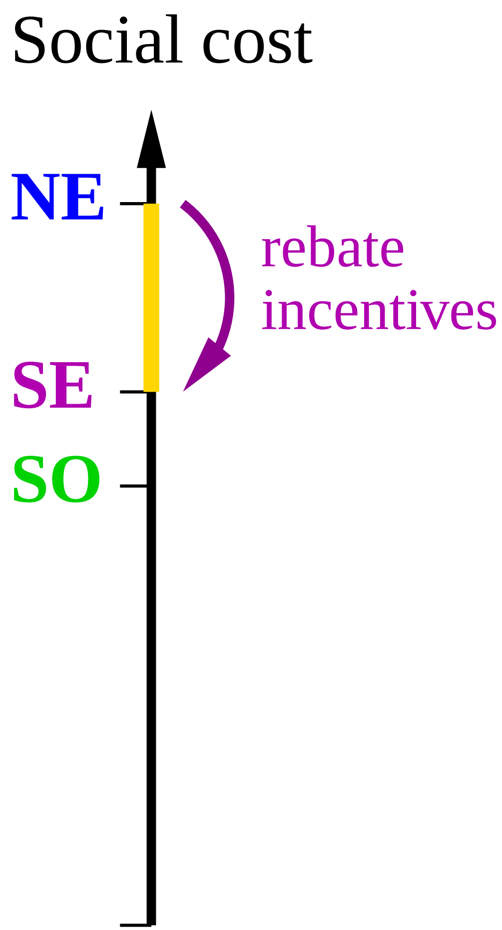

Stackelberg Game

- We focus on monetary incentives
- A mechanism where participants pay extra for popular resources may not be acceptable
- It may be more fair to offer rebates/subsidies for unpopular ones
  - system's owner offers rebate s_a for resource a
  - participants perceive cost $[c_a(x_a) - s_a]^+$
  - actual rebate paid $= \min(s_a, c_a(x_a))$
    (perceived costs can't be negative: rebates cannot exceed costs)

Eliciting Coordination with Rebates

- The system's owner (Stackelberg leader) offers rebates to participants

- The vector of rebates s induces a 2^{nd}-stage equilibrium x^s

The leader **modifies the participants' game** to elicit a more desirable outcome

- Owner computes optimal rebates s^* predicting participants' reactions. The pair (s^*, x^{s^*}) is a Stackelberg equilibrium (**SE**)

Social Cost

$$C_\rho(s) := \underbrace{\sum x_a^s \left[c_a(x_a^s) - s_a\right]^+}_{\text{participants' perceived cost}} + \rho \underbrace{\sum x_a^s \min(c_a(x_a^s), s_a)}_{\text{rebate cost}}$$

$$= \underbrace{\sum x_a^s c_a(x_a^s)}_{\text{participants' real cost}} + (\rho - 1) \underbrace{\sum x_a^s \min(c_a(x_a^s), s_a)}_{\text{rebate cost}}$$

Parameter ρ quantifies sensitivity of leader to cost of rebates

- $\rho \to 0$: unlimited rebate budget $\Rightarrow s = (c_a(x_a^{SO}))_{a \in A}$ and $x^s = x^{SO}$
- $\rho = 1$: two terms of social cost equally important $\Rightarrow C(\textbf{SO}) \leq C_\rho(s) \leq C(\textbf{NE})$
- $\rho \to \infty$: the leader cannot afford rebates $\Rightarrow s = 0$ and $x^s = x^{NE}$

Is the SO a relevant reference?

- We wanted to quantify the cost of lacking coordination

- By comparing **NE** to **SO**, we implicitly acknowledge that the latter is attainable

But this is not necessarily true!

- E.g., prices for different participants under a **SO** may vary:
  its unfairness may prevent its implementation

- We propose to compare to an achievable coordinated solution

Price of Anarchy Revisited

We redefine the price of anarchy in the Stackelberg game as:

$$\textbf{SPOA} := \frac{C(\textbf{NE})}{C(\textbf{SE})} = \frac{C_\rho(0)}{\min_{s\geq 0} C_\rho(s)}$$

As this compares an equilibrium to what can be achieved with rebates, **SPOA** is a less pessimistic estimate for the efficiency-loss

If **POA** is small, don't do anything! Otherwise:

$$\text{If } \textbf{SPOA} \text{ is } \begin{cases} \text{large, then rebates good to improve system} \\ \text{small, then additional coordination is beneficial} \end{cases}$$

Case $\rho \leq 1$

Participants' perceived cost is more important than cost of rebates

$$C_\rho(s) = \underbrace{\sum x_a^{\mathsf{s}} \left[c_a(x_a^{\mathsf{s}}) - s_a\right]^+}_{\text{participants' perceived cost}} + \rho \underbrace{\sum x_a^{\mathsf{s}} \min(c_a(x_a^{\mathsf{s}}), s_a)}_{\text{rebate cost}}$$

- The optimal rebate is to offer $c_a(x_a^{\mathsf{SO}})$
- In the 2^{nd}-stage game, the unique equilibrium is the **SO**
- The inefficiency is **SPOA** $= \frac{C_\rho(0)}{\min_s C_\rho(s)} = \frac{C(\mathbf{NE})}{\rho C(\mathbf{SO})} = \frac{\mathbf{POA}}{\rho}$

Stackelberg Equilibrium for $\rho \geq 1$

- Consider substitutable resources (a.k.a. parallel-link networks)
  – each user selects a single resource

Theorem. (A nonlinear complementary problem characterization)

In a **SE**, $c(x) + xc'(x)$ is minimum and equal for all subsidized arcs

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Contrast that to a well-known result: Beckmann et al.'56

In a **SO**, $c(x) + xc'(x)$ is minimum and equal for all used arcs

Stackelberg Equilibrium for $\rho \geq 1$

Theorem. (Characterization of optimal rebates)

Sort resources wrt $c(x) - xc'(x)$. In a **SE**:

– no need to subsidize the 'smallest' arcs because they are cheap

– the 'largest' arcs are not used because they are expensive

– those in the middle are subsidized

For affine costs $c(x) = ax + b$, we have $c(x) - xc'(x) = b$

Rebates are $\frac{1}{2}[b - D_\rho]^+$, where constant D_ρ can be found efficiently

SPOA for $\rho \geq 1$

Theorem. For substitutable resources and affine costs:

$$\textbf{SPOA} \leq \frac{4\rho}{4\rho - 1} \qquad \text{(tight bound)}$$

SPOA for $\rho \geq 1$

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- If $\rho \to 1$, then **SPOA** $\to$ **POA** $= 4/3$

  small ρ: system benefits from rebate mechanism

- If $\rho \to \infty$, then **SPOA** $\to 1$

  large ρ: coordination not useful

Final Remarks: Other Situations To Consider

Look for additional insights for computing optimal rebates efficiently and on the price of anarchy for:

- Participants: **price-takers** / price-setters
- Market structure: **substitutable resources** / single / **multiple commodities**
- Cost functions: **affine** / general
- Participants have different price sensitivities
- Rebates and taxes together
- Other definitions of social cost