The MIMO Broadcast channel: From Information Theory Principles to Transmission schemes

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Outline

MIMO Broadcast Channel

Assumption

- MIMO broadcast channel: K receivers(Rx) with N antennas and a transmitter (Tx) with $M > 1$ antennas.
- Characterize the error event at the Rx side when
  - No CSIT is assumed at the Tx side
  - Full CSIT is assumed at the Tx side

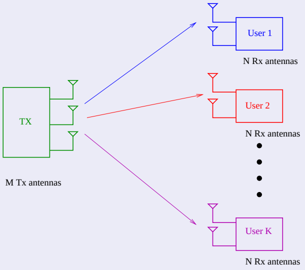

Figure: A MIMO broadcast channel

✔ In Point-to-Point MIMO system, the capacity is independent from the availability of CSI at the transmitter side at the high SNR regime.

✔ In the Multiuser BC, the CSIT assumption affects largely the capacity region that can be achieved.

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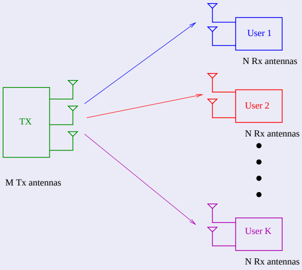

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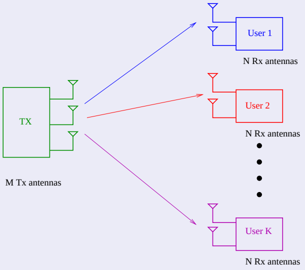

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Error events in the MIMO BC

- Question : Is it possible to propose a strategy that allows simultaneous transmission and outperforms time sharing when no CSIT is assumed?

  - Answer : Diversity Mutiplexing Tradeoff analysis

- The detection error event at user k occurs when user k is unable to decode his own message.

$$P_e^{(k)} = \text{Prob}\left\{\hat{X}_k \neq X_k \mid X \text{ is sent}\right\} = \text{Prob}\left\{R_k \leq \bar{R}_k | R \text{ is transmitted}\right\}$$

- In general, a BC is characterized by the system error probability, $P_{e,\text{sys}}$, and refers to the event that at least one user is in error. $P_{e,\text{sys}}$ can be bounded by

$$P_e^{(k)} \leq P_{e,\text{sys}} \leq \sum_{i=1}^{K} P_e^{(i)} \quad \text{for all } k$$

- When $\bar{R}_k$ scales as $r_k \log_2 \text{SNR}$, DMT can be defined accordingly.

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Outage Analysis without CSIT

- ✔ The DMT is a powerful tool to evaluate strategy's performance in MIMO channel [Zheng and Tse, 2003], MAC channel [Tse et al., 2004], cooperative networks [El Gamal].
- ✔ It was first introduced in [Zheng and Tse, 2003] to characterize error event over a $M \times N$ MIMO.
  The tradeoff curve is a piecewise linear function connecting the points $(r, (N - r)(M - r))$, with $r = 1 \ldots \min(M, N)$;

Theorem

The Diversity Multiplexing Tradeoff (DMT) of a MIMO Broadcast Channel when no CSI is available at the transmitter side is given by

$$d_{\mathrm{BC}}(r) = d_{M \times N}(Kr_u)$$

where r_u denotes the multiplexing gain per user, $r = Kr_u$ is the total multiplexing gain, and $d_{M,N}(r)$ is the DMT of a $M \times N$ MIMO Rayleigh channel

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Outer bound (1/2)

2-users BC is considered. The generalization to the K-users case is immediate.

- Some definitions ...

  A two-users BC is said to be physically degraded if $X \rightarrow Y_1 \rightarrow Y_2$ forms a Markov chain.

  For a MIMO degraded BC,

$$\boxed{I_d(X;Y_1,Y_2) = I(X;Y_1)}$$

Lemma

The outage probability of a physically degraded BC is a lower bound of the general broadcast channel case.

Proof:

- The outage probability of a general BC is

$$P_o = \text{Prob}\left\{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_{\{1,2\}}\right\}$$

  with $\mathcal{O}_1$ is the outage event that occurs when user 1 is in outage, but user 2 is not in outage.
- For a degraded BC : $I(X,Y_1) \geq I(X,Y_2)$.

  If user 1 is in outage, then user 2 is mandatory in outage. Then, $\mathcal{O}_1 = \varnothing$.
- Then,

$$P_{o,d} = \text{Prob}\left\{\mathcal{O}_2 \cup \mathcal{O}_{\{1,2\}}\right\} \leq P_o$$

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Outer bound (2/2)

To find an outer bound, the degraded cooperative BC is considered,

- The cooperative degraded BC is in outage, if the equivalent single-user channel is in outage

$$P_{o,cd} = \text{Prob}\left\{ \underbrace{I_d(X;Y_1,Y_2)}_{I(X;Y_1)} \leq R_t = KR_u \right\}$$
$$\dot{\geq} \ \text{SNR}^{-d_{M\times N}(Kr_u)}$$

- Then,

$$\text{SNR}^{-d_{M\times N}(Kr_u)} \doteq P_{o,cd} \leq P_{o,d} \leq P_o \doteq \text{SNR}^{-d(r_u)}$$

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Inner bound (1/2)

☞ Objective : To propose a coding scheme that achieves the outer bound.

- By superposing the K-data streams at the Tx side, the received signal at user k is

$$\boldsymbol{y}_k = \boldsymbol{H}_k \, \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K) + \boldsymbol{n}_k$$

$\psi(.)$ represents a linear combination of the K-codewords.

- Outage event occurs if at least one user is in outage, and is defined by

$$\mathcal{O} \triangleq \bigcup_{S} \mathcal{O}_S$$

with $S \subseteq \{1, \ldots, K\}$, and $\mathcal{O}_S \triangleq \left\{ H \in \mathbb{C}^{N \times KM} : I(\boldsymbol{x}_S; \boldsymbol{y} | \boldsymbol{x}_{S^c}, \boldsymbol{H} = H) \leq \sum_{i \in S} R_i \right\}$

- The outage event $\mathcal{O}_S$ at user k occurs when
  - ✔ User k is able to successfully decode data symbols in $S^c = \{1, \ldots, K\} - S$
  - ✔ User k is unable to decode the target rate of the streams in the set S.

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- Using a SIC decoder to cancel the contribution of signal in S^c, the equivalent channel model is

$$\boldsymbol{y}_{k,S} = \boldsymbol{H}_S \, \boldsymbol{x}_S + \boldsymbol{n}_k, \quad \boldsymbol{H}_S = [\boldsymbol{H}_k \dots \boldsymbol{H}_k] \in \mathbb{C}^{N \times M|S|}$$

  Outage probability of the equivalent model is

$$P(\mathcal{O}_S) \doteq \text{SNR}^{-d_{M\times N}(\sum_{i\in S} r_i)}$$

- For the BC, outage probability is such that

$$P(\mathcal{O}) = P\left(\bigcup_S \mathcal{O}_S\right) \leq \sum_S P(\mathcal{O}_S)$$

  The dominant event corresponds to jointly decode the K-data streams at each receiver side, then

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Delay Requirement and achievable scheme

Example: $K = 3$ users with $N = 2$ Rx antennas and a Tx with $M = 2$ Tx antennas.

- The optimal DMT that can be achieved by each user is $d_{2,2}(3r)$.
- First intuitive scheme: Time sharing between the 3 users by sending a Golden codeword for each user.
- Second scheme allows simultaneous transmission and minimizes the delay, such that :
- ✗ The transmitted signal is the superposition of the 3 data streams, such that

$$X = \psi(X_1, X_2, X_3) = \sqrt{\frac{\text{SNR}}{3}}(X_1 + X_2 + X_3)$$
$$= \sqrt{\frac{\text{SNR}}{3}} \begin{bmatrix} x_1 & x_2 & x_3 \\ \gamma\sigma(x_3) & \sigma(x_1) & \sigma(x_2) \end{bmatrix}$$

- ✗ Each receiver decodes jointly the 3 data streams.
- ☞ Note that for the 2×2 MIMO configuration, X is a code with non-vanishing determinant, and achieves therefore the DMT $d_{2\times2}(r_t)$, with $r_t = 3r_u$ is the total multiplexing gain.

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&= \sqrt{\frac{\text{SNR}}{3}} \begin{bmatrix} x_1 & x_2 & x_3 \\ \gamma\sigma(x_3) & \sigma(x_1) & \sigma(x_2) \end{bmatrix}
\end{aligned}
$$

- ✘ Each receiver decodes jointly the 3 data streams.
- ☞ Note that for the 2×2 MIMO configuration, X is a code with non-vanishing determinant, and achieves therefore the DMT $d_{2\times 2}(r_t)$, with $r_t = 3r_u$ is the total multiplexing gain.

Motivations

☞ What does this error event corresponds to?

➠ With full CSIT, maximal throughput that can be achieved depends naturally on the channel's quality. In this case, error can be reduced to zero at the Rx side.

✘ In practical system, achieving a minimal rate at the receiver side independently of the channel's quality is the goal of the system designer.

At user k, error occurs if

$$P_e = \text{Prob}\left\{ R_k \leq \bar{R}_k | \bar{R} \text{ is transmitted} \right\}$$

To characterize the system error event, the probability that at least one user in error is considered.

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☞ [Weingarten et al., 2006] : With full CSIT, DPC achieves the capacity region of the MIMO Broadcast channel and is the optimal scheme.

Theorem : [Lee and Jindal, 2007]

At high SNR , the rate achieved by user k when a DPC precoder is used, with $M \geq KN$ is such as,

$$R_k \doteq \log |\boldsymbol{I} + \boldsymbol{P}_k \boldsymbol{F}_k \boldsymbol{F}_k^\dagger|$$

where F_k, $k = 1 \ldots K$ is the projection of H_k onto the null-space of $\{H_j\}_{j=1}^{k-1}$, which is equivalent to a $N \times (M - (k-1)N)$ i.i.d Rayleigh channel.

☞ MIMO interpretation:

User k is equivalent in term of its achievable rate to a $N \times (M - (k-1)N)$ MIMO system

BC is equivalent to K parallel independent MIMO channel with different antenna configurations $N \times (M - (k-1)N)$.

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DPC and Rx thresholds (1/4)

First assumption: Symmetrical constraint, all the thresholds are equal $\bar{R}_1 = \ldots = \bar{R}_K = \frac{\bar{R}}{K}$.

- ➥ Error event is dominated by the less conditioned user : User that sees all other users' signals as interference.
- ✘ The error probability decays as $\text{SNR}^{-d(\bar{r})}$, where

$$d(\bar{r}) = d_{N \times [M-(K-1)N]}\left(\frac{\bar{r}}{K}\right)$$

Second assumption: Optimal DPC scheme with adequate thresholds allocation

- Objective : To improve the system error performance by allocating lower data rate threshold at the most unreliable user.
- Thresholds $\bar{r}_k$ at the receiver side should be that the global system error probability is minimized.

$$P_e \doteq \min_{\bar{r}_1,\ldots,\bar{r}_K} \text{SNR}^{-\min_{i=1\ldots K} d_i(\bar{r}_i)}$$

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➥ The optimization problem is

$$\text{Objective: } \max_{\bar{r}_1,\ldots,\bar{r}_K} \left[\min_{k:r_k>0} (N-\bar{r}_k)(M-(k-1)N-\bar{r}_k) \right]$$

$$\text{Such that: } \quad \bar{r}_1+\ldots+\bar{r}_K=\bar{r}; \quad \bar{r}_k \in [0,N]$$

➥ The optimal solution of this problem depends on the decreasing order of β_k^i, $k=1\ldots K$ and $i=0\ldots N$ with

$$\beta_k^i=(N-i)(M-(k-1)N-i)=(N-i)(M_k-i)$$

➥ The optimal DMT is the piecewise linear function connecting the points (α_i,δ_i), $i=1,\ldots,KN$, where

- ✔ $(\delta_1,\ldots,\delta_{KN},0)$ is the ordered combination of β_k^i
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Example 1 : Single user case - Ordering of δ_k is immediate

Optimal DMT with single antenna users

The DMT is the piecewise linear function connecting the points (α_k, δ_k), $k = 1 \dots K$, with

$$\delta_k = M - k + 1, \qquad \alpha_k = \sum_{j=0}^{k-2} \frac{k-j-1}{M-j}$$

Proof:

- $\alpha_1 = 0$, $\delta_1 = M$
- $\bar{r} \in [\alpha_1, \alpha_2]$, then user 1 is only active, and $\bar{r} = \bar{r}_1$.

$$d(\bar{r}) = d_1(\bar{r}_1) = M(1 - r_1)$$

$d(\alpha_2) = M - 1 \Rightarrow \alpha_2 = \frac{1}{M}$

- $\bar{r} \in [\alpha_2, \alpha_3]$, then user 1 and 2 are active. The DMT curve is such that

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α_3 is such that $\alpha_3 = \alpha_3^1 + \alpha_3^2$, with $d_1(\alpha_3^1) = M - 2$ and $d_2(\alpha_3^2) = M - 2$,

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DPC and Rx thresholds (4/4)

Example 1 : MIMO BC with $M = 5$ Tx antennas and $K = 2$ Rx with $N = 2$ antennas.

User 1 : 5×2 MIMO channel and User 2: 3×2 MIMO channel

- The tradeoff curve is the piecewise linear function connecting $(0, 10)$, $(\alpha_2, 6)$, $(\alpha_3, 4)$, $(\alpha_4, 2)$, $(4, 0)$
- The four pieces of the tradeoff curve are : :

✔ $0 \leq r \leq \alpha_1$:

User 1 is only active, $r = r_1$ and α_2 is such that $d_1(\alpha_2) = 6$.

✔ $\alpha_2 \leq r \leq \alpha_3$:

User 1 and 2 are both active with $r = r_1 + r_2$, $0 \leq r_1, r_2 \leq 1$

$\alpha_3 = \alpha_3^1 + \alpha_3^2$, such that :

$d_1(\alpha_3^1) = d_2(\alpha_3^2) = 4 \Rightarrow \alpha_3 = 3/2$

✔ And so on ...

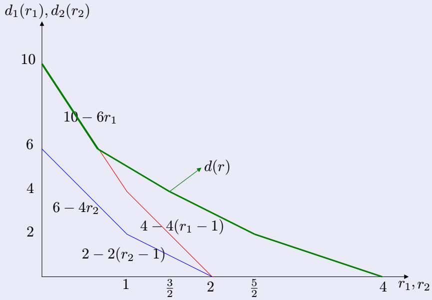

Figure: DMT of User 1 (5×2), and user 2(3×2)

Numerical example

MIMO BC with 5T

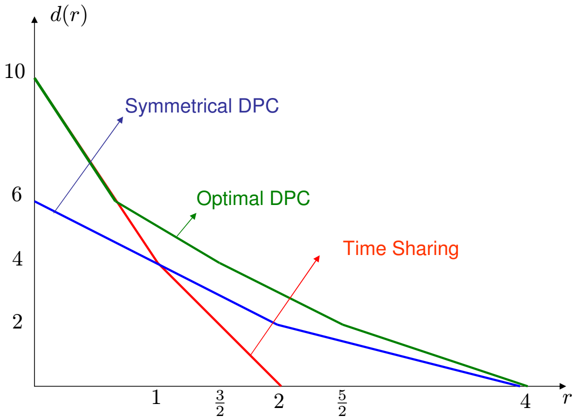

Figure: DMT comparison with full CSIT and without CSIT

Conclusion

In this talk ...

- ☞ Analysis of the DMT for a Fading MIMO BC without CSIT and optimal schemes proposal.
- ☞ Characterization of the optimal error event with full CSIT and Rx threshold constraint.

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