Network Coding Rates and Edge-Cut Bounds

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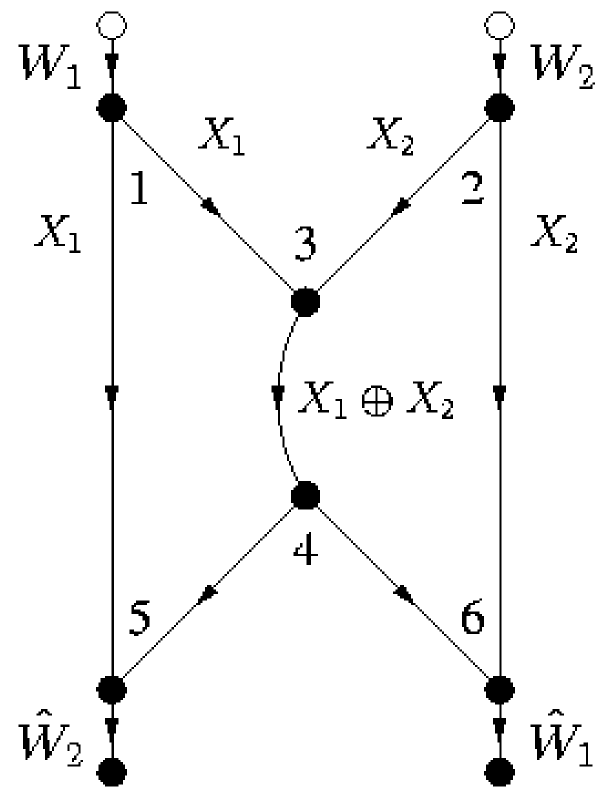
Outline

1) Network Coding for Classical Networks
   - Achievable rates
   - Non-achievable rates
2) Functional dependence graphs (FDGs) and progressive d-separating edge-set (PdE) bounds
3) Networks with Interference
   - Two-way, Transmit & Receive Interference
4) Strengthened bounds for bidirectional rings

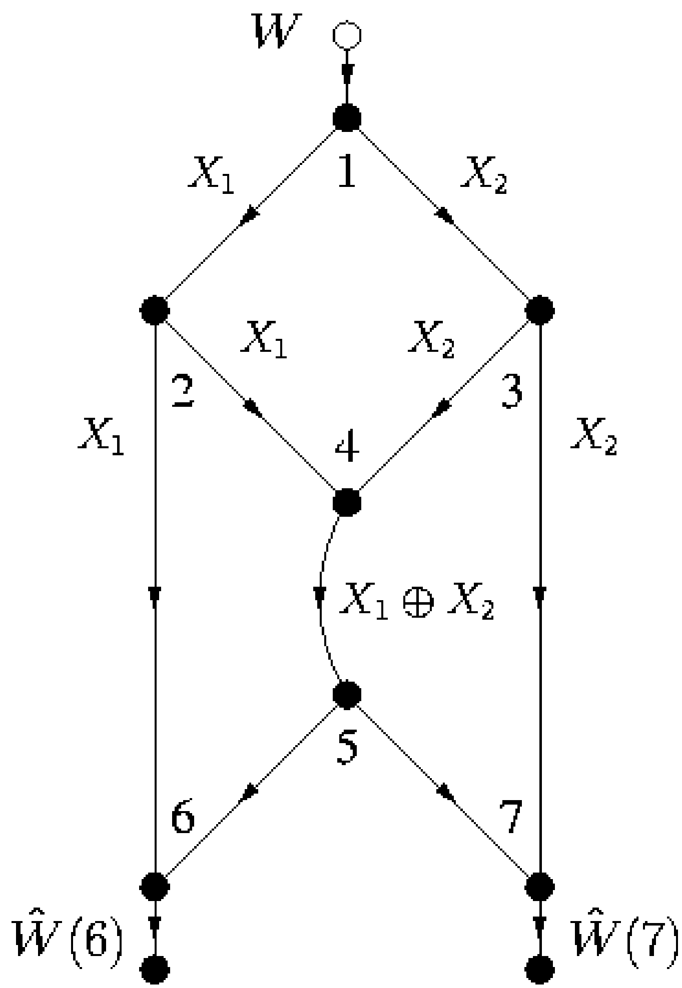
1) Network Coding for Classical Networks

- Standard example:
  a (directed) "butterfly" network with unit edge capacities and 2 source-destination pairs
- Routing (store & forward) achieves $R_1=R_2=1/2$
- Network coding (store, combine, copy & forward) achieves $R_1=R_2=1$
- Achieve gains by combining packets through bottlenecks and using more edges



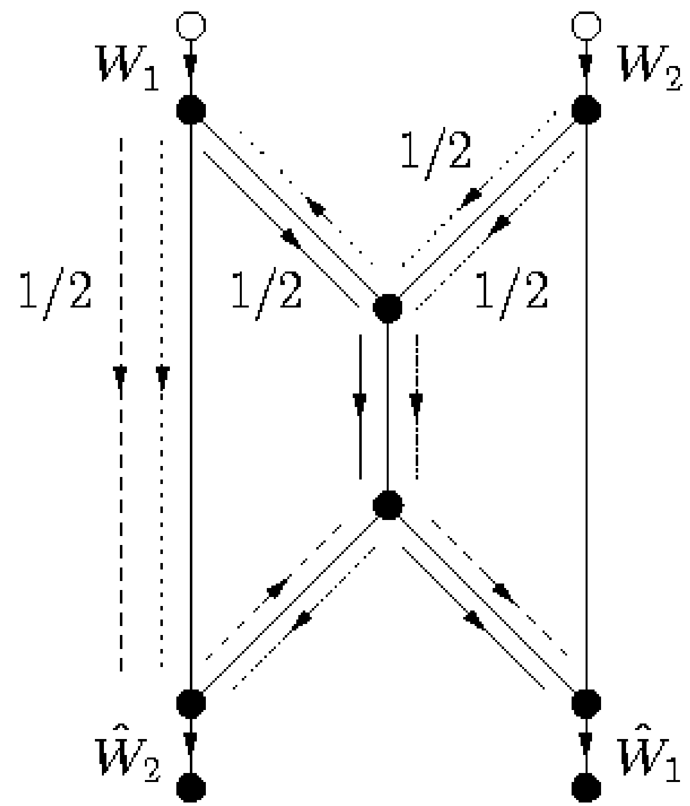
Discussion

- Largest gains for <u>multicast</u>
- Coding methods:
  - Linear coding
  - Random coding
- Can gain <u>rate</u> & <u>reliability</u>
- Cost: must use extra links to "correct" combining
- Gains are sensitive to network topology



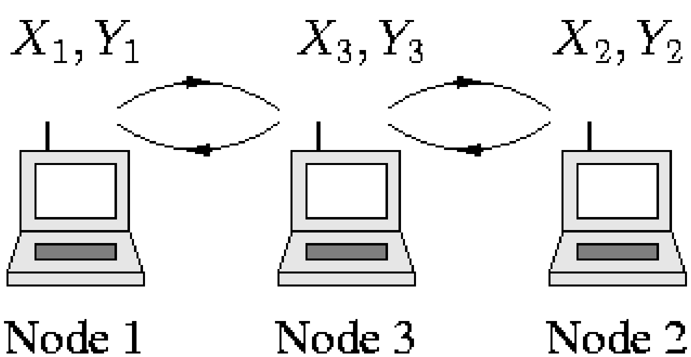
Discussion (cont'd)

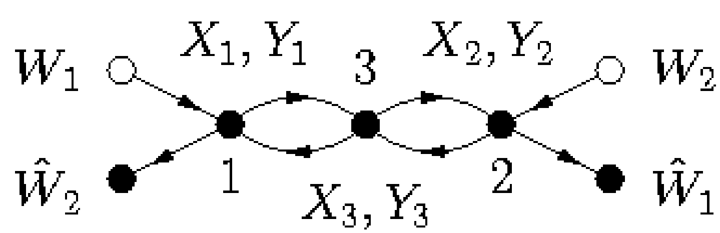
- For undirected networks, routing has more freedom and gains are smaller
- Example: achieve a cut bound for 2 source-destination pairs
- Open problem: does network coding improve routing for multiple unicast sessions in undirected networks?



Discussion (cont'd)

- For wireless networks, the gains can also be large. Wireless nodes can only
  - broadcast symbols, thereby causing interference
  - operate under half-duplex constraints
- Example: suppose node 3 decodes X_1 and X_2 and broadcasts X_1+X_2. Result: achieve $R_1=R_2=1/2$



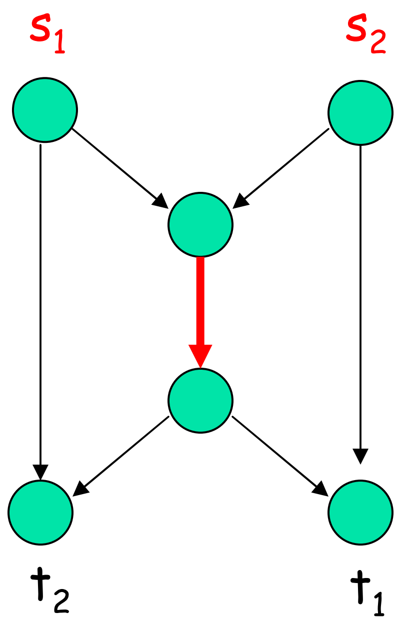


Non-achievable Rates

- Standard rate outer bound:
  - vertex-partitioning edge-cut bound
- Our focus here: edge-cut bounds that do not necessarily partition the vertices
  - Develop a systematic method that applies broadly (multi-message multicast, undirected edges, noise)
  - Bonus: the method generalizes to more complex problems (broadcasting, interference)
- Goal: an edge cut bound similar to the routing one, namely

$$\Sigma_{k \in S} R_k \leq \Sigma_{e \in E} C_e$$

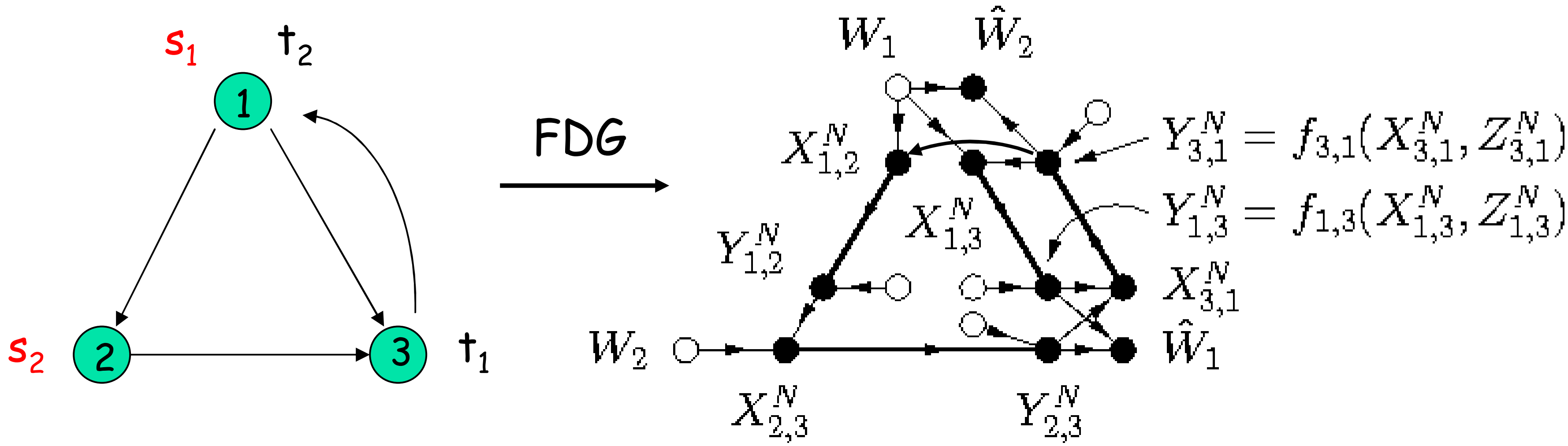
- Problem: a straight-forward approach doesn't work (see below)
- Related work: K. Jain et al., N. Harvey et al. 2005/2006



- The red edge disconnects both sources from both sinks, so the best routing rates satisfy $R_1+R_2\leq 1$
- But network coding achieves $(R_1,R_2)=(1,1)$
- Note: the vertex-partitioning edge-cut bound gives the capacity here; the edge-cut bound to follow is at least as good and sometimes better

2) Functional Dependence Graphs (FDGs)

- An FDG is a (possibly cyclic) directed graph where
  - Vertices represent random variables (RVs)
  - Edges represent the functional relations between the RVs
- Example: an FDG for a noisy classical network
  (W_k: messages, $X_{i,j}$: channel inputs, $Y_{i,j}$: channel outputs)

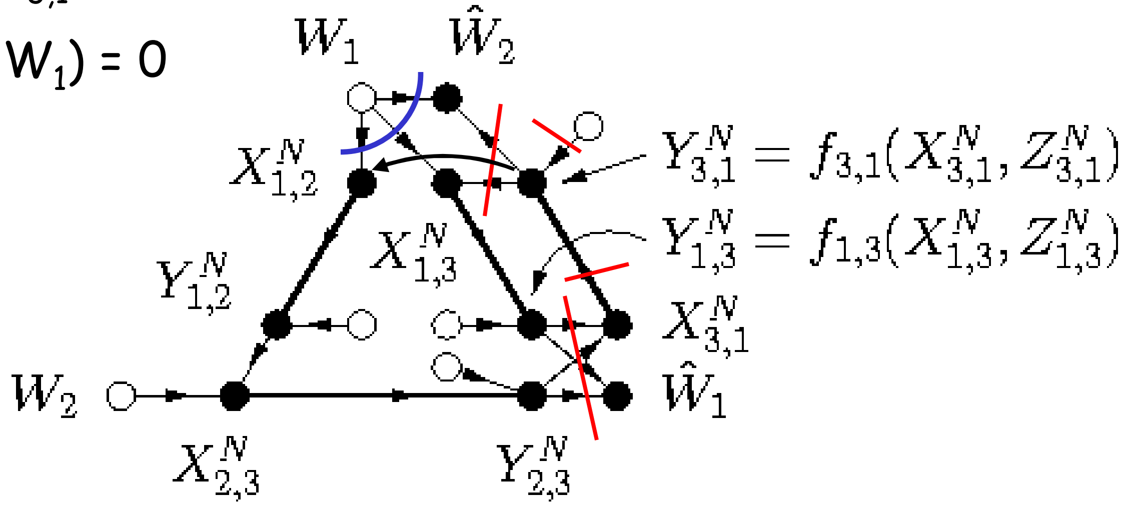


d-Separation and fd-Separation

- Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be vectors whose entries are RVs (vertices) of a FDG
- Success after the following implies $I(\underline{A} ; \underline{B} \mid \underline{C}) = 0$ (Pearl 1988)
  - Consider only the vertices and edges met when moving backward from the vertices in $\underline{A}$, $\underline{B}$, or $\underline{C}$ ("causality")
  - Remove the outgoing edges of the vertices in $\underline{C}$ and successively remove outgoing edges of vertices and on cycles w/o incoming edges
  - Check if there is no undirected path from "$\underline{A}$" to "$\underline{B}$"

- Ex: $I(W_1 ; \hat{W}_1 \mid \underline{Y}_{2,3}\ \underline{Y}_{1,3}\ \underline{Z}_{3,1}) = 0$

  $I(W_2 ; \hat{W}_2 \mid \underline{Y}_{2,3}\ \underline{Y}_{1,3}\ \underline{Z}_{3,1}\ W_1) = 0$



Progressive d-Sep. Edge-Set (PdE) Bounds

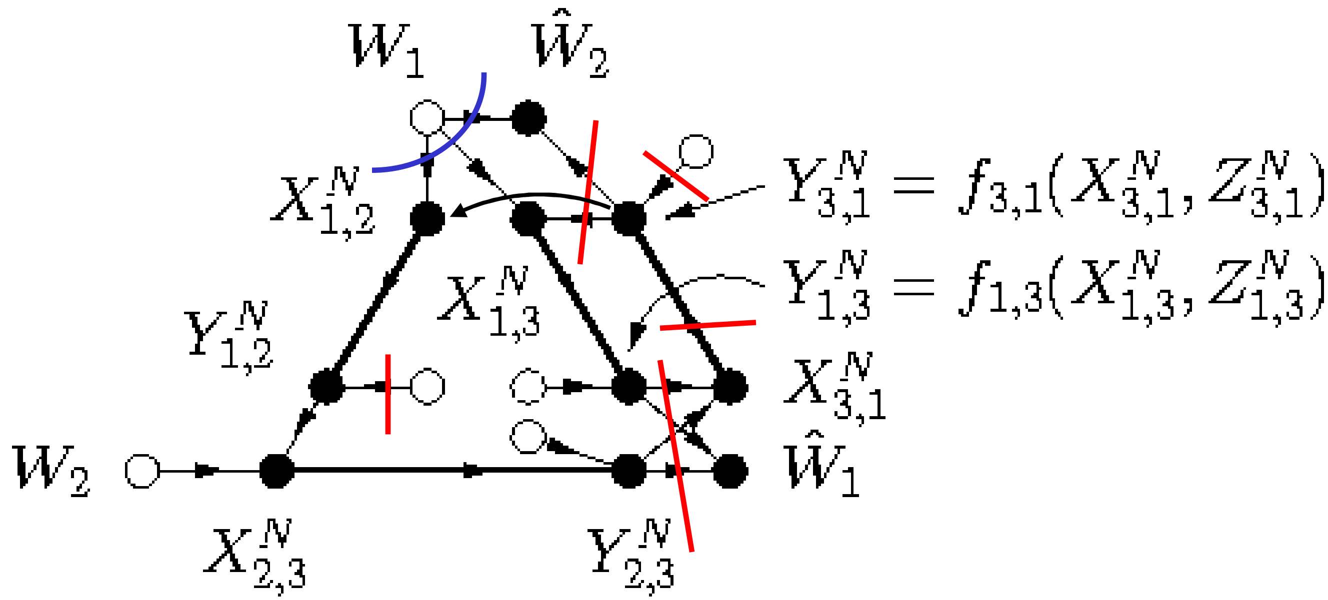
- Consider a set E of edges and an ordered set S of K sources:
  - Initialization (of the network FDG)
    - Set i=1. Remove edges of sources not in S and noise not "in" E
    - Remove outgoing edges of the FDG vertices Y_e with e ϵ E.
    - Remove outgoing edges of vertices/cycles w/o incoming edges
  - Iterations (progressive edge removal)
    - If S_i is connected in an undirected sense from T_i then stop without a bound
    - Else remove the edges coming out of S_i
    - Remove outgoing edges of vertices/cycles w/o incoming edges
  - Termination and Bound
    - Increment i. If i ≤ K go to the previous step.
    - Else one has the desired bound.

Example 1: A 3-Node Network

- Using E={(1,3),(2,3)} and S={1,2}, we have the PdE bound

$$R_1 + R_2 \leq C_{1,3} + C_{2,3}$$

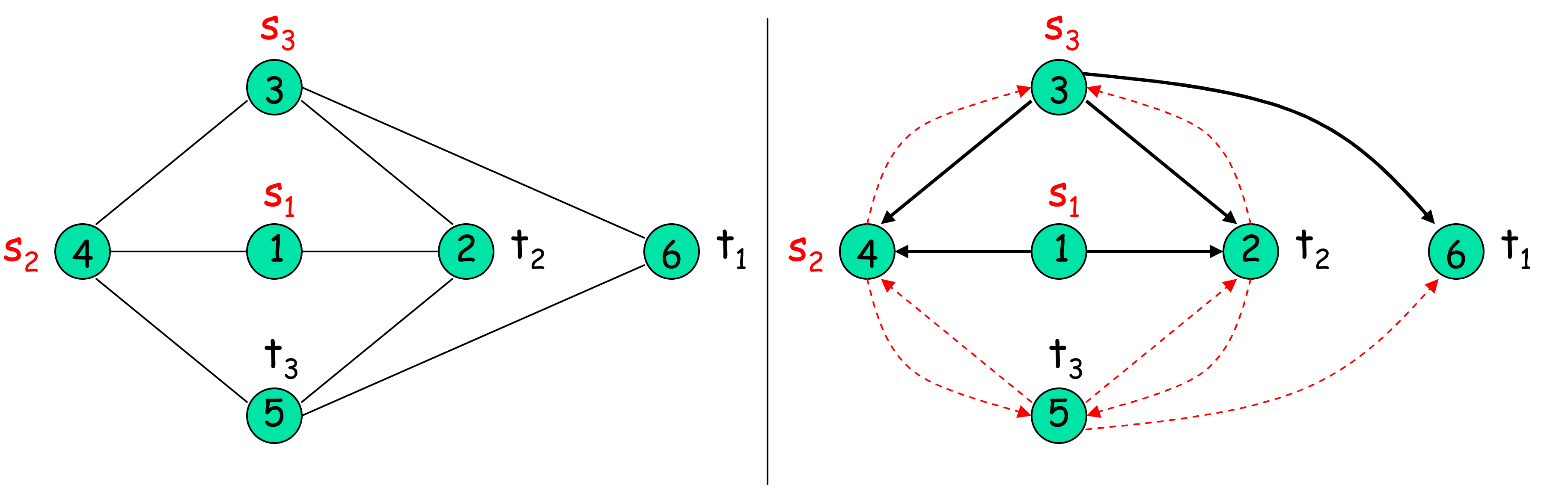
- The standard cut-set bound does not give this bound.
- The bound applies to network coding as well as routing.



Example 2: Hu's 3-Commodity Problem

- Consider the undirected network below with edge capacities 2
- $(R_1,R_2,R_3)=(4,2,1)$ is impossible with routing, but the "routing" edge-cut bound permits it.
- Note: PdE bound for $R_1=4$ requires S_1/T_1 have only out/in-edges
- Check: use E={(2,3),(4,3),(2,5),(4,5)} and S={3,1,2} to get

$$4+R_2+R_3 \leq R_{23}+R_{43}+R_{25}+R_{45}$$

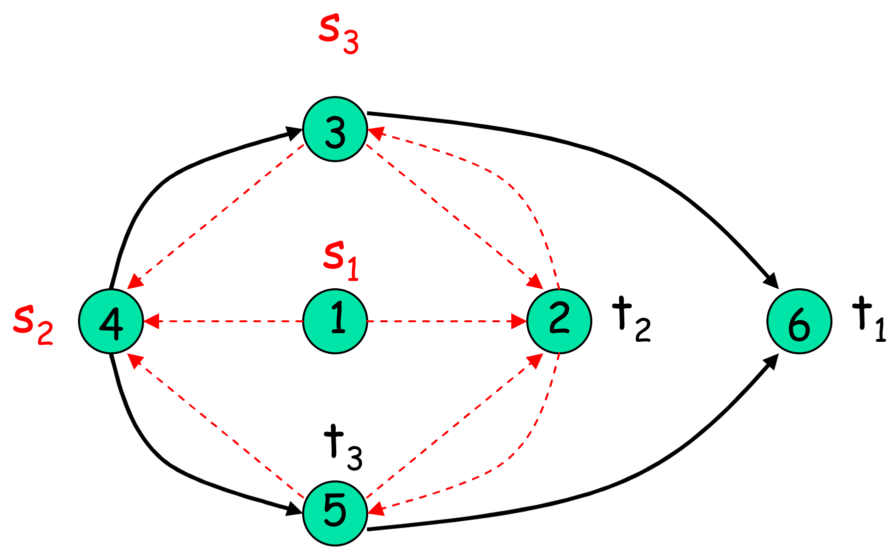


Hu's Problem (continued)

- We have $4+R_2+R_3 \le R_{23}+R_{43}+R_{25}+R_{45}$
- Next: use E={(3,2),(3,4),(5,2),(5,4)} and S={2,3} to get

$$R_2+R_3 \le R_{32}+R_{34}+R_{52}+R_{54}$$

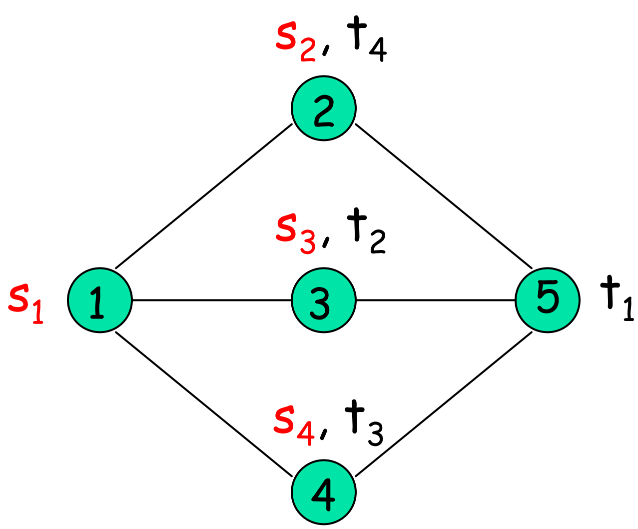
- We thus have $4+2(R_2+R_3) \le 8$ so that $(R_1,R_2,R_3)=(4,2,1)$ is not possible even with network coding

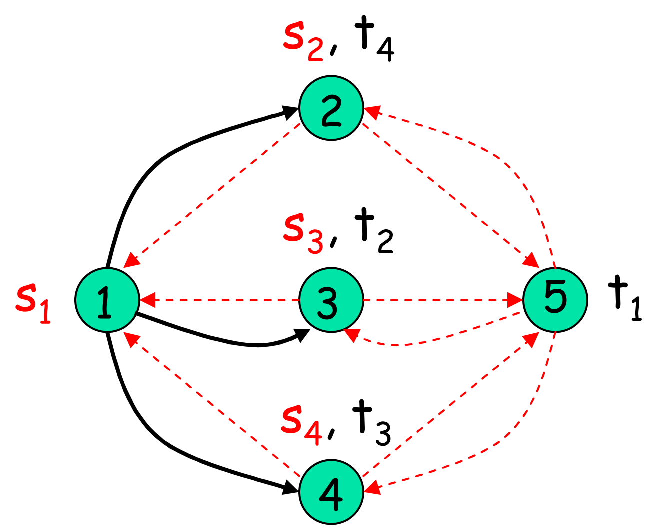


Example 3: Okamura & Seymour's Problem

- Consider the undirected network below with edge capacities 1
- Result: $(R_1,R_2,R_3,R_4)=(1,1,1,1)$ is impossible with routing, but is permitted by the vertex-partitioning edge-cut bound
- Use $E=\{(2,1),(3,1),(4,1),(2,5),(3,5),(4,5)\}$ and $S=\{1,2,3,4\}$ to get

$$R_1+R_2+R_3+R_4 \le R_{21}+R_{31}+R_{41}+R_{25}+R_{35}+R_{45}$$





Okamura & Seymour's Problem (continued)

- Combining this and similar bounds gives us

  $$R_1+R_2+R_3+R_4 \leq 3$$

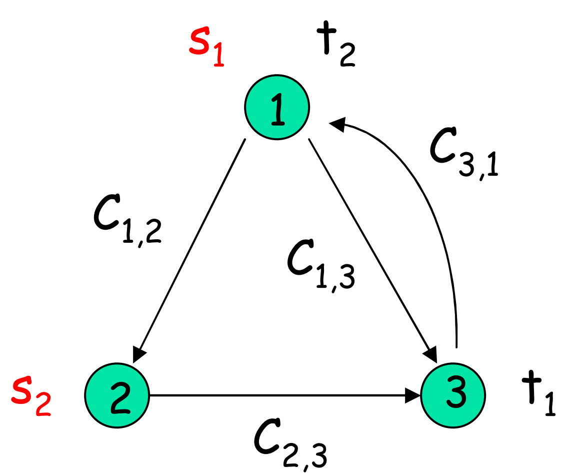
  with or without network coding
- In particular, this implies that the best symmetric rate is $R_i=3/4$ for all i (see Jain et al., Lehman et al. 2005)
- The above steps, combined with certain vertex-partitioning edge-cut bounds, give the capacity region for this problem.
- Result: routing is optimal for this problem

3) Networks with Interference

- Can one generalize the PdE bound to more complex networks?

$$\Sigma_{k \varepsilon S} R_k \leq \Sigma_{e \varepsilon E} f_e(?)$$

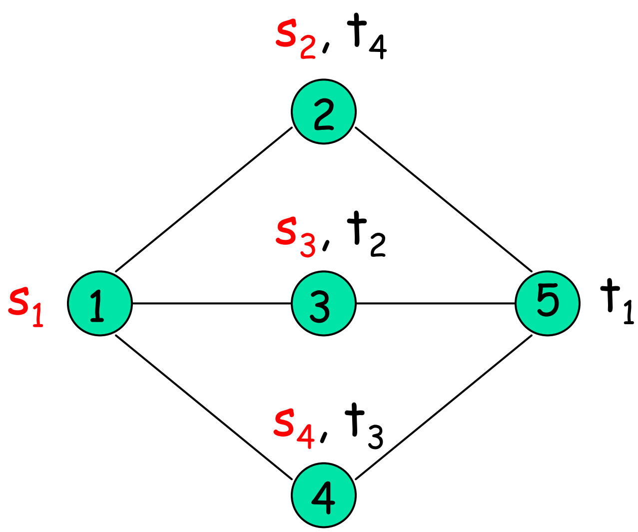
- Consider the following types of communication networks

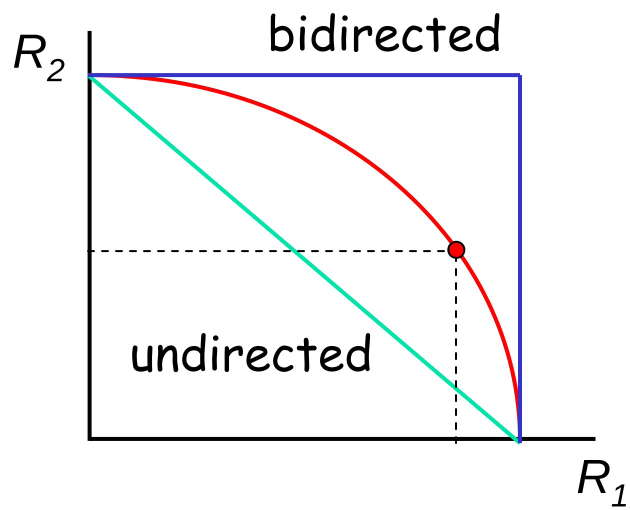


- Classical networks: edges represent non-interfering channels
- Two-way channel edges
- Aref networks: every vertex broadcasts a common input into its outgoing edges. One can further add independent edge noise.
- Networks with interference and independent vertex noise

Example 1: Two-Way Channel Edges

- Suppose every undirected edge is a two-way channel.
- A natural approach (as done for undirected networks): split every edge into a pair of bidirected edges with capacities (R_1,R_2) on the boundary of that edge's capacity region
- Is the network capacity characterized by all such splittings? (I.e., can one separate channel and network coding?)



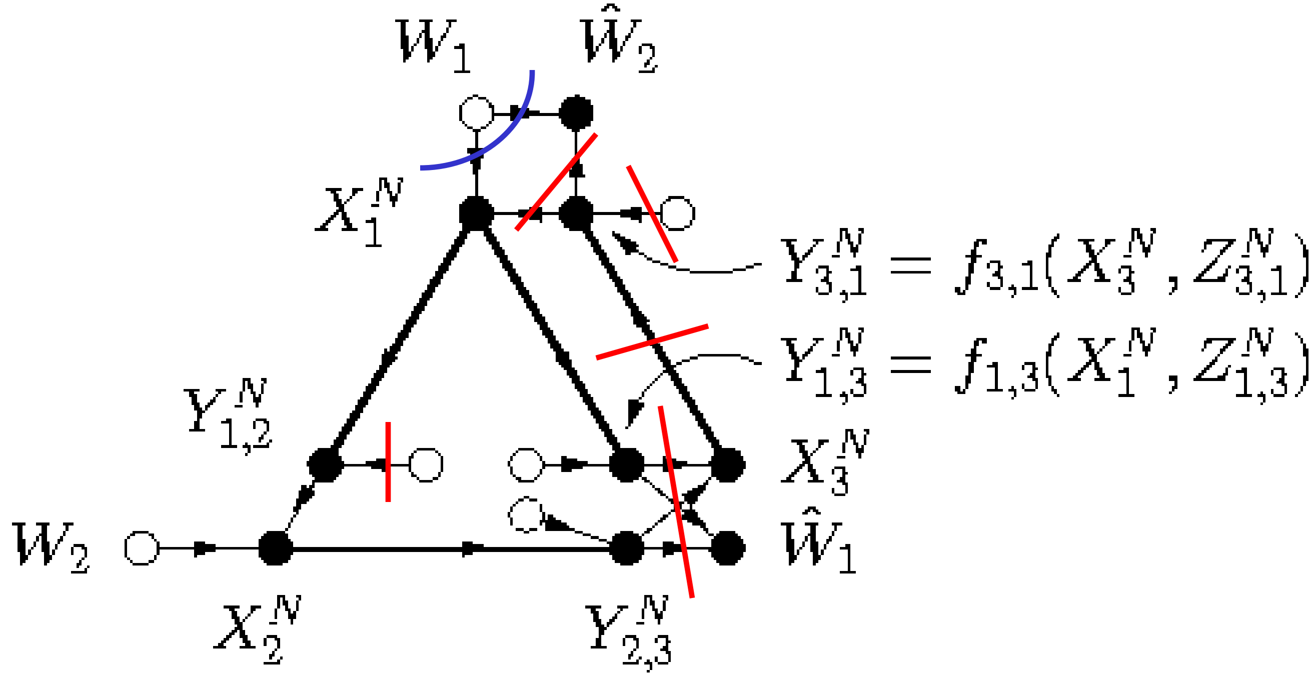


Example 2: Noisy Aref Networks

- Using E={(1,3),(2,3)} and S={1,2}, we have the PdE bound

$$R_1 + R_2 \leq I(X_1 ; Y_{1,3}) + I(X_2 ; Y_{2,3})$$

- The standard cut-set bound does not give this bound.
- Finally, form the union of rates over all (independent) inputs.

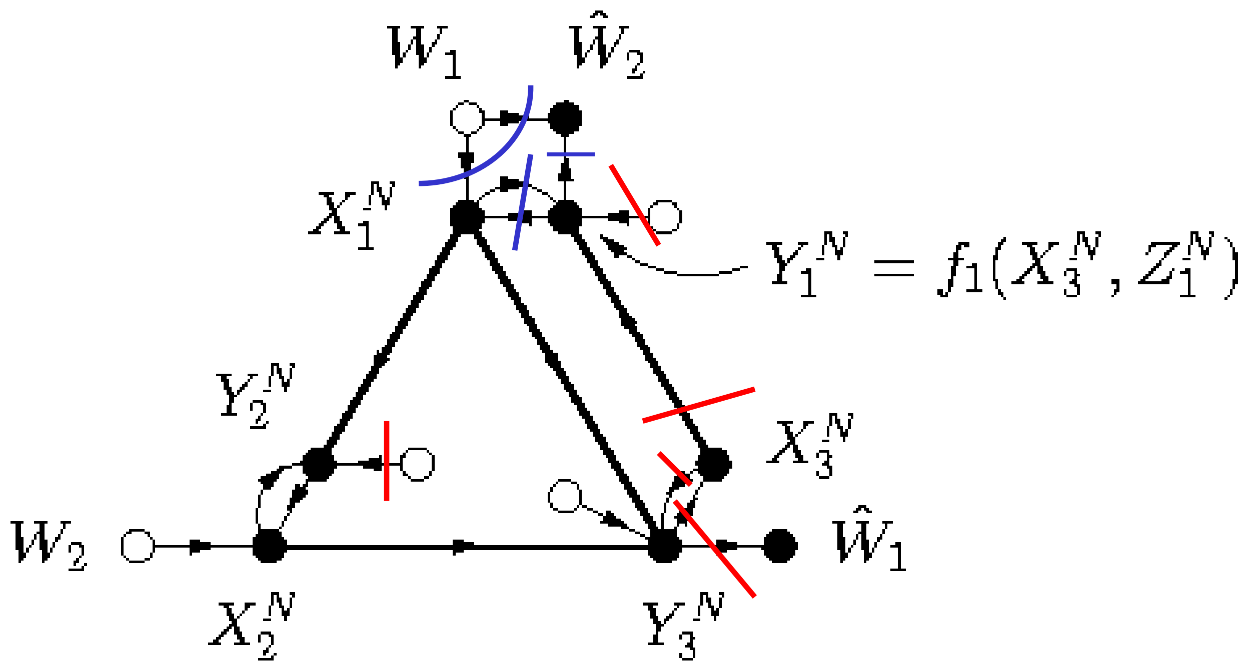


Example 3: Networks with Interference

- Using E={(1,3),(2,3)} and S={1,2}, we have the PdE bound

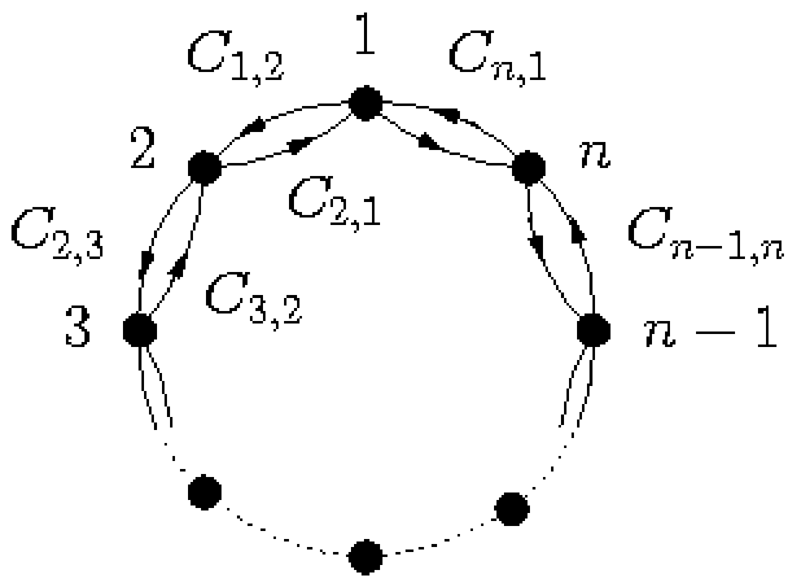
$$R_1 + R_2 \le I(X_1\ X_2\ ;\ Y_3 \mid X_3)$$

- The standard cut-set bound does not give this bound.
- Finally, form the union of rates over all inputs.



4) Strengthened Bounds

- Suppose we have a multi-message unicast problem on a bidirectional ring network. Does network coding help?
- Result: routing is rate optimal via 1) cut bounds, 2) PdE bounds, 3) new generalizations of PdE bounds

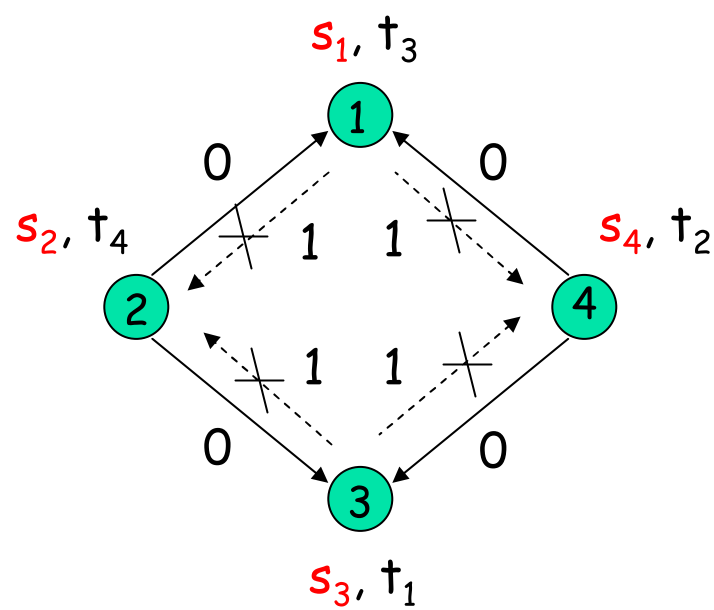


Proof Outline

- The routing region is char. by a linear program and/or its dual.
- For the dual inequalities, weights are assigned to certain edges and one computes the min. path lengths (sum of the edge weights) from the sources to their destinations.
- Example: for the ring below, suppose (1,2), (1,4), (3,2), (3,4) are assigned weight 1 (later cut); the other edges have weight 0. The rate bound is:

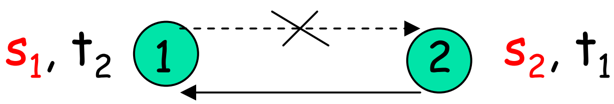
$$\Sigma_{s\varepsilon S}\, d_s R_s \leq \Sigma_{e\varepsilon E}\, w_e C_e$$

Here: $R_1 + R_2 + R_3 + R_4 \leq C_{1,2} + C_{1,4} + C_{3,2} + C_{3,4}$

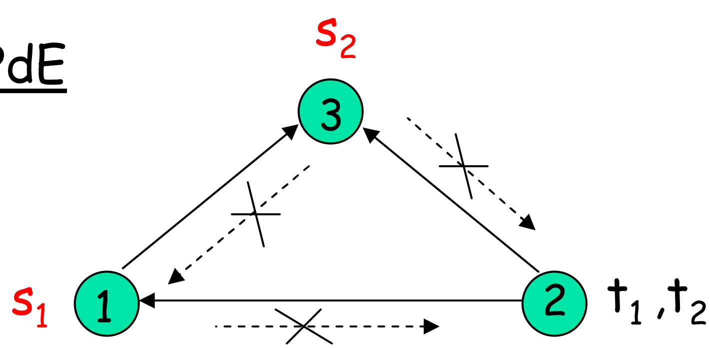


- Next, show one need consider only d_s and w_e that are 0 or 1 (one can use a geometric argument due to Han/Kobayashi)
- Show there are 3 types of dual bounds: cut, PdE, other
- Finally, use Fano's ineq. to get the same converse bounds.

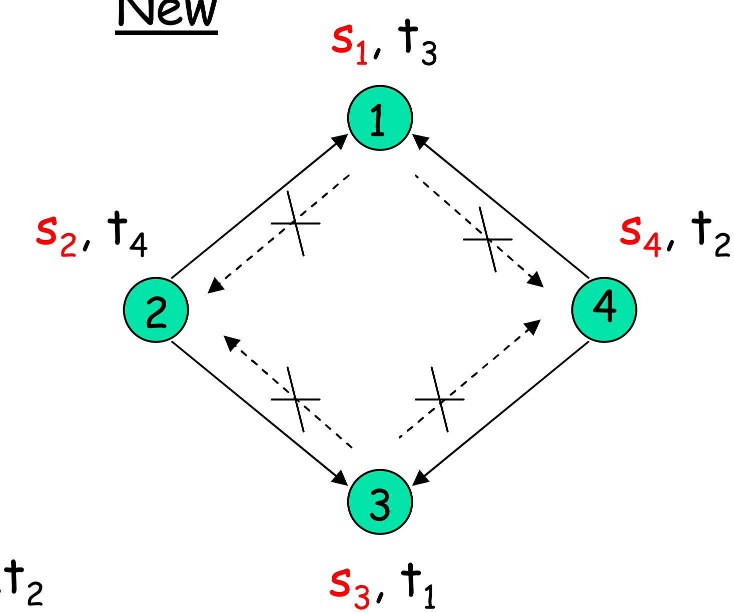
Cut-Set



PdE



New



Summary

- Edge-Cut Bounds:
  - Useful for many networks other than classical ones
  - Apply to general multi-message multicast
  - Generalize the standard cut-set bound (without auxiliary random variables)
- Outlook:
  - Can one extend edge-cut bounds to networks with dependent vertex noise?
  - Understand the new bounds for ring networks.