Network Calculus

A General Framework for Interference Management and Resource Allocation

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Outline

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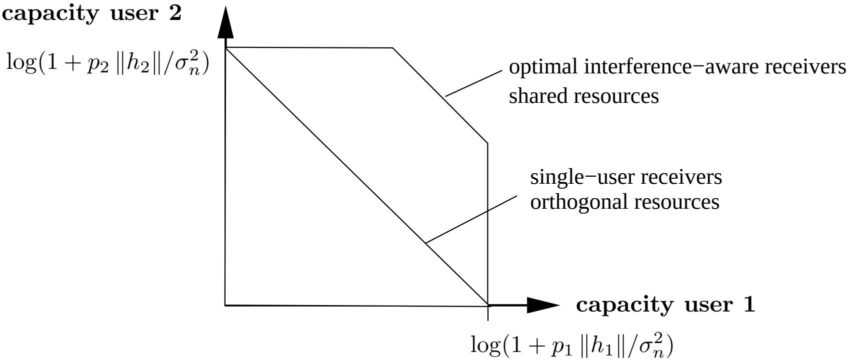
Interference in Multiuser Wireless Networks

- evolution of wireless networks:
  - high-rate services
  - densely populated user environments
- interference between users puts a limit on how many users per cell can be served at a certain data rate
- countermeasure: adaptive signal processing and resource allocation/scheduling strategies

Example: Multiple Access Channel (MAC)

"... letting transmitted signals interfere with each other (in a controled way) increases capacity provided that the receivers take into account the multiaccess interference."

[Verdu, "Fifty Years of Shannon Theory"]



Discussion

- objective: modelling of performance tradeoffs caused by interference
- in the past, results were mainly derived for special system layouts (e.g. MIMO MAC), are there more general principles?
- generally difficult due to complicated interdependencies between system functionalities ("cross-layer problem")
- chosen approach: abstract framework ("network calculus")
  - ➠ focus on core properties
  - ➠ rigorous, allows to handle problems analytically
  - ➠ provides intuition and roadmap for implementation

Some Interference Models

- classical linear model

$$\mathcal{I}_k(\mathbf{p}, \sigma_n^2) = \mathbf{v}^T \mathbf{p} + \sigma_n^2$$

- interference in a multiuser MIMO channel with optimum antenna combining

$$\mathcal{I}_k(\mathbf{p}, \sigma_n^2) = \frac{1}{\mathbf{h}_k^H \left(\sigma_n^2 \mathbf{I} + \sum_{l \neq k} p_l \mathbf{h}_l \mathbf{h}_l^H\right)^{-1} \mathbf{h}_k}$$

- generalization: adaptive receive strategy z_k

$$\mathcal{I}_k(\mathbf{p}, \sigma_n^2) = \min_{z_k \in \mathcal{Z}_k} \Big(\underbrace{\mathbf{p}^T \mathbf{v}(z_k)}_{\color{red}{\textit{Interference}}} + \underbrace{\sigma_n^2 n_k(z_k)}_{\color{red}{\textit{Noise}}} \Big) , \quad k = 1, 2, \ldots, K$$

Axiomatic Approach: Interference Functions

Definition

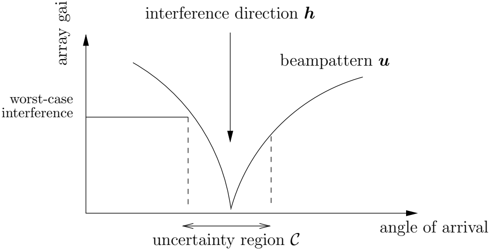
A function $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is said to be an interference function if the following axioms are fulfilled:

A1	(positivity)	$\mathcal{I}(\mathbf{p}) > 0$ if $\mathbf{p} > 0$
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p}) \quad \forall\alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}') \quad$ if $\mathbf{p} \geq \mathbf{p}'$

Another Example: Robust Nullsteering Beamforming

interference can be reduced by nullsteering beamforming:

- assume that the interference direction is only known up to an uncertainty c from a region $\mathcal{C}$



- the beamformer $\mathbf{u}$ minimizes the worst-case interference power:

$$\mathcal{I}(\mathbf{p}) = \min_{\|\mathbf{u}\|=1} \left(\max_{c \in \mathcal{C}} \sum_l p_l |\mathbf{u}^H \mathbf{h}_l(c)|^2 \right)$$

$\Rightarrow$ this is also an interference function (A1–A3 fulfilled)

Outline

1. Interference Functions
2. Application to SIR-Based Utility Sets
3. Application to Game Theory
4. Structure of Interference Functions and Utility Sets
5. Conclusions

QoS Model for Wireless Systems

- signal-to-interference ratio

$$\mathsf{SIR}(\mathbf{p}) = \frac{p_k}{\mathcal{I}_k(\mathbf{p})}$$

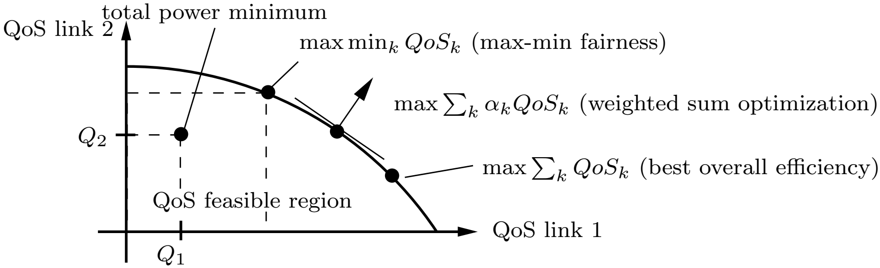
- the QoS is a strictly monotonic function of the SIR

$$QoS(\mathbf{p}) = \phi\big(\mathsf{SIR}(\mathbf{p})\big)$$

examples:		
	$\phi(x) = x$	SIR
	$\phi(x) = \log(x)$	SIR in dB
	$\phi(x) = 1/(1+x)$	Min. Mean Squared Error (MMSE)
	$\phi(x) = x^{-\alpha}$	BER slope, diversity order α
	$\phi(x) = \log(1+x)$	capacity for Gaussian signals
	...	

Resource Allocation

- for multiuser systems, the transmission strategy is typically a tradeoff between efficiency and fairness requirements



Fixed-Point Iteration

For standard interference functions it was shown [Yates'95]

If target SIR $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]$ are feasible, i.e., $C(\boldsymbol{\gamma}) \leq 1$, under a sum-power constraint, then for an arbitrary initialization $\mathbf{p}^{(0)} \geq 0$, the iteration

$$p_k^{(n+1)} = \gamma_k \cdot \mathcal{I}_k(\mathbf{p}^{(n)}), \quad k = 1, 2, \ldots, K$$

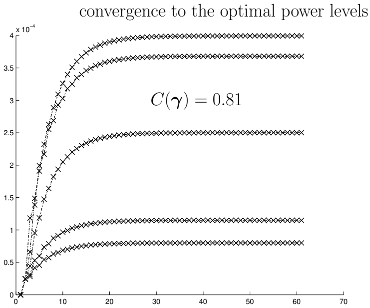
converges to the optimum of the power minimization problem

$$\inf_{\mathbf{p}>0} \sum_{k=1}^{K} p_k \quad \text{s.t.} \; \frac{p_k}{\mathcal{I}_k(\mathbf{p})} \geq \gamma_k, \; \forall k \; ,$$

Properties of the Fixed-Point Iteration

The fixed-point iteration has the following properties:

- component-wise monotonicity
- optimum achieved iff $p_k^{(n+1)} = \gamma_k \mathcal{I}_k(\mathbf{p}^{(n)}), \forall k$
- optimizer $\lim_{n\to\infty} \mathbf{p}^{(n)}$ is unique



Exploiting Concavity

- consider concave interference functions

$$\mathcal{I}_k(\mathbf{p}) = \min_{z_k \in \mathcal{Z}_k} \Big(\underbrace{\mathbf{p}^T \mathbf{v}(z_k)}_{\textit{Interference}} + \underbrace{n_k(z_k)}_{\textit{Noise}} \Big) , \quad k = 1, 2, \ldots, K$$

- the parameter z_k can be interpreted as a receive strategy
- for K users, we have an interference coupling matrix

$$\mathbf{V}(z) = [\mathbf{v}_1(z_1), \ldots, \mathbf{v}_K(z_K)]^T$$

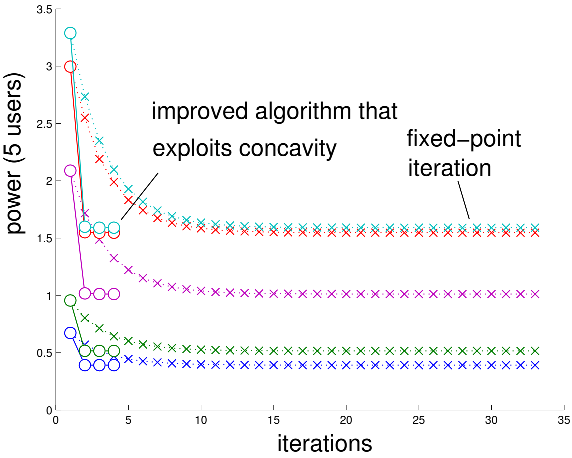
Exploiting Concavity: Matrix-Based Iteration

By exploiting the special structure of concave interference functions, a new iteration is obtained:

Alternating optimization of receive strategies $z^{(n)}$ and power allocation $\mathbf{p}^{(n)}$

1. $z_k^{(n)} = \arg\min_{z_k \in \mathcal{Z}_k} \Big[\mathbf{V}(z)\mathbf{p}^{(n)} + \mathbf{n}(z)\Big]_k \,, \quad k \in \{1, 2, \dots, K\}$
2. $\mathbf{p}^{(n+1)} = (\mathbf{I} - \mathbf{\Gamma}\mathbf{V}(z^{(n)}))^{-1} \cdot \mathbf{\Gamma} N(z^{(n)})$

Convergence Behavior

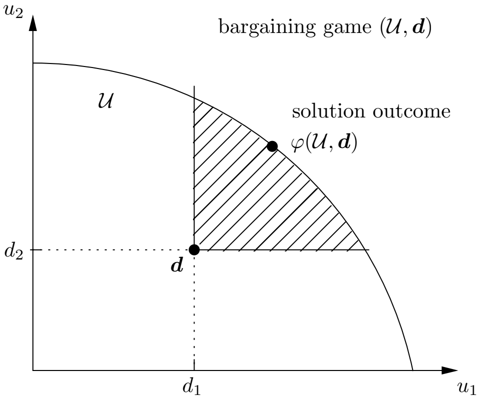


Outline

Cooperative Bargaining

K players try to reach an unanimous agreement on utilities $\mathbf{u} = [u_1, \ldots, u_k]$

- the utility region $\mathcal{U} \subset \mathbb{R}^K_{++}$ is convex, comprehensive, closed, bounded
- depending on the chosen strategy, the solution outcome φ results
- if the bargaining fails, the disagreement outcome $\mathbf{d}$ results



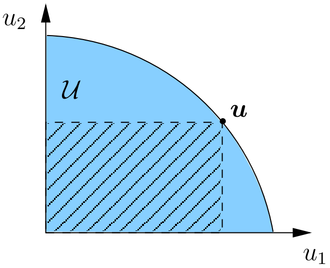
Standard Properties of Utility Sets

- downward-comprehensive

  for all $\mathbf{u} \in \mathcal{U}$ and $\mathbf{u}' \in \mathbb{R}^K_{++}$

  $\mathbf{u}' \leq \mathbf{u} \implies \mathbf{u}' \in \mathcal{U}$

- closed (contains its boundary)
- convex
- upper-bounded



Utility Sets and Interference Calculus

Theorem

Every compact comprehensive utility set from $\mathbb{R}^K_{++}$ can be expressed as a sub-level set

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^K_{++} : C(\mathbf{u}) \leq 1\}$$

depending on an interference function $C(\mathbf{u})$.

The sub-level set $\mathcal{U}$ is convex if and only if $C(\mathbf{u})$ is a convex interference function.

Example: The SIR Feasible Set

- example: K interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ and weighting factors $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]$ (e.g. SIR requirements). The optimum of the weighted SIR balancing problem is

$$C(\boldsymbol{\gamma}) = \inf_{\mathbf{p}>0} \Big(\max_{1 \leq k \leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k} \Big)$$

- SIR feasible region

$$\mathcal{S} = \{\boldsymbol{\gamma} : C(\boldsymbol{\gamma}) \leq 1\}$$

➠ $C(\boldsymbol{\gamma})$ satisfies A1–A3

➠ every SIR region is a level set of an interference function

Axiomatic Framework for Symmetric Nash Bargaining

- **WPO** Weak Pareto Optimality . The players should not be able to collectively improve upon the solution outcome.
- **IIA** Independence of Irrelevant Alternatives. If the feasible set shrinks but the solution outcome remains feasible, then the outcome is also the solution of the smaller set.
- **SYM** Symmetry: If the region is symmetric, then the outcome does not depend on the identities of the users.
- **STC** Scale Transformation Covariance. The outcome is component-wise scale-invariant.

The Nash Product

For convex comprehensive sets the unique Nash bargaining solution fulfilling the axioms WPO, IIA, SYM, STC is obtained as the solution of

$$\max_{\{\mathbf{u}\in\mathcal{U}:\mathbf{u}>\mathbf{d}\}} \prod_{k=1}^{K} (u_k - d_k)$$

Often, the choice of the zero of the utility scales does not matter, so we can choose $\mathbf{d} = \mathbf{0}$

$$\max_{\mathbf{u}\in\mathcal{U}} \prod_{k=1}^{K} u_k$$

Nash Bargaining and Proportional Fairness

- the product optimization approach is equivalent to proportional fairness [Kelly'98]

$$\hat{\mathbf{u}} = \arg\max_{\mathbf{u}\in\mathcal{U}} \prod_{k=1}^{K} u_k = \arg\max_{\mathbf{u}\in\mathcal{U}} \log \prod_{k=1}^{K} u_k = \arg\max_{\mathbf{u}\in\mathcal{U}} \sum_{k=1}^{K} \log u_k$$

- if the region $\mathcal{U}$ is convex closed comprehensive and bounded, then symmetric Nash bargaining and proportional fairness are equivalent

Bargaining over SIR Feasible Sets

- for wireless systems, an important performance measure is the signal-to-interference ratio

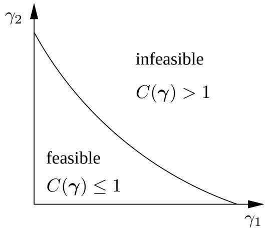
$$\mathsf{SIR}_k(\mathbf{p}) = \frac{p_k}{\mathcal{I}_k(\mathbf{p})} \qquad \begin{array}{l} \leftarrow \text{useful power} \\ \leftarrow \text{interference (+noise) power} \end{array}$$

- indicator of feasibility: $C(\gamma) = \inf_{\mathbf{p}>0}\Big(\max_k \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k}\Big)$

- the SIR region

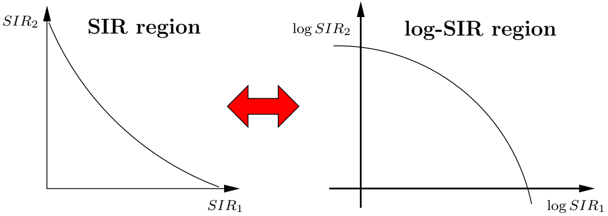
$$\mathcal{S} = \{\gamma \in \mathbb{R}_+^K : C(\gamma) \leq 1\}$$

is generally not convex, so results from classical bargaining theory cannot be applied directly



Convex log-SIR Region

- if the underlying interference functions are log-convex, then the SIR region is log-convex

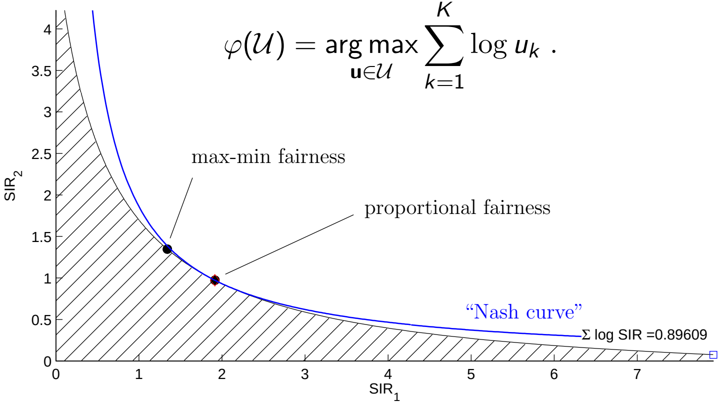


- SIR region has special properties which can be exploited for bargaining (closed, comprehensive, log-convex)

Extension of the Classical Nash Bargaining Framework

- If the region $\mathcal{U}$ is strictly convex after a log-transformation ("log-convex"), then the Nash axioms WPO, IIA, SYM, STC characterize a single-valued solution outcome

$$\varphi(\mathcal{U}) = \arg\max_{\mathbf{u}\in\mathcal{U}} \sum_{k=1}^{K} \log u_k \ .$$



Outline

Representation of General Interference Functions

Theorem

Let $\mathcal{I}$ be an arbitrary interference function, then

$$\mathcal{I}(\mathbf{p}) = \min_{\hat{\mathbf{p}} \in \underline{L}(\mathcal{I})} \max_k \frac{p_k}{\hat{p}_k} = \max_{\hat{\mathbf{p}} \in \overline{L}(\mathcal{I})} \min_k \frac{p_k}{\hat{p}_k}$$

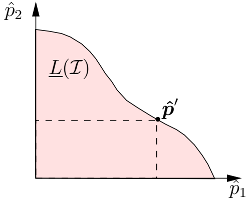
- $\mathcal{I}(\mathbf{p})$ can always be represented as the optimum of a weighted max-min (or min-max) optimization problem
- The weights $\hat{\mathbf{p}}$ are elements of convex/concave level sets

$$\underline{L}(\mathcal{I}) = \{\hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \leq 1\}$$
$$\overline{L}(\mathcal{I}) = \{\hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \geq 1\}$$

Interference Functions and Utility/Cost Regions

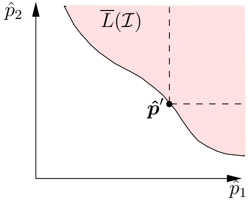
- the set $\underline{L}(\mathcal{I})$ is closed bounded and monotonic decreasing

$$\hat{\mathbf{p}} \leq \hat{\mathbf{p}}' , \quad \hat{\mathbf{p}}' \in \underline{L}(\mathcal{I}) \quad \Longrightarrow \quad \hat{\mathbf{p}} \in \underline{L}(\mathcal{I})$$



- the set $\overline{L}(\mathcal{I})$ is closed and monotonic increasing

$$\hat{\mathbf{p}} \geq \hat{\mathbf{p}}' , \quad \hat{\mathbf{p}}' \in \overline{L}(\mathcal{I}) \quad \Longrightarrow \quad \hat{\mathbf{p}} \in \overline{L}(\mathcal{I})$$



➡ every interference function can be interpreted as an optimum of a utility/cost resource allocation problem

Definition

We say that $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is a concave interference function if it fulfills the axioms:

A1	(non-negativeness)	$\mathcal{I}(\mathbf{p}) \geq 0$
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p}) \quad \forall\alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}') \quad$ if $\mathbf{p} \geq \mathbf{p}'$
C1	(concavity)	$\mathcal{I}(\mathbf{p})$ is concave on $\mathbb{R}^K_+$

Examples for Concave Interference Functions

- beamforming:

$$\mathcal{I}_k(\mathbf{p}) = \frac{1}{\mathbf{h}_k^H \left(\sigma_n^2 \mathbf{I} + \sum_{l \neq k} p_l \mathbf{h}_l \mathbf{h}_l^H\right)^{-1} \mathbf{h}_k}$$

- generalization: receive strategy z_k

$$\mathcal{I}_k(\mathbf{p}, \sigma_n^2) = \min_{z_k \in \mathcal{Z}_k} \Big(\underbrace{\mathbf{p}^T \mathbf{v}(z_k)}_{\textit{Interference}} + \underbrace{\sigma_n^2 n_k(z_k)}_{\textit{Noise}} \Big) , \quad k = 1, 2, \ldots, K$$

Representation of Concave Interference Functions

Theorem

Let $\mathcal{I}(\mathbf{p})$ be an arbitrary concave interference function, then

$$\mathcal{I}(\mathbf{p}) = \min_{\mathbf{w} \in \mathcal{N}_0(\mathcal{I})} \sum_{k=1}^{K} w_k p_k \ , \quad \textit{for all } \mathbf{p} > 0.$$

where

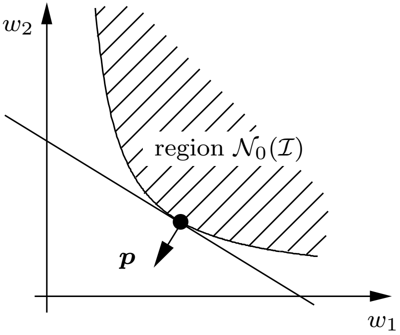
$$\mathcal{N}_0(\mathcal{I}) = \{\mathbf{w} \in \mathbb{R}_+^K : \underline{\mathcal{I}}^*(\mathbf{w}) = 0\}$$

and $\underline{\mathcal{I}}^*(\mathbf{w}) = \inf_{\mathbf{p}>0}\Big(\sum_{l=1}^{K} w_l p_l - \mathcal{I}(\mathbf{p})\Big)$ *is the conjugate of* $\mathcal{I}$.

Interpretation of Concave Interference Functions

$$\mathcal{I}(\mathbf{p}) = \min_{\mathbf{w} \in \mathcal{N}_0(\mathcal{I})} \sum_{k=1}^{K} w_k p_k$$

- the set $\mathcal{N}_0(\mathcal{I})$ is closed, convex, and upward-comprehensive
- any concave interference function can be interpreted as the solution of a loss/cost minimization problem



Convex Interference Functions

Definition

We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is a convex interference function if it fulfills the axioms:

A1	(non-negativeness)	$\mathcal{I}(\mathbf{p}) \geq 0$	
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p})$	$\forall\alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}')$	if $\mathbf{p} \geq \mathbf{p}'$
C2	(convexity)	$\mathcal{I}(\mathbf{p})$ is convex on $\mathbb{R}_+^K$	

Example: Robustness

- An example is the worst-case model

$$\mathcal{I}_k(\mathbf{p}) = \max_{c_k \in \mathcal{C}_k} \mathbf{p}^T \mathbf{v}(c_k), \quad \forall k \ ,$$

  where the parameter c_k models an 'uncertainty' (e.g. caused by channel estimation errors or system imperfections).
- the optimization is over a compact uncertainty region $\mathcal{C}_k$
- $\mathcal{I}_k(\mathbf{p})$ is a convex interference function

Representation of Convex Interference Functions

Theorem

Let $\mathcal{I}(\mathbf{p})$ be an arbitrary convex interference function, then

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k \; , \quad \text{for all } \mathbf{p} > 0.$$

where

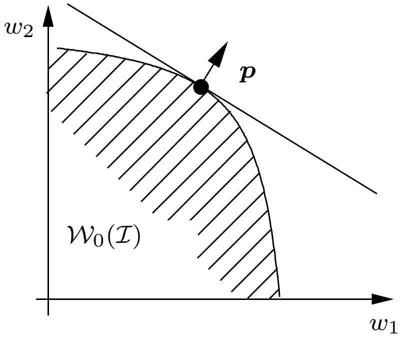
$$\mathcal{W}_0(\mathcal{I}) = \{\mathbf{w} \in \mathbb{R}^K_+ : \bar{\mathcal{I}}^*(\mathbf{w}) = 0\}$$

and $\bar{\mathcal{I}}^*(\mathbf{w}) = \sup_{\mathbf{p}>0}\Big(\sum_{l=1}^{K} w_l p_l - \mathcal{I}(\mathbf{p})\Big)$ *is the conjugate of* $\mathcal{I}$.

Interpretation of Convex Interference Functions

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k$$

- the set $\mathcal{W}_0(\mathcal{I})$ is closed, convex, and downward-comprehensive
- any convex interference function can be interpreted as the solution of a utility maximization problem



Log-Convex Interference Functions

Definition

We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is a log-convex interference function if it fulfills the axioms:

A1	(non-negativeness)	$\mathcal{I}(\mathbf{p}) \geq 0$	
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p})$	$\forall\alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}')$	if $\mathbf{p} \geq \mathbf{p}'$
C3	(log-convexity)	$\mathcal{I}_k(\mathrm{e}^{\mathbf{s}})$ is log-convex on $\mathbb{R}^K$	

Example: SIR Set Based on Log-Convex Interference

- Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be log-convex interference functions, then the SIR-balancing optimum

$$C(\gamma) = \inf_{\mathbf{p}>0} \Big(\max_{1 \leq k \leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k} \Big)$$

  is a log-convex interference function, i.e., $C(\exp \mathbf{q})$ is a log-convex (thus convex) function.
- the SIR feasible set $\mathcal{S} = \{\gamma : C(\gamma) \leq 1\}$ is convex on a logarithmic scale
- this "hidden convexity" can be exploited for designing resource allocation algorithms

Representation of Log-Convex Interference Functions

Theorem

Every log-convex interference function $\mathcal{I}(\mathbf{p})$*, with* $\mathbf{p} > 0$*, can be represented as*

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{L}(\mathcal{I})} \Big(f_{\mathcal{I}}(\mathbf{w}) \cdot \prod_{l=1}^{K} (p_l)^{w_l} \Big) .$$

where $\quad f_{\mathcal{I}}(\mathbf{w}) = \inf_{\mathbf{p}>0} \frac{\mathcal{I}(\mathbf{p})}{\prod_{l=1}^{K} (p_l)^{w_l}} , \quad \mathbf{w} \in \mathbb{R}_+^K, \quad \sum_k w_k = 1$

$$\mathcal{L}(\mathcal{I}) = \big\{ \mathbf{w} \in \mathbb{R}_+^K : f_{\mathcal{I}}(\mathbf{w}) > 0 \big\}$$

Connection between Convex and Log-Convex Functions

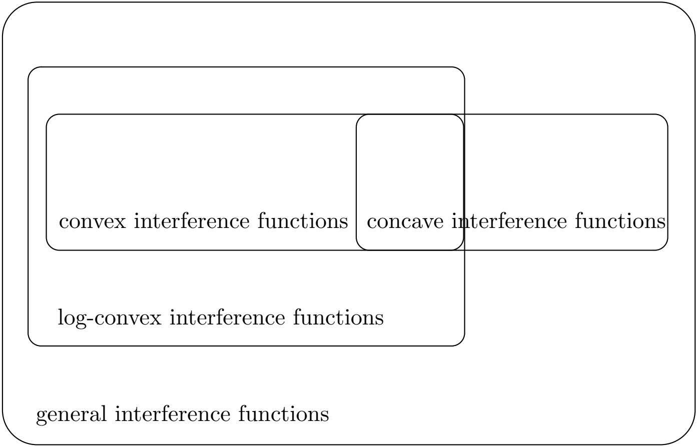
- every convex function $\mathcal{I}(\mathbf{p})$ can be expressed as

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0} \sum_k w_k p_k$$

$\log \sum_k w_k \mathrm{e}^{s_k}$ is convex
$\implies \log \max_{\mathbf{w} \in \mathcal{W}_0} \sum_k w_k \mathrm{e}^{s_k}$ is convex
$\implies \mathcal{I}(\mathrm{e}^{\mathbf{s}})$ is log-convex

- if $\mathcal{I}(\mathbf{p})$ is convex then $\mathcal{I}(\mathrm{e}^{\mathbf{s}})$ is log-convex
(but the converse is not true)

Categories of Interference Functions



Outline

Conclusions

- the framework of interference functions is applicable to different areas in wireless communications:
  - physical layer design
  - medium access control
  - resource allocation and utility optimization
- results provide intuition about the behavior of coupled multiuser systems
- useful for characterizing operating points of the system, design of algorithms
- many interesting open questions