Randomly Designed Space-Time Block Codes for the DF Relay Channel

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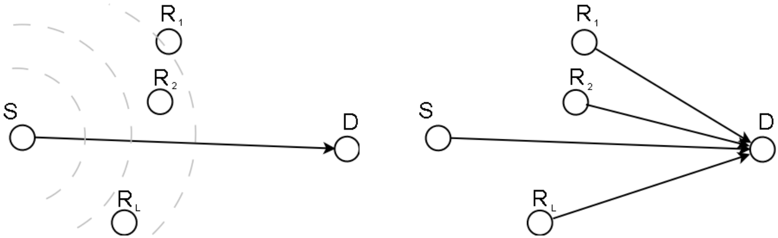
SUPÉLEC: Gif sur Yvette, December 10, 2009

Outline

Cooperative communications



- In most communications systems, information is spread to all terminals.
- Exploit this fact to introduce diversity.

Simple relays and STCodes

Relays should be kept simple (complexity, power consumption...)

⇒NO Channel State Information.

⇒From MIMO experience, STCodes seems the best way to achieve diversity under these assumptions.

STCodes and Dynamic Systems

Classical STCodes are not suitable for dynamic systems where the number of relays is not known beforehand and can change at any time.

The problem is already known in literature, e.g.:

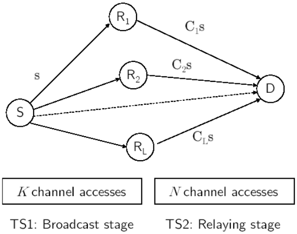
- LW03 suggests block codes from orthogonal design;
- JH06 employs LD-STBCs with unitary matrices;
- SMS07 proposes a partially random code.

[LW03] J. N. Laneman and G. W. Wornell. Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks. *IEEE Trans. Inf. Theory*, 49(10):2415–2425, October 2003.

[JH06] Y. Jing and B. Hassibi. Distributed space-time coding in wireless relay networks. *IEEE Trans. Wireless Commun.*, 5(12):3524–3536, December 2006.

[SMS07] B. Sirkeci-Mergen and A. Scaglione. Randomized space-time coding for distributed cooperative communications. *IEEE Trans. Signal Process.*, 55(10):5003–5017, October 2007.

Transmission scheme



- We consider a simple relay communication system where a set of L half-duplex relays aids a point-to-point communication.
- Two transmission phases:
  1. the source broadcasts K symbols;
  2. relays encode the K source symbols into N new ones to be forwarded.
- Relays employ randomly designed linear-dispersion space-time block coding (STBC) to introduce diversity.
- The destination estimates the information from the the signals received in both phases.

Randomized LD-STBC: what and why

The l-th relay is assigned a randomly generated matrix $\mathbf{C}_l$, which left multiplies the column vector of source symbols.

STC Mapping:

$$\mathbf{s} \rightarrow \underbrace{\left[\begin{array}{c|c|c|c|c} \mathbf{s} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_1\mathbf{s} & \mathbf{C}_2\mathbf{s} & \cdots & \mathbf{C}_L\mathbf{s} \end{array}\right]}_{\text{transmitter}} \Big\} \text{time}$$

The coding matrices are independently generated for each relay and they do NOT depend on their number, channels, ...

Signal Model

ASSUMPTION: All relays $\{R_l\}_{l=1}^{L}$ could decode the source message $\mathbf{s}$. The signal received by the destination is:

$$\mathbf{d} = \begin{bmatrix} h_s \mathbf{I}_K \\ \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{C}} \end{bmatrix} \mathbf{s} + \mathbf{n} = \begin{bmatrix} h_s \mathbf{I}_K \\ \sum_{l=1}^{L} g_l h_{dl} \mathbf{C}_l \end{bmatrix} \mathbf{s} + \mathbf{n},$$

where

$$\widetilde{\mathbf{\Psi}} = \begin{bmatrix} g_1 h_{d1} & \dots & g_L h_{dL} \end{bmatrix} \otimes \mathbf{I}_N = \begin{bmatrix} g_1 h_{d1} \mathbf{I}_N & \dots & g_L h_{dL} \mathbf{I}_N \end{bmatrix}$$

$$\widetilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_L \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \mathbf{D} \end{bmatrix}$$

$$\mathbb{E}[\mathbf{s}] = \mathbf{0}, \qquad \mathbb{E}[\mathbf{s}\mathbf{s}^H] = P_s \mathbf{I}_K, \qquad \mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_d^2 \mathbf{I}_N)$$

The $N \cdot K$ entries of the coding matrices $\{\mathbf{C}_l\}$ are i.i.d. random variables with zero mean and variance $1/N$.

Linear Minimum-Mean-Square-Error Receiver

ASSUMPTION: The destination knows all channels and codes.

The linear MMSE estimate of s_1 is

$$\hat{s}_1 = \mathbb{E}[s_1 \mathbf{d}^H]\left(\mathbb{E}[\mathbf{d}\mathbf{d}^H]\right)^{-1}\mathbf{d}$$

which results in the following SNIR

$$SNIR = \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s}{\sigma_d^2}\mathbf{c}_1^H \widetilde{\mathbf{\Psi}}^H \left(\frac{P_s/\sigma_d^2}{1 + P_s|h_s|^2/\sigma_d^2}\widetilde{\mathbf{\Psi}}\mathbf{D}\mathbf{D}^H\widetilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1}\widetilde{\mathbf{\Psi}}\mathbf{c}_1.$$

Note that this is a random quantity because it depends on the random codes $\{\mathbf{C}_l\}$!!

For a $N \times N$ square matrix $\mathbf{A}$ with real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$, the *empirical eigenvalue distribution* is

$$F_N(x) = \frac{1}{N} |\{\lambda_n, n = 1 \ldots N : \lambda_n < x\}|.$$

Now, for example,

$$\frac{1}{N} \mathrm{tr}\{\mathbf{A}\} = \frac{1}{N} \sum_{n=1}^{N} \lambda_n = \int x \, \mathrm{d}F_N(x).$$

For some ensembles of random matrices, there exist a probability distribution $F(x)$ such that

$$F_N(x) \to F(x), \text{in some way, when } N \to +\infty.$$

Often, it is more convenient to show the convergence of the *Stieltjes* transform.

For the distribution $F(x)$:

$$m_F(z) = \mathbb{E}\left[\frac{1}{z - X}\right] = \int \frac{1}{z - x}\,\mathrm{d}F(x), \text{for } z \in \mathbb{C}^+.$$

For the empirical eigenvalue distribution $F_N(x)$:

$$m_N(z) = \int \frac{1}{z - x}\,\mathrm{d}F_N(x) = \frac{1}{N}\mathrm{tr}\left\{(z\mathbf{I} - \mathbf{A})^{-1}\right\}.$$

Other interesting results: given $\mathbf{x}$ (such that $\mathbb{E}[\mathbf{x}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \frac{1}{N}\mathbf{I}_N$) and $\mathbf{y}$ (similar to and independent of $\mathbf{x}$), independent of $\mathbf{A}$, then

$$\mathbf{x}^H\mathbf{A}\mathbf{x} \to \frac{1}{N}\mathrm{tr}[\mathbf{A}] \qquad \text{and} \qquad \mathbf{x}^H\mathbf{A}\mathbf{y} \to 0 \qquad \text{almost surely.}$$

Asymptotic SNIR and S.E.

ASSUMPTION: $K, N \to +\infty$ but $K/N = \alpha$, the coding rate. Then

$$SNIR = \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s}{\sigma_d^2}\mathbf{c}_1^H \widetilde{\mathbf{\Psi}}^H \left(\frac{P_s/\sigma_d^2}{1 + P_s|h_s|^2/\sigma_d^2} \widetilde{\mathbf{\Psi}}\mathbf{D}\mathbf{D}^H \widetilde{\mathbf{\Psi}}^H + \mathbf{I}_N \right)^{-1} \widetilde{\mathbf{\Psi}}\mathbf{c}_1$$

tends almost surely to the deterministic quantity

$$SNIR^{\mathrm{iid}} = \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s}{\sigma_d^2}\frac{\sum_{l=1}^L |g_l h_{dl}|^2}{\beta}, \text{with } \beta = 1 + \alpha\beta \frac{\chi \sum_{l=1}^L |g_l h_{dl}|^2}{\beta + \chi \sum_{l=1}^L |g_l h_{dl}|^2}.$$

For symmetry and logarithm continuity, the spectral efficiency tends to

$$I_{\mathrm{LMMSE}}^{\mathrm{iid}} = \frac{K}{K+N} \ln\left(1 + SNIR^{as}\right) = \frac{\alpha}{1+\alpha} \ln\left(1 + SNIR^{as}\right).$$

Asymptotic S.E. for the ML receiver

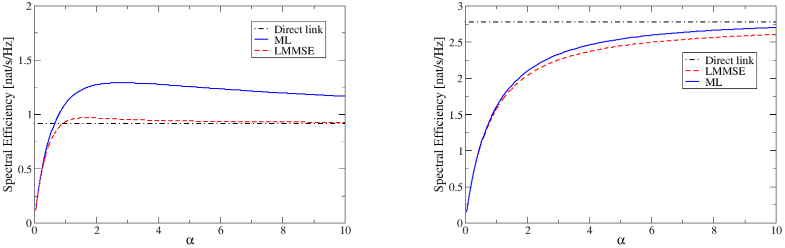
Similarly, we can show that the ML spectral efficiency

$$I_{\mathrm{ML}} = \frac{1}{K+N} \ln\det\left(\mathbf{I}_{K+N} + \frac{P_s}{\sigma_d^2} \begin{bmatrix} h_s \mathbf{I}_K \\ \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{C}} \end{bmatrix} \begin{bmatrix} h_s^* \mathbf{I}_K & \widetilde{\mathbf{C}}^H \widetilde{\mathbf{\Psi}}^H \end{bmatrix} \right)$$

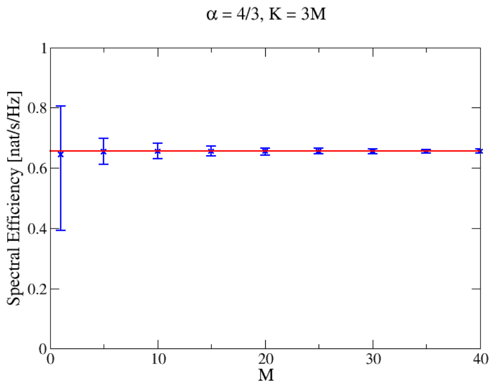
tends almost surely to

$$I_{\mathrm{ML}}^{\mathrm{iid}} = \frac{\alpha}{1+\alpha} \ln\left(1+\rho|h_s|^2\right) + \frac{1}{1+\alpha}\left[\alpha \ln\left(1+\frac{\chi}{\beta}\sum_{l=1}^{L}|g_l h_{dl}|^2\right) + \ln\beta + \frac{1}{\beta} - 1\right].$$

Spectral efficiency vs. α



We can try to find a condition that tells us whether relaying is superior or not to the simple direct link in terms of *instantaneous* spectral efficiency. However, it is better to look to *outage probability*.



- The limits for $K = \alpha N \rightarrow +\infty$ are excellent approximations of the finite reality, even for "not so long" codes.
- We approximate the outage probability from the asymptotic expressions

Outage probability

ASSUMPTIONS: All the channels undergo quasi static, frequency flat, Rayleigh fading and $|g_l|^2 = 1/\alpha, l = 1 \dots L$.

FEELING: In the asymptotic expressions, we identify a component for each transmitter, e.g.

$$SNIR^{\mathrm{iid}} = \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s}{\sigma_d^2}\frac{\sum_{l=1}^{L}|h_{dl}|^2}{\alpha\beta}.$$

PROBLEM: The coefficient β depends on all the channels and correlates all the components.

Given a set $\mathcal{L}$ of relays, only the relays in the decoding subset $\mathcal{L}' \subseteq \mathcal{L}$ decode the source message and participate in the second phase.

For a target rate R, $P_{out}(R) = \Pr[I \leq R]$ can be computed as

$$P_{out}(R) = \sum_{\mathcal{L}' \subseteq \mathcal{L}} \Pr[I \leq R | \mathcal{L}'] \Pr[\mathcal{L}' \text{ is the decoding set}]$$

with $\Pr[\mathcal{L}' \text{ is the decoding set}] = \prod_{l \in \mathcal{L}'} \Pr[I_{ul} > R] \prod_{l \in \mathcal{L} \setminus \mathcal{L}'} \Pr[I_{ul} \leq R]$.

Since $\beta = 1 + \alpha\beta \frac{\chi \sum_{l=1}^{L} |h_{dl}|^2}{\alpha\beta + \chi \sum_{l=1}^{L} |h_{dl}|^2}$, one has

$$\beta < z \Longleftrightarrow z > 1 + \alpha z \frac{\chi \sum_{l=1}^{L} |h_{dl}|^2}{\alpha z + \chi \sum_{l=1}^{L} |h_{dl}|^2}.$$

For large SNR $\rho = P_s/\sigma_d^2$, we want to approximate the outage probability as $P_{out}(R) \sim \kappa\rho^{-d}$, where

- d is the diversity order ($\sim$ number of independent paths);
- κ is the outage gain.

Assume the target rate increases with the SNR, i.e. $R = r \ln \rho$, where

- r is multiplexing gain.

Then, the diversity order as a function of the multiplexing gain, namely

$$d(r) = -\lim_{\rho \to +\infty} \frac{\ln P_{out}(r \ln \rho)}{\ln \rho},$$

is the diversity–multiplexing tradeoff.

For the LMMSE receiver, the diversity order is a function of α:

$$d_{\mathrm{LMMSE}} = \begin{cases} L+1 & \text{for } \alpha \leq \alpha_{th} \\ 1 & \text{for } \alpha > \alpha_{th} \end{cases},$$

where

$$\alpha_{th} = 1 + \frac{1}{Q(\alpha_{th})}, \qquad Q = Q(\alpha) = \exp\left(\frac{1+\alpha}{\alpha} R\right) - 1.$$

The respective outage gains are

$$\kappa_{\mathrm{LMMSE}} = \begin{cases} \zeta_s \sum_{l=0}^{L} \binom{L}{l} \left(\frac{\zeta_u Q}{z}\right)^{L-l} \frac{(\alpha \zeta_d)^l}{l!} \mathcal{P}_l \\ \zeta_s \left(Q - \frac{Q+1}{\alpha}\right). \end{cases}$$

We also find two different DMTs:

$$d(r) = \begin{cases} (L+1)\left(1 - \dfrac{1+\alpha}{\alpha} r\right) & \text{for } \alpha \leq 1, \\ 1 - \dfrac{1+\alpha}{\alpha} r & \text{for } \alpha > 1. \end{cases}.$$

For the ML receiver, the diversity order is always maximum: $d = L + 1$.
The outage gain can be computed only numerically.

[LW03] analyzes TDMA and STBC from orthogonal design:

$$I_{rep}^{(\mathcal{L}')} = \frac{1}{L+1} \ln\left(1 + \rho|h_s|^2 + \rho \sum_{l \in \mathcal{L}'} |h_{dl}|^2\right),$$

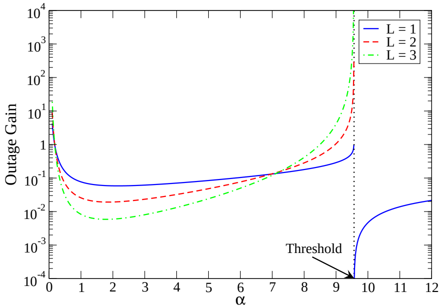
$$I_{stc}^{(\mathcal{L}')} = \frac{1}{2} \ln\left(1 + \rho|h_s|^2\right) + \frac{1}{2} \ln\left(1 + \rho \sum_{l \in \mathcal{L}'} |h_{dl}|^2\right).$$

Both achieve full diversity and

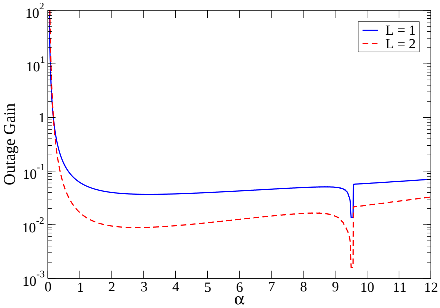
$$\kappa_{rep} = \left(e^{(L+1)R} - 1\right)^{L+1} \zeta_s \sum_{l=0}^{L} \binom{L}{l} \frac{\zeta_d^l (\zeta_u z^{-1})^{L-l}}{(l+1)!} \qquad d_{rep}(r) = (L+1)[1 - (L+1)r]$$

$$\kappa_{stc} = \left(e^{2R} - 1\right)^{L+1} \zeta_s \sum_{l=0}^{L} \binom{L}{l} \zeta_d^l (\zeta_u z^{-1})^{L-l} A_l\left(e^{2R} - 1\right) \qquad d_{stc}(r) = (L+1)(1 - 2r).$$

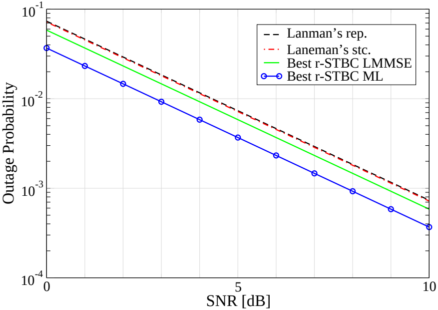
[LW03] J. N. Laneman and G. W. Wornell. Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks. *IEEE Trans. Inf. Theory*, 49(10):2415–2425, October 2003.


Outage Gain
α
L = 1
L = 2
L = 3
Threshold
10^4
10^3
10^2
10^1
1
10^-1
10^-2
10^-3
10^-4
0 1 2 3 4 5 6 7 8 9 10 11 12

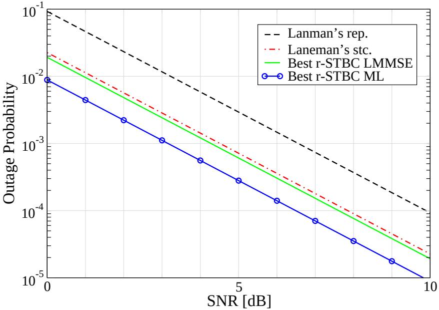
LMMSE


Outage Gain
α
L = 1
L = 2
10^2
10^1
1
10^-1
10^-2
10^-3
0 1 2 3 4 5 6 7 8 9 10 11 12

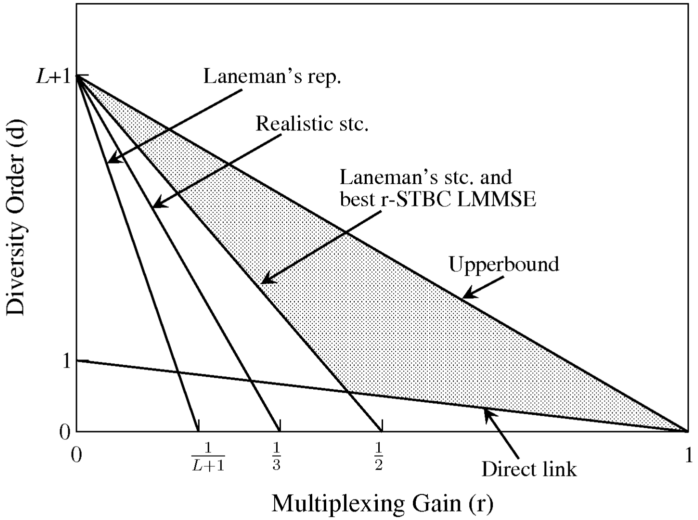
ML

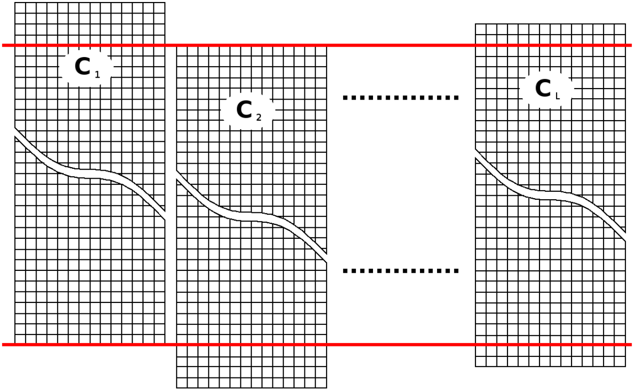


1 relay



2 relays


L+1
Laneman's rep.
Realistic stc.
Laneman's stc. and
best r-STBC LMMSE
Upperbound
Diversity Order (d)
1
0
0
$\frac{1}{L+1}$
$\frac{1}{3}$
$\frac{1}{2}$
Direct link
1
Multiplexing Gain (r)



Let δ be the maximum delay

If $\delta \ll N$
$\Rightarrow \alpha = \frac{K}{N-\delta} \approx \frac{K}{N}$ and everything the same.

If δ comparable with N
$\Rightarrow \alpha = \frac{K}{N-\delta}$, we can compute a lower bound

$$I \geq \frac{\alpha}{1 + \alpha\left(1 + 2\delta/N\right)} \ln \ldots$$

Codes for Synchronous relays

Then, for synchronous relays, we can think to some more structured codes and see how much we can gain.

The signal in the second transmission phase, due to the relays, is

$$\widetilde{\boldsymbol{\Psi}}\widetilde{\mathbf{C}}\mathbf{s} = \sum_{l=1}^{L} g_l h_{dl} \mathbf{C}_l \mathbf{s} = \sum_{l=1}^{L} g_l h_{dl} \left(\sum_{k=1}^{K} \mathbf{c}_{l,k} s_k \right) = \sum_{k=1}^{K} \left(\sum_{l=1}^{L} g_l h_{dl} \mathbf{c}_{l,k} \right) s_k$$

For a perfect multiplexing within the relays, we suggest random *isometric* codes (orthonormal columns).

Linear Minimum-Mean-Square-Error Receiver

Same signal model:

$$\mathbf{d} = \begin{bmatrix} h_s \mathbf{I}_K \\ \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{C}} \end{bmatrix} \mathbf{s} + \mathbf{n} = \begin{bmatrix} h_s \mathbf{I}_K \\ \sum_{l=1}^{L} g_l h_{dl} \mathbf{C}_l \end{bmatrix} \mathbf{s} + \mathbf{n},$$

and same SNIR for symbol 1:

$$SNIR = \frac{P_s}{\sigma_d^2} |h_s|^2 + \frac{P_s}{\sigma_d^2} \mathbf{c}_1^H \widetilde{\mathbf{\Psi}}^H \left(\chi \widetilde{\mathbf{\Psi}} \mathbf{D}\mathbf{D}^H \widetilde{\mathbf{\Psi}}^H + \mathbf{I}_N \right)^{-1} \widetilde{\mathbf{\Psi}} \mathbf{c}_1.$$

Since the columns of the new codes are orthonormal, $\mathbf{c}_1$ and $\mathbf{D}$ are not independent anymore. The RMT approach does not hold!!!

For two *non-commutative* random variables a, b, the joint distribution is characterized by all the moments

$$\phi(a), \phi(b), \phi(a^2), \phi(b^2), \phi(ab), \phi(a^2 b), \phi(ab^2), \phi(a^2 b^2), \phi(abab), \ldots$$

Note that, e.g., $\phi(a^2 b^2) \neq \phi(abab)$!!

$N \times N$ random matrices can be seen as non-commutative random variables. The moments are computed with the functional $\phi(\cdot) = \frac{1}{N}\mathbb{E}[\mathrm{tr}\{\cdot\}]$.

If a, b, c are three free random variables with zero mean ($\phi(a) = 0$), then, e.g., $\phi(abcabacbcabc) = 0$ but, in general, $\phi(abcab^2c) \neq 0$

Take two free non-commutative random variables a and b with distributions μ_a and μ_b, respectively. Let us name

$\mu_{a+b} = \mu_a \boxplus \mu_b$ the distribution of $a + b$ and

$\mu_{ab} = \mu_a \boxtimes \mu_b$ the distribution of ab.

There exist two transformations $R_\mu(z)$ and $S_\mu(z)$ of generic distribution μ such that

- $R_{\mu_{a+b}} = R_{\mu_a} + R_{\mu_b}$ and
- $S_{\mu_{ab}} = S_{\mu_a} S_{\mu_b}$.

Our random code matrices are *almost surely asymptotically free* when $K = \alpha N \to +\infty$ and we can use the R- and S-transforms to characterize the eigenvalue distribution of sums and product of matrices.

Asymptotic SNIR and S.E.

The code-dependent term of the SNIR is

$$\gamma = \mathbf{c}_1^H \widetilde{\mathbf{\Psi}}^H \left(\chi \widetilde{\mathbf{\Psi}} \mathbf{D}\mathbf{D}^H \widetilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1} \widetilde{\mathbf{\Psi}} \mathbf{c}_1 = \frac{\eta}{1 - \chi\eta},$$

where

$$\eta = \frac{1}{K} \mathrm{tr}\left\{ \left(\chi \widetilde{\mathbf{\Psi}} \widetilde{\mathbf{C}} \widetilde{\mathbf{C}}^H \widetilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1} \widetilde{\mathbf{\Psi}} \widetilde{\mathbf{C}} \widetilde{\mathbf{C}}^H \widetilde{\mathbf{\Psi}}^H \right\} \to \int \frac{t}{1 + \chi t} \nu^2(\mathrm{d}t) = -\frac{1}{\chi} M_{\nu^2}(-\chi).$$

for $K = \alpha N \to +\infty$. ν^2 is the limit eigenvalue distribution of $\widetilde{\mathbf{\Psi}} \widetilde{\mathbf{C}} \widetilde{\mathbf{C}}^H \widetilde{\mathbf{\Psi}}^H$ and $M_{\nu^2}(z)$ its moment generating function

$$\begin{aligned} M_{\nu^2}(z) &= \int \frac{zt}{1 - zt} \nu^2(\mathrm{d}t) \\ &= \frac{1}{z} G_{\nu^2}\left(\frac{1}{z}\right) - 1 && G_{\nu^2}(z) \text{ is the Stieltjes transform} \\ &= \sum_{n=1}^{+\infty} \mathbb{E}_{\nu^2}[X^n] z^n, && \text{for } |z| \text{ small enough.} \end{aligned}$$

Asymptotic SNIR and S.E. (cont'd)

For large $K, N, K/N = \alpha$, we can compute a deterministic approximation of the SNIR and the spectral efficiency

$$\eta \Rightarrow \gamma \Rightarrow SNIR \Rightarrow I.$$

Unfortunately, the function $M_{\nu^2}(z)$ is only known as the solution to

$$M_{\nu^2}(z) = C_\nu[zT(M_{\nu^2}(z))]$$

in a neighborhood of zero. At this point, most of the times we are only able to obtain the following approximation:

$$M_n(z) = z \sum_{j=1}^{n} \frac{\lambda_{j,n}\gamma_{j,n}}{1 - z\lambda_{j,n}},$$

the moment generating function of $\nu_n = \sum_{j=1}^{n} \gamma_{j,n}\delta_{\lambda_{j,n}}$, with the same $2n - 1$ first moments of ν^2.

Similarly, the deterministic limit of the spectral efficiency in the ML case is

$$\begin{aligned} I_{\mathrm{ML}}^{\mathrm{Haar}} &= \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) - \frac{\alpha}{1+\alpha} \int_0^{\chi} \frac{M_{\nu^2}(-z)}{z} \, \mathrm{d}z \\ &\approx \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) - \frac{\alpha}{1+\alpha} \int_0^{\chi} \frac{M_n(-z)}{z} \, \mathrm{d}z \\ &\approx \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) + \frac{\alpha}{1+\alpha} \sum_{j=1}^{n} \gamma_{j,n} \ln(1 + \chi\lambda_{j,n}). \end{aligned}$$

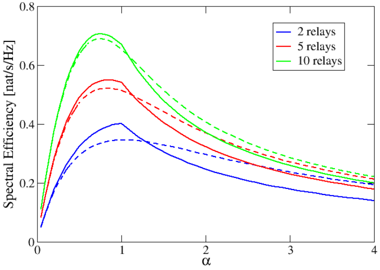
We choose $\{g_l\}$ such that $|g_l h_{dl}|^2 = 1, l = 1 \ldots L$.

$M_{\nu^2}(z)$ can be expressed in closed form and

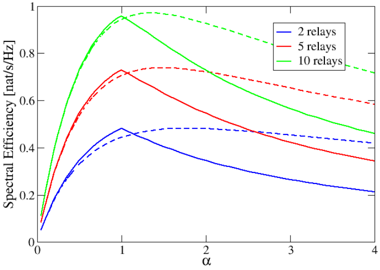
$$\eta^{\text{Haar}} = \frac{L}{2\alpha\chi} \frac{1 + L(1+\alpha)\chi - \sqrt{(1 - L(1-\alpha)\chi)^2 + 4(L-\alpha)\chi}}{1 + L^2\chi}$$

whereas

$$\eta^{\text{i.i.d.}} = \frac{1}{2\alpha L\chi^2}\left[1 + L(1+\alpha)\chi - \sqrt{(1 - L(1-\alpha)\chi)^2 + 4L\chi}\right].$$


0.8
0.6
0.4
0.2
0
Spectral Efficiency [nat/s/Hz]
0
1
2
3
4
α
2 relays
5 relays
10 relays

LMMSE


1
0.8
0.6
0.4
0.2
0
Spectral Efficiency [nat/s/Hz]
0
1
2
3
4
α
2 relays
5 relays
10 relays

ML

Low-power approximation as in [Ver02]:

$$\frac{E_b}{N_0}_{\min} = \frac{\ln 2}{\dot{I}(0)} \qquad \text{and} \qquad S_0 = -\frac{2[\dot{I}(0)]^2}{\ddot{I}(0)} \text{ in bit/s/Hz/(3 dB).}$$

For the isometric codes

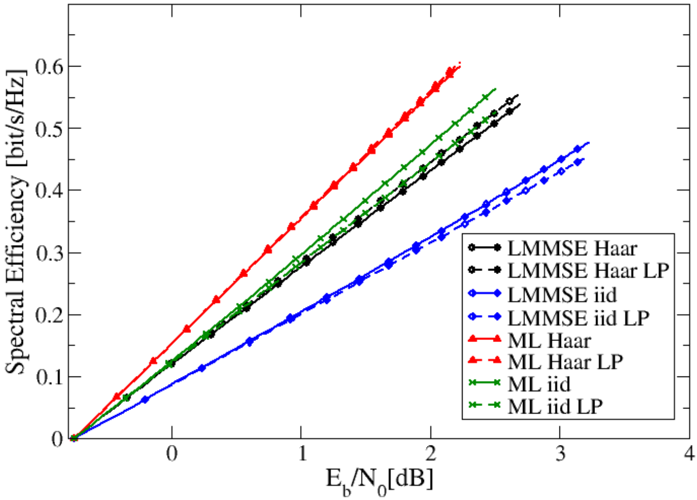
$$\left.\frac{\partial I}{\partial \rho}\right|_{\rho=0} = \frac{\alpha}{1+\alpha}\left(|h_s|^2 + m_1\right)$$

$$\left.\frac{\partial^2 I}{\partial \rho^2}\right|_{\rho=0} = \begin{cases} -\dfrac{\alpha}{1+\alpha}\left(|h_s|^4 + 2|h_s|^2 m_1 + 2m_2 - m_1^2\right) & \text{LMMSE} \\ -\dfrac{\alpha}{1+\alpha}\left(|h_s|^4 + 2|h_s|^2 m_1 + m_2\right) & \text{ML.} \end{cases}$$

Note

$$\frac{S_0^{\mathrm{ML}}}{S_0^{\mathrm{LMMSE}}} = 1 + \frac{m_2 - m_1^2}{|h_s|^4 + 2|h_s|^2 m_1 + m_2}.$$

[Ver02] S. Verdú. Spectral efficiency in the wideband regime. *IEEE Trans. Inf. Theory*,


Spectral Efficiency [bit/s/Hz]
0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
E_b/N_0[dB]
LMMSE Haar
LMMSE Haar LP
LMMSE iid
LMMSE iid LP
ML Haar
ML Haar LP
ML iid
ML iid LP

Rectangular R-transform

We need the singular-value distribution of a sum of matrices $K \times N$, namely $\widetilde{\mathbf{C}}^H \widetilde{\mathbf{\Psi}}^H = \sum_{l=1}^{L} g_l^* h_{dl}^* \mathbf{C}_l^H$: *rectangular R-transform* of ratio α.

Given two symmetric ($\mu(B) = \mu(-B)$) distributions μ_a and μ_b, we have

$$C_{\mu_a \boxplus_\alpha \mu_b}(z) = C_{\mu_a}(z) + C_{\mu_b}(z).$$

The singular law of $g_l^* h_{dl}^* \mathbf{C}_l^H$ is $\mu_l = \frac{1}{2}(\delta_{-|g_l h_{dl}|} + \delta_{|g_l h_{dl}|})$, meaning that

$$C_\nu(z) = \frac{1}{2\alpha} \sum_{l=1}^{L} \left[\left(1 + 4\alpha |g_l h_{dl}|^2 z\right)^{1/2} - 1 \right], |z| \text{ small enough}$$

[BG09] F. Benyach-Georges. Rectangular random matrices, related convolution. *Probability Theory and Related Fields*, 144(3-4):471–515, July 2009.

$$M_{\mu^2}(z) = \int \frac{zt}{1-zt}\mu^2(\mathrm{d}t) = \int \frac{zt^2}{1-zt^2}\mu(\mathrm{d}t), \qquad \text{mom. gen. fct.}$$

$$T(z) = (\alpha z + 1)(z+1)$$

$$H_\mu(z) = zT \circ M_{\mu^2}(z) = z(\alpha M_{\mu^2}(z)+1)(M_{\mu^2}(z)+1) \qquad \text{rect. Cauchy transf.}$$

$$U(z) = \frac{-(\alpha+1) + [(\alpha+1)^2 + 4\alpha z]^{1/2}}{2\alpha} \qquad \text{inverse of } T(z)-1$$

$$C_\mu(z) = U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right)$$

everything defined for $z \in \mathbb{C} \setminus \mathbb{R}_+$ and small enough.

To invert the procedure and recover the probability measure μ, we need to find the moment generating function which is solution to

$$M_{\mu^2}(z) = C_\mu[zT(M_{\mu^2}(z))], \quad |z| \text{ small enough.}$$

Conclusions

- The relay channel need flexible and dynamic STCs.
- The presented randomized LD-STBCs are full-diversity achieving.
- They are also robust to small relay asynchronicities.
- Isometric codes bring only little benefit.
- Mathematical side-effect: rectangular R-transform.

THANK YOU!

Moments of ν^2

Even though we cannot reverse the rectangular R-transform, we can compute all the moments of ν^2. Indeed, in a neighborhood of zero,

$$M_{\nu^2}(z) = \sum_{k=1}^{+\infty} \mathbb{E}_{\nu^2}[X^k] z^k = \sum_{k=1}^{+\infty} m_k z^k$$

and

$$C_\nu(z) = \sum_{k=1}^{+\infty} c_k z^k, \text{with } c_k = \frac{(2\alpha)^{k-1}}{k!} \left(\prod_{i=0}^{k-1} (1-2i) \right) \sum_{l=1}^{L} |g_l h_{dl}|^{2k},$$

Hence, from

$$M_{\nu^2}(z) = \sum_{k=1}^{+\infty} c_k z^k \left[\alpha M_{\nu^2}^2(z) + (\alpha+1) M_{\nu^2}(z) + 1 \right]^k$$

one can recover the moments by comparing coefficients of terms in z^k.

$$m_1 = \sum^{L} |g_l h_{dl}|^2, \quad m_2 = (\alpha+1) \left(\sum^{L} |g_l h_{dl}|^2 \right)^2 - \alpha \sum^{L} |g_l h_{dl}|^4, \quad m_3 = \ldots$$

Gauss-Jacobi mechanical quadrature

We want to approximate $G_\mu(z) = \int \frac{1}{z-t}\mu(\mathrm{d}t)$, by $\bar{G}_n(z) = \sum_{k=1}^{n} \frac{\gamma_{k,n}}{z-\lambda_{k,n}}$, the Stieltjes transform of $\bar{\mu}_n = \sum_{k=1}^{n} \gamma_{k,n}\delta_{\lambda_{k,n}}$, so that the first $2n-1$ moments of μ and $\bar{\mu}_n$ coincide.

We can build a family of polynomials $\{p_k\}$ such that

- $p_k(\lambda)$ has degree k and positive leading coefficient;
- they are orthonormal, i.e. $\langle p_k, p_m\rangle = \int p_k(\lambda)p_m(\lambda)\mu(\mathrm{d}\lambda) = \delta_{k,m}$,
- they satisfy

$$\lambda p_k(\lambda) = b_{k-1}p_{k-1} + a_k p_k(\lambda) + b_k p_{k+1}(\lambda)$$

with $a_k = \langle \lambda p_k(\lambda), p_k(\lambda)\rangle > 0$ and $b_k = \langle \lambda p_k(\lambda), p_{k+1}(\lambda)\rangle > 0$.
The recurrence formula is initiated by $p_0(\lambda) = 1$ and $p_1(\lambda) = \frac{\lambda - a_0}{b_0}$.

The first coefficients are

$$a_0 = m_1, \quad b_0 = m_2 - m_1^2, \quad a_1 = \frac{m_3 - 2m_1 m_2 - m_1^3}{\left(m_2 - m_1^2\right)^2}, \quad b_1 = \dots$$

The approximating Stieltjes transform is

$$\bar{G}_n(z) = \sum_{k=1}^{n} \frac{\gamma_{k,n}}{z - \lambda_{k,n}} = -\cfrac{1}{z - a_0 - \cfrac{b_0^2}{z - a_1 - \cfrac{b_1^2}{\dots - \cfrac{b_{n-2}^2}{z - a_{n-1}}}}}.$$

$\{\lambda_{k,n}\}$ are the n roots of $p_n(\lambda)$.
Christoffel-Darboux formula

$$\gamma_{k,n} = \frac{1}{\sum_{i=0}^{n-1} \left|p_i(\lambda_{k,n})\right|^2}.$$

with $\sum_{k=1}^{n} \gamma_{k,n} = 1$.