Gegenbauer polynomials and positive definiteness

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Abstract

Gaussian random fields $Z(\xi, u)$ defined over the unit sphere cross time, $\mathbb{S}^d \times \mathbb{R}$, and which are isotropic with respect to $\xi \in \mathbb{S}^d$ and stationary with respect to time $u \in \mathbb{R}$ are important in connection with time dependent measurements on the surface of the earth. The covariance of the random variables $Z(\xi, u), Z(\eta, v)$ can be written as

$$\mathrm{cov}\,(Z(\xi, u), Z(\eta, v)) = f(\xi \cdot \eta, u - v), \qquad (\xi, u), (\eta, v) \in \mathbb{S}^d \times \mathbb{R}, \quad (1)$$

for a continuous function $f : [-1, 1] \times \mathbb{R} \to \mathbb{R}$, where $\xi \cdot \eta$ is the scalar product of the unit vectors.

Recall that the Gegenbauer polynomials $C_n^{(\lambda)}$ are given by the generating function ($\lambda > 0$ with modifications if $\lambda = 0$)

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) r^n, \quad |r| < 1, x \in \mathbb{C}$$

and define the normalized ultraspherical polynomials by

$$c_n(d, x) = C_n^{(\lambda)}(x)/C_n^{(\lambda)}(1), \text{ for } \lambda = (d-1)/2, d = 1, 2, \ldots$$

In the talk I shall report on a recent paper [1] characterizing the functions f entering in Equation (1) as the functions having a representation

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad (x, u) \in [-1, 1] \times \mathbb{R},$$

where $\varphi_{n,d}$ is a sequence of real-valued continuous positive definite functions on $\mathbb{R}$. The series is uniformly convergent, which is equivalent to $\sum_{n=0}^{\infty} \varphi_{n,d}(0) < \infty$.

We also prove a far reaching generalization, where $\mathbb{R}$ is replaced by an arbitrary locally compact groups G (written multiplicatively and with neutral element e). The class denoted $\mathcal{P}(\mathbb{S}^d, G)$ consists of the continuous functions $f : [-1, 1] \times G \to \mathbb{C}$ which are positive definite in the sense that for any $n \in \mathbb{N}$ and any $(\xi_1, u_1), \ldots (\xi_n, u_n) \in \mathbb{S}^d \times G$ the $n \times n$-matrix

$$\left[f(\xi_k \cdot \xi_l, u_k^{-1} u_l) \right]_{k,l=1}^n$$

is hermitian and positive semi-definite.

These functions are precisely those with expansion

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x),$$

where $\varphi_{n,d}$ is a sequence of continuous positive definite functions on G satisfying $\sum_{n=0}^{\infty} \varphi_{n,d}(e) < \infty$.

When G is the trivial group with one element this reduces to a famous result [2] by Schoenberg about positive definite functions on spheres.

References

[1] C. Berg and E. Porcu, *From Schoenberg coefficients to Schoenberg functions.* ArXiv1505.05682

[2] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J. **9** (1942), 96–108.