Gegenbauer polynomials and positive definiteness

Christian Berg

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Abstract

Gaussian random fields \(Z(\xi, u)\) defined over the unit sphere cross time, \(S^d \times \mathbb{R}\), and which are isotropic with respect to \(\xi \in S^d\) and stationary with respect to time \(u \in \mathbb{R}\) are important in connection with time dependent measurements on the surface of the earth. The covariance of the random variables \(Z(\xi, u), Z(\eta, v)\) can be written as

\[
\text{cov} (Z(\xi, u), Z(\eta, v)) = f(\xi \cdot \eta, u - v), \quad (\xi, u), (\eta, v) \in S^d \times \mathbb{R},
\]

for a continuous function \(f : [-1, 1] \times \mathbb{R} \to \mathbb{R}\), where \(\xi \cdot \eta\) is the scalar product of the unit vectors.

Recall that the Gegenbauer polynomials \(C_n^\lambda\) are given by the generating function \((\lambda > 0\) with modifications if \(\lambda = 0\))

\[
(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) r^n, \quad |r| < 1, x \in \mathbb{C}
\]

and define the normalized ultraspherical polynomials by

\[
c_n^{(d)}(x) = C_n^{(\lambda)}(x)/C_n^{(\lambda)}(1), \quad \text{for } \lambda = (d - 1)/2, d = 1, 2, \ldots
\]

In the talk I shall report on a recent paper [1] characterizing the functions \(f\) entering in Equation (1) as the functions having a representation

\[
f(x, u) = \sum_{n=0}^{\infty} \phi_n^{d}(u)c_n^{d}(x), \quad (x, u) \in [-1, 1] \times \mathbb{R},
\]

where \(\phi_n^{d}\) is a sequence of real-valued continuous positive definite functions on \(\mathbb{R}\). The series is uniformly convergent, which is equivalent to \(\sum_{n=0}^{\infty} \phi_n^{d}(0) < \infty\).

We also prove a far reaching generalization, where \(\mathbb{R}\) is replaced by an arbitrary locally compact groups \(G\) (written multiplicatively and with neutral element \(e\)). The class denoted \(\mathcal{P}(S^d, G)\) consists of the continuous functions \(f : [-1, 1] \times G \to \mathbb{C}\) which are positive definite in the sense that for any \(n \in \mathbb{N}\) and any \((\xi_1, u_1), \ldots (\xi_n, u_n) \in S^d \times G\) the \(n \times n\)-matrix

\[
[f(\xi_k \cdot \xi_l, u_k^{-1} u_l)]_{k,l=1}^{n}
\]

is hermitian and positive semi-definite.

These functions are precisely those with expansion

\[
f(x, u) = \sum_{n=0}^{\infty} \phi_n^{d}(u)c_n^{d}(x),
\]
where $\varphi_{n,d}$ is a sequence of continuous positive definite functions on $G$ satisfying $\sum_{n=0}^{\infty} \varphi_{n,d}(e) < \infty$.

When $G$ is the trivial group with one element this reduces to a famous result [2] by Schoenberg about positive definite functions on spheres.

References
