Nonlinear noninteracting control with stability in discrete time: a dynamic solution

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Abstract

The geometric approach proposed in [2] for nonlinear noninteracting control with stability via static state feedback in discrete time, is now used to find a dynamic solution for systems characterized by an invertible drift. Geometric necessary and sufficient conditions are given. A constructive procedure for the computation of the dynamic compensator solving the problem is proposed.

keywords. Discrete–time systems, nonlinear decoupling, stability, dynamic feedback.

1 Introduction

Nonlinear noninteracting control with stability has been widely studied in continuous time (see [7], [8], [14], [17], [1] and the references therein) and different approaches have been pursued in discrete time (see [11], [6], [10], [3], [2]).

Following [2], where noninteraction with stability via static state feedback was investigated, the present paper addresses the problem through dynamic compensations.

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When dealing with this problem in a geometric context, it is efficient to preliminarily express the property of noninteraction in terms of some invariant distributions. In [7], dealing with continuous-time input-affine systems, it is in fact shown that the solution to the problem of noninteraction with stability via regular static-state feedback can be linked to the stability properties of a subdynamics of the zero dynamics - the $R^*$-dynamics - which can be computed from the controllability distributions and which defines the so-called fixed modes. This result generalizes to the nonlinear continuous-time context a well known linear result [18].

In the discrete-time context, a major difficulty stands in the characterization and computation of these distributions. A preliminary result to the decoupling problem with stability was given in [3] where the linearization of the $R^*$-dynamics was considered. More recently, thanks to the apparatus developed for discrete-time dynamics in [12], [13], an explicit characterization of the distributions involved in the solution of the problem has been given in [2]. In the present paper we generalize this study to dynamic-state feedback laws.

For continuous-time systems, a necessary condition for solving the problem of noninteraction with stability via dynamic-state feedback is given in [17]. It is shown that the $R^*$-dynamics contains a subdynamics - the $\Delta_{mix}$-dynamics - , which is invariant under dynamic feedback, so that its stability properties cannot be modified by any dynamic feedback preserving noninteraction. In the linear case, this subdynamics is zero dimensional so that the problem always admits a solution, provided stabilization is achievable through dynamic feedback. In [1], under some regularity assumptions, the asymptotic stability of the $\Delta_{mix}$-dynamics is shown to be also sufficient to solve the problem and the structure of a dynamic compensator solving the problem is given.

Following the lines set in [8], [17] and [1], we study the problem in discrete time and give necessary conditions and sufficient conditions for its solvability under the assumption that the given system is drift invertible and that such a property is maintained by the whole feedback system. To this end, we introduce, in the discrete-time context, the distribution $\Delta_{mix}$ and we show that the solution is linked to the asymptotic stability of the residual dynamics associated with it. We give an algorithm to compute the over-mentioned distribution and we specify the structure of a dynamic compensator solving the problem.
The original contribution stands in the complete development of the geometrical framework for the over-mentioned discrete time dynamic right to the definition of a procedure for the computation of the dynamic compensator. An example illustrates the proposed procedure.

The paper is organized as follows. The results on decoupling with stability via static feedback set in [2] are summarized in Section 2. On these bases, in Section 3, the geometric structure of decoupled dynamics together with the characterization of regular dynamic feedback laws preserving noninteraction and drift invertibility are investigated. Necessary and sufficient conditions are given in Section 4 where a regular dynamic feedback solving the problem is also proposed. An example shows the technical aspects.

2 Recalls and preliminaries

The following standard notations will be used throughout the paper. Given two vector fields $\tau_1(x), \tau_2(x)$, a real valued function $\lambda(x)$ and a diffeomorphism $\phi(x)$ on $\mathbb{R}^n$, we denote by $L_{\tau_1}\lambda(x) := \frac{\partial \lambda(x)}{\partial x} \tau_1(x)$, the Lie derivative of $\lambda$ with respect to $\tau_1$, by $ad_{\tau_1}\tau_2(x) := [\tau_1, \tau_2](x) = L_{\tau_1} \circ L_{\tau_2}(Id)|_x - L_{\tau_2} \circ L_{\tau_1}(Id)|_x$, the Lie bracket of vector fields and by $Ad_{\phi}\tau_1$ the transport of $\tau_1$ along $\phi(x)$; i.e. $Ad_{\phi}\tau_1 := \left( \frac{\partial \phi}{\partial x} \tau_1 \right)|_{\phi^{-1}}$. By construction, $\tau_1$ and $Ad_{\phi}\tau_1$ are $\phi$-related; i.e. $\phi_{*}\tau_1 := (Ad_{\phi}\tau_1) \circ \phi$. Moreover, $Id$ denotes the identity function, $I$ the identity operator, $J\phi := \frac{\partial \phi(x)}{\partial x}$ the Jacobian of $\phi(x)$, $\phi^{-1}(x)$ the inverse of $\phi(x)$ and $(.)|_x$ the evaluation at $x$ of the function into parentheses.

2.1 Recalls on static-state feedback decoupling

Consider the nonlinear discrete-time system

\begin{align}
    x_{k+1} &= f(x_k, u_k) \\
    y_k^i &= h^i(x_k) & i = (1, \cdots, m)
\end{align}

where the state $x$ belongs to $\mathbb{R}^n$, the input and output vectors $u$ and $y$ belong to $\mathbb{R}^m$; $(x, u) = (0, 0)$ is an equilibrium pair for (1-2); the functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and
$h^i : \mathbb{R}^n \to \mathbb{R}$ are analytic in their arguments. As usual, when dealing with dynamic decoupling with stability, we assume system (1–2) already decoupled and with strong relative degree [12] equal to $r = (r_1, \ldots, r_m)$, around $(0, 0)$; i.e., each output $y^j_{r_i+p}$ is influenced for any $p \geq 0$ by the corresponding input sequence $u^i_0 \cdots u^i_p$ only, with a delay of $r_i \geq 1$ time-steps and the decoupling matrix has full rank $m$ locally around the origin. In this discrete time context we will assume that the Jacobian $Jf_0(x)$ of $f_0(.) := f(., 0)$, has full rank around 0 so that the dynamics (1) is locally invertible in $(X_0, U_0)$, a neighborhood of $(0, 0)$. This last property will be recalled as $H1$ when necessary.

In the sequel whenever system (1-2) is recalled, it is implicitly assumed that the overmentioned properties of noninteraction, strong relative degree and drift invertibility hold true.

Setting the study in the geometric framework introduced in [12], we recall that, due to $H1$, the $m$ analytic functions $(iG^0(., u))$, $i = (1, \ldots, m)$, locally defined by the equality

$$iG^0(f(x, u), u) = \frac{\partial (f(x, u))}{\partial u^i}, \quad (3)$$

can be uniquely computed according to

$$iG^0(x, u) := \frac{\partial (f(x, u))}{\partial u^i} \bigg|_{x=f^{-1}(x, u)} := iG^0_1(x) + \sum_{s \geq 1} \sum_{i_1, \ldots, i_s \geq 1} \frac{u^{i_1} \cdots u^{i_s}}{s!} (i_{i_1 \cdots i_s} G^0_{s+1}(x)).$$

Hereafter, we will denote by $\eta := i_1 \cdots i_s$, any generic sequence of indices of length $s$, taken in $(1, \ldots, m)$; the concatenation of two sequences $\eta_i$ and $\eta_j$, will be denoted by $\eta := \eta_i \cdot \eta_j$; $I$ will denote the sequence with all indices equal to $i$.

Due to $H1$, we can also define the transport of any $i_0G^0_{s+1}(\cdot)$ along $f_0(\cdot)$ - or along its iterated composition : $f^p_0(\cdot) := f_0(\cdot) \circ \cdots \circ f_0(\cdot)$, $p \geq 1$ - by

$$i_0G^p_{s+1}(\cdot) := \left( Jf_0(\cdot) \ i_0G^{p-1}_{s+1}(\cdot) \right)_{f_0^{-1}(\cdot)} := Ad_{f_0} (i_0G^{p-1}_{s+1}(\cdot)) = Ad_{f_0}^p (i_0G^0_{s+1}(\cdot)),$$

where $Ad_{f_0} := Ad_{f_0} \circ \cdots \circ Ad_{f_0}$; $p$-times.

Analogously, the transport of any vector field $iG^0(., u_k)$ along $f(., u_{k+p}) \circ \cdots \circ f(., u_{k+1})$, can be iteratively defined: for any input sequence $(u_k, \ldots, u_{k+p}) \in U_0^{p+1}$, for $p \geq 0$, we set

$$iG^p(u_{k+1}, \ldots, u_{k+p})(\cdot, u_k) := Ad_{f(., u_{k+p})} \circ \cdots \circ Ad_{f(., u_{k+1})} iG^0(\cdot, u_k).$$

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**Lemma 2.1** The properties of noninteraction and strong relative degree $r = (r_1, \cdots, r_m)$ of system (1-2) are equivalent to the following set of equalities:

- $L_{ij}G^p(u_{k+1}, \cdots, u_{k+p})(h^i)|_x = 0$, $\forall p \geq 0, \forall j \neq i$  
- $\frac{\partial}{\partial u_k}L_{ij}G^p(u_{k+1}, \cdots, u_{k+p})(h^i)|_x = 0$, $\forall p \geq 0, \forall j \neq i$  
- $L_{ij}G^p(u_{k+1}, \cdots, u_{k+p})(h^i)|_x = 0$, $0 \leq p < r_i - 1$, for all $x \in X_0$ and $i = (1, \ldots, m)$. Equivalently
  - for any sequence $\eta$ of length $s \geq 1$, $L_{\eta G^s}(h^i)|_x = 0$, $\forall p < r_i - 1$
  - for any sequence $(\eta_1 \cdots \eta_r) \neq I$ of length $(s_1 + \cdots + s_r) \geq 1, r \geq 1$,
    $$L_{\eta_1 G^s_1} \cdots L_{\eta_r G^s_r}(h^i)|_x = 0, \quad k_j \geq 0, s_j \geq 1$$

and in addition
  - $L_{i G^{r_i-1}}(h^i)|_x = L_{i G^0}(h^i \circ f_0^{r_i-1})|_x \neq 0$.  

**Lemma 2.2** A distribution $\Delta$ is locally invariant under (1) iff

$$Ad_{f_0}(\Delta) \equiv \Delta$$

$$[\eta G^0, \Delta] \subset \Delta; \forall s \geq 1, \forall \eta = i_1 \cdots i_s, i_r \in [1, m].$$

Let $\Delta^i_0$ be the distribution generated by the vector fields $\{\eta G^0_s, s \geq 1\}$; i.e. $\Delta^i_0 := \{\eta G^0_s, s \geq 1\}$ and let $R^*_i$ and $R^*$ be

- $R^*_i := < f_0, \eta G^0_s, s \geq 1 | \Delta^i_0 >$, the smallest distribution invariant under (1) and containing $\Delta^i_0$;
- $R^* = \sum_{i=1}^{m} R^*_j$.
The following algorithm can be used to compute the distribution \( R^*_i \) starting from the distribution \( \Delta^i_0 \). Let, for \( k > 0 \)

\[
\Delta^i_k := \Delta^i_{k-1} + Ad_{f_0} \Delta^i_{k-1} + \sum_{j=1}^{m} \sum_{\eta} \left[ \eta \cdot G^0_s, \Delta^i_{k-1} \right],
\]

(6)

and let \( k^* \) be the first index such that \( \Delta^i_{k^*} = \Delta^i_{k^*+1} \); then, \( R^*_i \equiv \Delta^i_{k^*} \). By construction, \( R^*_i \) is involutive and its generic element \( \theta \) takes the form

\[
\theta = ad_{n_1} G^1_{s_1} \circ \cdots \circ ad_{n_r} G^r_s (Id), \quad k_1 \leq \cdots \leq k_r.
\]

It could be shown, as in [2], that \( R^*_i \) is in fact the controllability distribution associated with \( u^i \); it is invariant under any regular static-state feedback which preserves noninteraction, so that it characterizes the class of feedback-equivalent decoupled systems. In the sequel we will assume, without any loss of generality, that \( \dim \left( \sum_{i=1}^{m} R^*_i \right) = n \), i.e. the system is strongly accessible. Moreover the following technical assumption is needed for developing a geometric study:

**H2:** the distributions \( R^*_i \), \( \sum_{j \neq i} R^*_j \) and \( R^* \) are nonsingular locally around the origin, i.e. they have constant dimension in a suitable neighborhood of the origin.

Denoting by \( S^* \) the integral submanifold of \( R^* \) containing the origin and by

\[
x_{k+1} = f_0(x_k) \bigg|_{S^*}
\]

(7)

the restriction to \( S^* \) of the free dynamics, the local asymptotic stability of (7) is clearly a necessary condition for noninteraction with stability via regular static-state feedback.

The dynamics (7) - the \( R^* \)-dynamics - can be transformed according to the coordinates change \( z = \Phi(x) \) deduced from \( R^*_i \) by setting, for \( i = (1, \cdots, m) \),

\[
(\sum_{j \neq i} R^*_j)^\perp = \text{span}\{dz^i(x)\}, \quad \text{with } z^i \text{ of dimension } \rho_i.
\]

(8)

Accordingly, \( (R^*)^\perp \) has dimension \( \sum_{i=1}^{m} \rho_i \) and is given by

\[
(R^*)^\perp = \left( \bigcap_{i=1}^{m} \sum_{j \neq i} R^*_j \right)^\perp = \text{span}\{dz^1(x), \cdots, dz^m(x)\}.
\]
In these new coordinates, system (1–2) takes the form

\[ z_{k+1}^i = F^i(z_k^i, u_k^i) \quad i = (1, \cdots, m) \quad (9) \]

\[ z_{k+1}^{m+1} = F^{m+1}(z_k, u_k) \]

\[ y_k^i = h^i(z_k^i) \quad i = (1, \cdots, m). \quad (10) \]

Since \( S^* = \{ x \in U^m : z^1(x) = 0, \cdots, z^m(x) = 0 \} \), the dynamics (7) has dimension \( \rho_{m+1} = n - \sum_{i=1}^{m} \rho_i \) and can be rewritten as

\[ z_{k+1}^{m+1} = F^{m+1}(0, \cdots, 0, z_k^{m+1}, 0) = F_0^{m+1}(0, \cdots, 0, z_k^{m+1}). \quad (11) \]

The following result was proven in [2].

**Theorem 2.1** Consider the strongly accessible system (1–2) and assume that it satisfies \( H2 \). The problem of noninteracting control with stability via regular static-state feedback is solvable if and only if

i) the dynamics (11) is asymptotically stable at \( z^{m+1} = 0 \),

ii) the dynamics (9) are stabilizable by regular static state feedback at the origin.

### 3 Geometric structure and properties of decoupled systems

In this section we will consider dynamic feedback laws. Let us preliminary note that condition ii) can be straightforwardly weakened. As a matter of fact if ii) holds true under the action of dynamic state feedback laws, a sufficient condition for the existence of a dynamic solution is found.

The major aspect concerns the possibility of weakening condition i) under the action of dynamic compensations. To this end it is first necessary to characterize the geometric

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\[ \text{The result presented in [2] was given in terms of another set of distributions, the } P_i^* \text{'s defined for } i = (1, \cdots, m) \text{ as the smallest distribution invariant under the dynamics } F(\cdot, u) \text{ and containing the vector fields } (j\eta) G_0^0, \text{ with } j \neq i. \text{ Under } H2, \text{ the set of distributions } P_i^* \text{ and } R_i^* \text{ are equivalent since } P_i^* = \sum_{j \neq i} R_j^*. \]
structure of a decoupled system in terms of some suitable distributions. The analysis recalled in the previous section will be used to enlighten a partition of the \( R^* \)-dynamics, which is fundamental to understand the action of a dynamic feedback. The discussion developed in Section 3.2 with respect to some properties of the system under dynamic compensations, leads to the definition of the announced necessary and sufficient conditions in Section 4.

### 3.1 A geometric tool for dynamic noninteraction with stability

Let us denote by \( \Delta_{\text{mix}} \), the distribution, contained in \( R^* \), and obtained from \( R^* \) by eliminating all the terms which depend on one input-channel only, as they do not represent an obstruction for achieving stabilization while preserving noninteraction. More precisely, let

\[
\Delta_{\text{mix}} = \text{span}\{\tau: \tau \in L_{\text{mix}}\}
\]

where

\[
L_{\text{mix}} := \{\tau: \tau = \text{ad}_{\eta_1 G_{s_1}^k} \cdots \text{ad}_{\eta_r G_{s_r}^k}(Id), 0 \leq k_1 \leq \cdots \leq k_r; \ \eta_1 \cdots \eta_r \neq I\}.
\]

Generalizing \( H_2 \), we set

\[
H_2^*: \text{the distributions } R^*_i, \sum_{j \neq i} R^*_j, R^*, \Delta_{\text{mix}} \text{ and } R^*_i + \Delta_{\text{mix}}, \text{associated with system (1–2), are nonsingular locally around the origin.}
\]

The following results can be proven.

**Proposition 3.1** \( \Delta_{\text{mix}} \) is the smallest distribution invariant under the dynamics (1) and containing the distribution

\[
\Delta_0 := \text{span}\left\{\eta G^0_s(x), [\eta_1 G^0_{s_1}(x), \eta_2 G^p_{s_2}(x)], p \geq 0; \eta \neq I, \ \eta_1 \cdot \eta_2 \neq I, \ s \geq 2, \ s_1 \geq 1, \ s_2 \geq 1\right\}.
\]

**Proof.** Let \( \Delta :=< f_0, \eta G^0_r | \Delta_0 > \) be the smallest distribution invariant under (1) and containing \( \Delta_0 \), which can be computed according to (6), so that its generic term is

\[
\tau = \text{ad}_{\eta_1 G^0_{s_1}} \cdots \text{ad}_{\eta_r G^0_{s_r}}(Id), \quad 0 \leq k_1 \leq \cdots \leq k_r, \ s_j \geq 1, \ \left\{\begin{array}{ll}
\eta_r \neq I, & \text{if } r = 1 \\
\eta_{r-1} \cdot \eta_r \neq I, & \text{if } r > 1
\end{array}\right.
\]

By construction, since \( \eta_{r-1} \cdot \eta_r \neq I \), then \( \Delta \subset \Delta_{\text{mix}} \). On the other side, any \( \bar{\tau} \in \Delta_{\text{mix}} \) has the form (12) with \( \eta_1 \cdots \eta_r \neq I \). Using the Jacobi identity, any \( \bar{\tau} \in \Delta_{\text{mix}} \) can be expressed...
as sum of elements of the form \(ad_{\eta}G_{t_1}^i \cdots ad_{\eta}G_{t_r}^i(Id)\), where the sequence \(\eta_{r-1} \cdot \eta_r \neq I\) and \(t_r \geq t_{r-1}\). Unfortunately, the set of indices \(t_1 \cdots t_{r-1}\) is not necessarily ordered which would immediately imply that \(\tau \in \Delta\). Let \(l\) be the first index such that \(t_{r-l+1} \leq \cdots \leq t_{r-1}\), whereas \(t_{r-1} > t_{r-l+1}\). By construction, \(\theta = ad_{\eta}G_{t_{r-l+1}}^{i-l} \cdots ad_{\eta}G_{t_{r-1}}^{i-1}G_{t_r}^i(Id) \in \Delta\) and, due to the invariance property, also \(\hat{\theta} = Ad_f^{-\tau} \theta \in \Delta\). Since \(ad_{\eta}G_{t_{r-l+1}}^{i-l} \cdots ad_{\eta}G_{t_{r-1}}^{i-1}G_{t_r}^i(Id)\) is an element of \(\Delta\), i.e. \(\Delta_{mix} \subset \Delta\).

The proof of the corollary below, detailed in the Appendix, follows the same arguments as Proposition 3.1.

**Corollary 3.1** \(\Delta_{mix}\) is involutive and contained in \(R^*\).

The results stated for the distribution \(\Delta_{mix}\) can be also extended, with similar arguments, to the \(m\) distributions \(R_t^* + \Delta_{mix}\) thus leading to the next result.

**Lemma 3.1** Let \(R_t^* + \Delta_{mix}\) be nonsingular locally around the origin, then, \(R_t^* + \Delta_{mix}\) is involutive and invariant under the dynamics (1).

From this we deduce that, due to the involutivity of \(\Delta_{mix}\) and since \(\Delta_{mix} \subset R^*\), in the coordinates \(z = \Phi(x)\) defined by (8), it is possible to further partition \(z^{m+1} = \phi^{m+1}(x)\) as

\[
 dz^{m+1} = (dz^{(m+1)a})^T dz^{(m+1)b})^T
\]

with \(\Delta_{mix} = \frac{\partial}{\partial z^{(m+1)a}}\) of dimension \(\rho_{(m+1)a}\),

so that

\[
 z = \Phi(x) := ((\phi^1(x))^T \cdots (\phi^m(x))^T, (\phi^{(m+1)a}(x))^T, (\phi^{(m+1)b}(x))^T)^T
\]

and

\[
 \Delta_{mix}^\perp = \text{span}\{dz^1, \cdots, dz^m, dz^{(m+1)b}\} \supset R^*\perp. \quad (13)
\]

By construction, for any \(\theta \in \Delta_{mix}\), \(dz^{(m+1)b}_k \theta = 0\) and due to the invariance of \(\Delta_{mix}\) - i.e. \(Ad_f(x_k, u_k) \theta \in \Delta_{mix}\) - we have

\[
 dz^{(m+1)b}_k \theta = [\frac{\partial \phi^{(m+1)b}_k}{\partial x} \circ f(x_k, u) \theta]_{\phi^{-1}} = \left[\left(\frac{\partial \phi^{(m+1)b}_k}{\partial x} \circ Ad_f(x_k, u) \theta\right)\right]_{f(x_k, u)} \theta^{-1} = 0
\]
which shows that \( F^{(m+1)b}(z_k, u_k) := \phi^{(m+1)b} \circ f(x_k, u_k) \circ \Phi^{-1}(z_k) \) is independent of \( z_k^{(m+1)a} \).

In conclusion, system (1–2) reads, in suitable coordinates,

\[
\begin{align*}
  z_{k+1}^i &= F^i(z_k^i, u_k^i) \quad i = (1, \ldots, m) \\
  z_{k+1}^{(m+1)a} &= F^{(m+1)a}(z_k, u_k) \\
  z_{k+1}^{(m+1)b} &= F^{(m+1)b}(z_k^1, \ldots, z_k^m, z_k^{(m+1)b}, u_k) \\
  y_k^i &= h^i(z_k^i) \quad i = (1, \ldots, m).
\end{align*}
\]

(14) must be compared with (9); it enlightens a partition of the dynamics \( F^{(m+1)}(z_k, u_k) \).

### 3.2 The properties of the extended decoupled system

Let us now discuss the properties of the system under the action of a dynamic control law of the form

\[
\begin{align*}
  \zeta_{k+1} &= \delta(x_k, \zeta_k, v_k) \\
  u_k &= \gamma(x_k, \zeta_k, v_k)
\end{align*}
\]

where \( \zeta \in \mathbb{R}^\nu, v \in \mathbb{R}^m \), which preserves noninteraction, strong relative degree and drift invertibility on the closed-loop system. Such a feedback will be called a regular dynamic noninteraction state feedback.

Setting \( x^e = (x^T, \zeta^T)^T \), the whole system \( x_{k+1}^e = f^e(x_k^e, v_k) \) is

\[
\begin{align*}
  x_{k+1} &= f(x_k, \gamma(x_k, \zeta_k, v_k)) \\
  \zeta_{k+1} &= \delta(x_k, \zeta_k, v_k) \\
  y_k^i &= h^i(x_k) \quad i = (1, \ldots, m),
\end{align*}
\]

and is characterized by the drift \( f_0^e = \begin{pmatrix} f(x_k, \gamma(x_k, \zeta_k, 0)) \\ \delta(x_k, \zeta_k, 0) \end{pmatrix} \).

**Remark.** Due to the class of feedback considered, the Jacobian \( Jf_0^e(x^e) \) has full rank around the origin so that the dynamics (1) is locally invertible in \( (X_0^e, U_0) \), a neighborhood of \( (0, 0) \). This property is the equivalent of \( \textbf{H1} \) over the extended system and will be denoted in the sequel by \( \textbf{H1}^e \).
As in the previous section, the following technical assumption is necessary in this geometric setup:

**H2**\(_{\text{ec}}\): the distributions \(R_i^e\), \(\sum_{j\neq i} R_i^e\), \(R_i^e\), \(\Delta_{\text{mix}}^e\) and \(R_i^e + \Delta_{\text{mix}}^e\), associated with (17) are nonsingular locally around the origin.

The following properties hold true for the extended system (17):

**Property 3.1** The canonical vector field \(iG^e(\cdot, v_k)\), \(i = (1, \ldots, m)\) is well defined as well its iterated transport \(iG^p(v_{k+1}, \ldots, v_{k+p})(\cdot, v_k)\), along \(x_{k+p+1}^e := f^e(\cdot, v_{k+p}) \cdots f^e(x_k^e, v_k)\), \(p \geq 0\).

For \(i = (1, \ldots, m)\),

\[
iG^e(x^e, v) := \left(\sum_{s=1}^{m} sG^e(x, \gamma((f^e)^{-1}(x^e, v))) \frac{\partial \gamma^s(x^e, v)}{\partial v^s} \right)_{x^e}^{(f^e)^{-1}(x^e, v)}.
\]

Its transport along the composed dynamics \(x_{k+p+1}^e\), \(p \geq 0\), takes the form

\[
iG^p(v_{k+1}, \ldots, v_{k+p})(x_k^e, v_k) = \left(\sum_{l=0}^{p} \sum_{s=1}^{m} sG^l(\bar{u}_{k+p-l+1}, \ldots, \bar{u}_{k+p})(x_k, \bar{u}_{k+p}) \frac{\partial \gamma^s(x_{k+p-l}^e, \bar{u}_{k+p})}{\partial v^s_k} \right)_{x_{k+p+1}^e}.
\]

where \(x_{k+p+1}^{e-1}\) denotes the composed reverse dynamics \((f^e)^{-1}(\cdot, v_k) \circ \cdots \circ (f^e)^{-1}(x_k^e, v_k)\), so that for any \(0 \leq j \leq p\), \(x_{k+j}^e \circ x_{k+p+1}^{e-1} = (f^e)^{-1}(\cdot, v_{k+j}) \circ \cdots \circ (f^e)^{-1}(x_k^e, v_k)\), while \(\bar{u}_{k+j}\) denotes \(\bar{u}_{k+j} := \gamma(x_{k+j}^e \circ x_{k+p+1}^{e-1}, v_{k+j})\).

**Property 3.2** \((\text{proven in the Appendix})\) Given system (1-2), the regular dynamic noninter- action state feedback (15-16) satisfies the following conditions for all \(x^e \in X_0^e\) and \(i = (1, \ldots, m)\):

\[
\frac{\partial \gamma^i(x_{k+p}^e, v_{k+p})}{\partial v_k^i} = 0, \quad j \neq i, \quad \forall p \geq 0
\]

and

\[
\frac{\partial \gamma^i(x_{k+p}^e, v_{k+p})}{\partial v_k^i} = 0, \quad \forall p < \varrho_i
\]

\[
\frac{\partial \gamma^i(x_{k+p}^e, v_{k+p})}{\partial v_k^i} \neq 0, \quad \forall p = \varrho_i
\]

where \(\varrho_i = r_i^e - r_i\) denotes the strong relative degree of the feedback.
From Property 3.2 and (18), we have

**Property 3.3** The canonical vector field \( i \mathcal{G}^p(v_{k+1}, \ldots, v_{k+p})(\cdot, v_k) \), \( i = (1, \ldots, m) \), associated with the extended system (17), which is noninteractive, has strong relative degree and satisfies \( H1^e \), is given by

\[
i \mathcal{G}^p(v_{k+1}, \ldots, v_{k+p})(x_k, v_k) = \sum_{l=0}^{p-q_i} i G^l(\bar{u}_{k+p-l+1} \cdots \bar{u}_{k+p})(x_k, \bar{u}_{k+p-l}) \left. \frac{\partial \gamma^l(x_k^{p-l}, v_k^{p-l})}{\partial v_k^l} \right|_{x_k^{p-l}}
\]

where \( \bar{u}_{k+l} := \gamma(x_k^l \circ x_{k+p+1}^{e-l}, v_k) \).

Property 3.3 states that for \( i = (1, \ldots, m) \), the vector field \( i \mathcal{G}^p(v_{k+1}, \ldots, v_{k+p})(\cdot, v_k) \), associated with the \( i \)-th input on the extended system is characterized by the vector field \( i \mathcal{G}^s(u_{k+1}, \ldots, u_{k+s})(\cdot, u_k) \) associated with the same \( i \)-th input on the original system. Correspondingly there is a precise link between the distributions \( R_{e}^* \), \( i = (1, \ldots, m) \) and \( \Delta_{mix}^e \), defined on the extended system and the \( R_{i}^* \), \( i = (1, \ldots, m) \) and \( \Delta_{mix} \), defined on the original system, as enlightened hereafter.

**Property 3.4** The distribution \( \Delta_{mix} \) is invariant, in the projective sense, under any regular dynamic noninteraction state feedback.

This property is stated in the sequel as Lemma 3.2 and is proven in the Appendix. It will be used in the statement of the main result.

**Lemma 3.2** Under \( H2^e \), consider the regular dynamic noninteraction state feedback (15–16), and assume that \( \Delta_{mix}^e \) has constant dimension locally around the origin. Let \( \pi \) be the canonical projection \( \pi : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n : (z, \zeta) \to z \), then \( \Delta_{mix} \) and \( \Delta_{mix}^e \) are \( \pi \)-related, i.e. locally

\[
\pi_\ast \Delta_{mix}^e = \Delta_{mix} \circ \pi.
\]

Denoting by \( S_{\Delta_{mix}}^e \), the maximal integral submanifold of \( \Delta_{mix}^e \) containing the origin, \( \gamma(x^e, 0) = 0 \), for \( x^e \in S_{\Delta_{mix}}^e \). Moreover if \( R_{e}^* \) is regularly computable around the origin, then \( R_{i}^* + \Delta_{mix}^e \) and \( R_{i}^* + \Delta_{mix} \) are \( \pi \)-related, i.e. locally

\[
\pi_\ast (R_{e}^* + \Delta_{mix}^e) = (R_{i}^* + \Delta_{mix}) \circ \pi.
\]
Denoting by $S_{\Delta_{\text{mix}}}^\prime$, the maximal integral submanifold of $\Delta_{\text{mix}}$ containing the origin, the restriction of the dynamics $F_0(z_k)$ to its invariant submanifold $S_{\Delta_{\text{mix}}}^\prime$ is given by
\[
z_{k+1}^{(m+1)a} = F^{(m+1)a}(0, \ldots, 0, z_k^{(m+1)a}, 0) = F_0^{(m+1)a}(0, \ldots, 0, z_k^{(m+1)a}).
\] (25)

Lemma 3.3 Let the projection
\[
\hat{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-\nu_{m+1}} : (z^1 \ldots z^m, z^{(m+1)a}, z^{(m+1)b}) \to (z^1 \ldots z^m, z^{(m+1)b}),
\]
and denote by $\hat{F}(\cdot,u)$, $\hat{F}_0(\cdot)$, and for $i = (1, \ldots, m)$, $\hat{G}_i^0(\cdot,u)$ and $\hat{h}_i(\cdot)$, the $\hat{\pi}$-related functions respectively to $F(\cdot,u)$, $F_0(\cdot)$, $G_0(\cdot,u)$ and $h(\cdot)$, and by $\hat{\Delta}_{\text{mix}}$ and $\hat{R}_i^*$, the distributions associated with $\hat{F}(\cdot,u)$. Then, for all $x \in X_0$
\[
\hat{\pi}_* \Delta_{\text{mix}} = \hat{\Delta}_{\text{mix}} \circ \hat{\pi} = 0
\]
\[
\hat{\pi}_*(\hat{R}_i^* + \Delta_{\text{mix}}) = \hat{R}_i^* \circ \hat{\pi} = \hat{\pi} \big< \hat{F}_0, \eta \hat{G}_i^0 \big| \hat{\Delta}_0 \big>
\]
Moreover the $\hat{\pi}$-related system is strongly accessible, has the same strong vector relative degree $r$ as the original one, and the associated distributions $(\hat{R}_i^*, \sum_{j \neq i} \hat{R}_j^*, \hat{R}_i^*)$ satisfy $\textbf{H1}$ and $\textbf{H2}$.

The proof of Lemma 3.3 is omitted as it can be deduced by showing that $\hat{\pi}_* \eta \hat{G}_i^p = \eta \hat{G}_i^p(\hat{\pi})$, and iteratively that
\[
\hat{\pi}_*(ad_{\eta \hat{G}_i^{k_1}} \circ \cdots \circ ad_{\eta \hat{G}_i^{k_{r-1}}} \circ ad_{\eta \hat{G}_i^{k_r}}(Id)) = (ad_{\eta \hat{G}_i^{k_1}} \circ \cdots \circ ad_{\eta \hat{G}_i^{k_{r-1}}} \circ ad_{\eta \hat{G}_i^{k_r}}(Id)) \circ \hat{\pi}.
\]
With reference to the $\hat{\pi}$-related system $\hat{F}(\hat{z},u)$, let us consider the distributions $\hat{R}_i^*$, $i = 1, \ldots, m$. We can further partition $z^{(m+1)b} := (z^{(m+1)b_1}, z^{(m+1)b_2})^T$ with
\[
\hat{R}_i^* = \text{span}\{ \frac{\partial}{\partial z^{(m+1)b_1}}, \frac{\partial}{\partial z^{(m+1)b_2}} \}
\]
\[
\hat{R}_i^* = \text{span}\{ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^{(m+1)b_2}} \}
\]
Accordingly we can consider $i = (1, \ldots, m)$ the maximal integral submanifold $S_i$ of $\hat{R}_i^*$ containing the origin; the restriction of $\hat{F}(\hat{z},u)$ to its invariant manifold computed for $u^j = 0$
and \( j \neq i \), is given by
\[
\begin{align*}
z_{k+1}^i &= F^i(z_k^i, u_k^i) \\
z_{k+1}^{(m+1)bi_2} &= F^{(m+1)bi_2}(0, \ldots, z_k^i, \ldots, 0, 0, z_k^{(m+1)bi_2}, u_k^i) \\
y_k^i &= h^i(z_k^i)
\end{align*}
\]

(26)

4 Necessary and sufficient conditions

Theorem 4.1 Consider the strongly accessible system (1–2). Under \( H_2^* \) the problem of noninteracting control with stability is solvable via a regular dynamic noninteraction state feedback (15–16) if

\begin{itemize}
  \item[i)] (25) is locally asymptotically stable at \( z^{(m+1)a} = 0 \),
  \item[ii)] the dynamics (26), \( i = (1, \ldots, m) \), are stabilizable by regular dynamic-state feedback at the origin.
\end{itemize}

Conversely assume that \( H_2^{*e} \) holds true on the extended system, then conditions i) and ii) are also necessary.

Let us briefly discuss the assumptions which characterize the results given in Theorem 2.1 and Theorem 4.1.

Remark. Let us note that due to the class of feedback laws considered, the whole control system is drift invertible (\( H_1^e \)). This technical assumption is used to define the distribution \( \Delta^{e}_{mix} \) on the extended system; more general solutions may exist which do not require \( H_1^e \). Let us however note that if conditions i) and ii) are satisfied a feedback which maintains drift invertibility on the whole control system can be computed as shown in the sufficiency part of the proof.

Remark. Assumption \( H_2^* \) required in Theorem 4.1 is stronger than assumption \( H_2 \) considered for the static solution given in Theorem 2.1, since it requires also the nonsingularity

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of the distributions $\Delta_{mix}$ and $R_i^* + \Delta_{mix}$, $i = 1, \cdots, m$. As a matter of fact, if the $R^*$-dynamics is unstable, so that a static solution does not exist, the nonsingularity of these additional distributions allows to seek for a dynamic solution by checking the stability of the $\Delta_{mix}$-dynamics which is contained in the $R^*$-dynamics, as well as the stabilizability of the $(R_i^* + \Delta_{mix})$-dynamics. In this sense the result stated in Theorem 4.1 is less restrictive than the result stated in Theorem 2.1, since condition i) in Theorem 2.1 is stronger than condition i) in Theorem 4.1.

The proof of Theorem 4.1 is developed in the next sections where the structure of the feedback is also given. An example shows the technical aspects.

4.1 The necessity

The necessity of i) is an immediate consequence of Lemma 3.2 which states that, for any initial state $z_0^e \in S_{\Delta_{mix}}^{<e}$, the first $n$ components of the state trajectory $z_k$ at time $k$, coincide with the components of the state trajectory $z_k$ at time $k$, starting from $\pi(z_0^e)$. Due to the invariance of $\Delta_{mix}$, necessarily, the restriction of $F_0^e(z_k, \zeta_k)$ to $S_{\Delta_{mix}}^{<e} + R_i^*$ is locally asymptotically stable only if the restriction of $F_0(z_k)$ to $S_{\Delta_{mix}}^*$, given by (25), is locally asymptotically stable too.

As for ii), $H2^{<e}$ implies the involutivity and invariance under $F^e(\cdot, v)$ of the concerned distributions. Thus we can consider the restriction of $F_0^e(\cdot)$ to its invariant manifold $S_{\Delta_{mix}}^{<e} + R_i^*$ which in the $z$-coordinates is given by

\[
\begin{align*}
    z_{k+1}^i &= F^i(z_k^i, \tilde{u}) \\
    z_{k+1}^{(m+1)a} &= F^{(m+1)a}(0, \cdots, z_k^i, \cdots, z_k^{(m+1)a}, 0, z_k^{(m+1)bi_2}, \tilde{u}) \\
    z_{k+1}^{(m+1)bi_2} &= F^{(m+1)bi_2}(0, \cdots, z_k^i, \cdots, 0, 0, z_k^{(m+1)bi_2}, \tilde{u}) \\
    \zeta_{k+1}^i &= \tilde{\delta}^i(z_k^i, z_k^{(m+1)a}, z_k^{(m+1)bi_2}, \zeta_k, 0)
\end{align*}
\]

where $\tilde{u} = [\tilde{u}^1, \cdots, \tilde{u}^m]^T$ with $\tilde{u}^j = 0$ for $j \neq i$ and $\tilde{u}_k^i = \tilde{\gamma}_i^i(z_k^i, z_k^{(m+1)a}, z_k^{(m+1)bi_2}, \zeta_k, 0)$. Due to the invariance projection property expressed in Lemma 3.2, reasoning as before, the above dynamics must be asymptotically stable, or equivalently, setting $\chi_k^i = z_k^{(m+1)a}$, the regular
dynamic noninteraction state feedback

\[ u^i_k = \tilde{\gamma}^i(z^i_k, \chi^i_k, z^{(m+1)\text{bi}2}_k, \zeta^i_k, 0) \]
\[ \chi^i_{k+1} = F^{(m+1)a}(0, \ldots, z^i_k, \ldots, \chi^i_k, z^{(m+1)\text{bi}2}_k, \tilde{u}) \]
\[ \zeta^i_{k+1} = \tilde{\delta}^i(z^i_k, \chi^i_k, z^{(m+1)\text{bi}2}_k, \zeta^i_k, 0) \]

must stabilize the subdynamics (26) thus proving the result.

### 4.2 The sufficiency

Let us preliminary point out the main features of the sufficiency part of the proof which is constructive starting from the reduced system:

\[ z^i_{k+1} = F^i(z^i_k, u^i_k) \quad i = (1, \ldots, m) \]
\[ z^{(m+1)\text{bi}}_{k+1} = F^{(m+1)\text{bi}}(z^1_k, \ldots, z^m_k, z^{(m+1)\text{bi}}_k, u_k) \] (27)
\[ y^i_k = h^i(z^i_k) \quad i = (1, \ldots, m). \]

System (27) is obtained from (14) through the canonical projection

\[ \tilde{\pi} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-p_{m+1}} : (z^1 \ldots z^m, z^{(m+1)a}, z^{(m+1)b}) \rightarrow (z^1 \ldots z^m, z^{(m+1)b}), \]

and is characterized, according to Lemma 3.3, by a zero dimensional $\Delta_{\text{mix}}$ dynamics.

This simplification, performed in the continuous-time case too, [1], is justified by the fact that, assuming i), and since $\Delta_{\text{mix}}$ is invariant under any regular dynamic noninteraction state feedback (Lemma 3.2) and it is contained in ker $dh$ (since $\Delta_{\text{mix}} \subset R^*$), then a regular dynamic noninteraction state feedback solving the problem for (27), solves the problem for (14) also.

The feedback control law is composed by a suitable extension of system (27) together with a stabilizing controller of the resulting system which exists because of ii). Its computation can be carried out in two different steps:

**Step 1** A dynamic extension is added on the $\tilde{\pi}$-related dynamics (27) in order to obtain a zero dimensional $R^*$–dynamics. The cited dynamic extension, which preserves the noninteraction properties of the original system as well as drift invertibility, is completely
defined by (27) and has the form

\[
\begin{align*}
\lambda_{k+1}^i &= F^i(z_k^i, u_k^i) + (z_k^i - \lambda_k^i) + w_k^{i1} \\
\mu_{k+1}^i &= F^{(m+1)b}(0, \ldots, 0, z_k^i, 0 \cdots 0, \mu_k^i, u_k^i) + w_k^{i2}
\end{align*}
\]

where \([(\lambda^{i}), (\mu^{i})]^T \in \mathbb{R}^{\rho_i} \times \mathbb{R}^{\rho_{(m+1)b}}\) and \(w^i := ((w^{i1})^T, (w^{i2})^T) \in \mathbb{R}^{\rho_i} \times \mathbb{R}^{\rho_{(m+1)b}}\) can be considered as fictitious new input variables. More precisely

- the dynamics \(\lambda_{k+1}^i\), of dimension \(\rho_i\), is essentially a copy of the dynamics associated with \(z_k^i\) plus two additive terms: \(z_k^i - \lambda_k^i\) which guarantees the invertibility of the drift and \(w_k^{i1}\) which behaves like a new input;

- the dynamics \(\mu_{k+1}^i\), of dimension \(\rho_{(m+1)b}\), is derived from the dynamics associated with \(z_k^{(m+1)b}\) by setting \(z_k^{(m+1)b} = \mu_k^i\) and \(z_k^j = 0\) for \(j \neq i\), plus an additive term \(w_k^{i2}\) which behaves like a new input.

\textbf{Step 2} This step deals with the stabilization problem. Assuming first \(w_k = 0\) an appropriate coordinates change linked to the controllability distributions of the extended system is derived. In these coordinates the extended system is split into \((m + 1)\) parts: it is first shown that one of these parts is unaccessible with respect to the real input \(u_k\) when \(w_k = 0\), while it can be stabilized by appropriately choosing the fictitious input vector \(w_k\); note that this choice does not affect the noninteractive properties of the extended system since the fictitious input vector \(w_k\) affects the dynamic extension only; the remaining \(m\) parts are each linked to one of the \(m\) subsystems (26) in such a way to inherit their stabilizability properties. As a consequence for each of these \(m\) parts it is possible to compute a stabilizing controller

\[
\begin{align*}
u_k^i &= \gamma^i(z_k, \lambda_k, \mu_k, \psi_k, v_k) \\
\psi_{k+1}^i &= \chi^i(z_k, \lambda_k, \mu_k, \psi_k^i, v_k)
\end{align*}
\]

which also preserves noninteraction.

In conclusion, the regular dynamic noninteraction state feedback (15-16) which solves the
problem is given in the $z$-coordinates by (28-29), that is

$$
\begin{align*}
\lambda^i_{k+1} &= F^i(z^i_k, u^i_k) + (z^i_k - \lambda^i_k) + w^{i1}_k \\
\mu^i_{k+1} &= F^{(m+1)b}(0, \ldots, 0, z^i_k, 0 \cdots 0, \mu^i_k, u^i_k) + w^{i2}_k \\
u^i_k &= \gamma^i(z_k, \lambda_k, \mu_k, \psi^i_k, v_k) \\
\psi^i_{k+1} &= \chi^i(z_k, \lambda_k, \mu_k, \psi^i_k, v_k)
\end{align*}
$$

with $w_k$ computed as pointed out in Step 2.

Let us now go inside the technical details. Consider the extended system (27–28), of dimension $2 \sum_{i} r_i + (m + 1) \rho (m+1)_b$, which is described by the equations

$$
\begin{align*}
z^i_{k+1} &= F^i(z^i_k, u^i_k), \quad i = (1, \ldots, m) \\
z^{(m+1)b}_{k+1} &= F^{(m+1)b}(z^1_k, \ldots, z^m_k, z^{(m+1)b}_k, u_k) \\
\lambda^i_{k+1} &= F^i(z^i_k, u^i_k) + (z^i_k - \lambda^i_k) + w^{i1}_k, \quad i = (1, \ldots, m) \\
\mu^i_{k+1} &= F^{(m+1)b}(0, \ldots, 0, z^i_k, 0 \cdots 0, \mu^i_k, u^i_k) + w^{i2}_k, \quad i = (1, \ldots, m) \\
y^i_k &= h^i(z^i_k), \quad i = (1, \ldots, m),
\end{align*}
$$

and consider, for $i = (1, \ldots, m)$, the distributions $\hat{R}^{e*}_i$ associated with (30) when $w_k = 0$.

By construction $\hat{R}^{e*}_i$ (Lemma A.1 in the Appendix) has the same dimension $s_i$ as $\hat{R}^*_i$, it is involutive and invariant under (30). Moreover, $\hat{R}^{e*}_i \subset \bigcap_{j \neq i} \ker \{ dh^j \}$ and $\hat{R}^{e*}_i \cap \sum_{j \neq i} \hat{R}^{e*}_j = 0$. We can then consider the coordinates change $((\xi^1)^T, \ldots, (\xi^m)^T, (\xi^{m+1})^T)^T$, defined by

$$
\left( \sum_{i=1}^{m} \hat{R}^{e*}_i \right)^\perp = \text{span} \{ d\xi^{m+1} \}, \quad \left( \sum_{j \neq i} \hat{R}^{e*}_j \right)^\perp = \text{span} \{ d\xi^i, d\xi^{m+1} \}, \quad i = (1, \ldots, m).
$$

Accordingly $\hat{R}^{e*}_i = \text{span} \{ \frac{\partial}{\partial \xi^i} \}$, and system (30) is transformed into

$$
\begin{align*}
\xi^i_{k+1} &= \varphi^i(\xi^i_k, \xi^{m+1}_k, w_k) + \partial^i(\xi_k, u_k, w_k)w_k \quad i = (1, \ldots, m) \\
\xi^{m+1}_{k+1} &= \varphi^{m+1}(\xi^{m+1}_k) + \partial^{m+1}(\xi_k, u_k, w_k)w_k \\
y^i_k &= h^i(\xi^i_k, \xi^{m+1}_k), \quad i = (1, \ldots, m),
\end{align*}
$$

where $w := (w^1, \ldots, w^m)^T$, dim$(\xi^i) = s_i$ for $i = (1, \cdots, m)$, and dim$(\xi^{m+1}) = s_{m+1}$.
In these new coordinates the extended system is thus split into \((m + 1)\) parts: the \(\xi_{k+1}^{m+1}\)–dynamics which is by construction unaccessible with respect to the real input \(u_k\) when \(w_k = 0\), and for \(i = (1, \cdots, m)\) the \(\xi_k^i\)–dynamics linked to the input \(u_k^i\) and the output \(y_k^i\).

As it will be shown hereafter:

a) the \(\xi_{k+1}^{m+1}\)–dynamics can be stabilized by appropriately choosing the fictitious input vector \(w_k\); note that this choice does not affect the noninteractive properties of the extended system since the fictitious input vector \(w_k\) affects the dynamic extension only;

b) the remaining \(m\) parts, are each related to one of the \(m\) subsystems (26) in such a way that the stabilizability condition ii) of (26) implies the stabilizability of the \(\xi_k^i\)–dynamics.

In order to prove a), let us first note that since \(\vartheta_{m+1}\) has full rank \(s_{m+1} = \dim(\xi^{m+1})\) around the equilibrium point, it is possible to choose \(w_k = \beta(\xi_k, u_k, \tilde{w}_k)\), with \(\beta(\xi_k, u_k, 0) = 0\), such that

\[
\vartheta_{m+1}(\xi_k, u_k, \beta(\xi_k, u_k, \tilde{w}_k)) = K \tilde{w}_k,
\]

with \(K = (I \ 0)\) and \(I\) is an identity matrix of dimension \(s_{m+1} \times s_{m+1}\). Accordingly (32) is transformed into

\[
\begin{align*}
\xi_{k+1}^i &= \varphi_i(\xi_k^i, \xi_{k+1}^{m+1}, u_k^i) + \vartheta_i(\xi_k, u_k, \tilde{w}_k) \tilde{w}_k, \quad i = (1, \cdots, m) \\
\xi_{k+1}^{m+1} &= \varphi_{m+1}(\xi_{k+1}^{m+1}) + K \tilde{w}_k \\
y_k^i &= h_i(\xi_k^i, \xi_{k+1}^{m+1}), \quad i = (1, \cdots, m).
\end{align*}
\]

To this end, denote by \(\Omega\) the subspace generated by the columns of \(E\); the result follows by simply noting that by construction \(\dim(\ker(\vartheta_{m+1}(0))) = \sum_{i=1}^{m} s_i - n\).

---

2Let us first rewrite the dynamics (30) as

\[
x_k^{i+1} = F^e(x^e, u, w) = F^e(x^e, u, 0) + Ew, \quad E = (0, I)^T
\]

is a \((n + \nu) \times \nu\) dimensional matrix and \(I\) is the identity matrix of dimension \(\nu \times \nu\).

Then \(\vartheta_{m+1}(0) = \frac{\partial \xi_{k+1}^{m+1}}{\partial w} \bigg|_{\xi_k = 0} = d\xi_{k+1}^{m+1}(0)E\), so that it has \(\nu\) columns and \(n + \nu - \sum_{i=1}^{m} s_i \leq \nu\) rows. Since \(\text{rank}(\vartheta_{m+1}(0)) = \nu - \dim(\ker(\vartheta_{m+1}(0)))\), it is sufficient to show that \(\dim(\ker(\vartheta_{m+1}(0))) = \sum_{i=1}^{m} s_i - n\). To this end, denote by \(\Omega\) the subspace generated by the columns of \(E\); the result follows by simply noting that by construction \(\dim(\ker(\vartheta_{m+1}(0))) = \dim(\sum_{i=1}^{m} \tilde{R}_i^e \cap \Omega) = \sum_{i=1}^{m} s_i - n\).
Choosing now \( \hat{w}_k = K_{m+1} \xi_k^{m+1} \), we can stabilize the \( \xi^{m+1} \)-dynamics still preserving the invertibility of the drift of the extended system (33). We thus have

\[
\begin{align*}
\xi_{k+1}^i &= \varphi_i(\xi_k^i, \xi_k^{m+1}, u_k^i) + \hat{\vartheta}_i(\xi_k, u_k) \xi_k^{m+1}, \quad i = (1, \ldots, m) \\
\xi_{k+1}^{m+1} &= \varphi_{m+1}(\xi_k^{m+1}) + \vec{K} \xi_k^{m+1} \\
y_k^i &= h^i(\xi_k^i, \xi_k^{m+1}), \quad i = (1, \ldots, m).
\end{align*}
\]

where \( \hat{\vartheta}_i(\xi_k, u_k) = \vartheta_i(\xi_k, u_k, K_{m+1} \xi_k^{m+1})K_{m+1} \) and \( \vec{K} = K_{m+1} \).

In order to prove b) let us consider \( S_1^e \), the integral manifold of \( \hat{R}_k^e \) containing the origin. In the new coordinates (31), \( S_1^e \) is the invariant manifold of (34) with \( \xi_{m+1}^i = 0 \) and \( \xi^j = 0, u^j \neq 0 \) for \( j \neq i \). The restriction of (34) to its invariant manifold \( S_1^e \) is then

\[ \xi_{k+1}^i = \varphi_i(\xi_k^i, 0, u_k^i) \]  

which, due to ii), is stabilizable via a regular dynamic noninteraction state feedback of the form

\[
\begin{align*}
u_k^i &= \hat{\gamma}_i(\xi_k^i, \psi_k^i, v_k^i) \\
\psi_{k+1}^i &= \hat{\chi}_i(\xi_k^i, \psi_k^i, v_k^i)
\end{align*}
\]

since the dynamics (35) is locally diffeomorphic to the dynamics (26). In fact, denoting by \( z^e = ((z^1)^T, \ldots, (z^m)^T, (z^{(m+1)b})^T, (\lambda^1)^T, (\mu^1)^T, \ldots, (\lambda^m)^T, (\mu^m)^T)^T, z^e \in \mathbb{R}^{n+\nu} \), consider the submanifold \( \mathcal{M}_i \) of \( \mathbb{R}^{n+\nu} \) given by

\[ \mathcal{M}_i = \left\{ z^e : (z^1, z^2, \ldots, z^{(m+1)b}) \in S_i, \lambda^i = z^i, \mu^i = z^{(m+1)b}, \lambda^j = \mu^j = 0, j \neq i \right\}. \]

Since \( S_i \) is invariant under \( F_0(\cdot) \) and \( G^0(\cdot, w^i) \) associated with the dynamics (27) when \( u_j = 0 \) for \( j \neq i \), then \( \mathcal{M}_i \) is invariant under \( F_0^e(\cdot) \) and \( G^{0e}(\cdot, u^i) \) associated with the extended dynamics (30) when \( w = 0 \) and \( u_j = 0 \) for \( j \neq i \). Moreover, \( \mathcal{M}_i \) has dimension \( s_i = \dim(\hat{R}_k^e) \) and therefore is the maximal integral manifold of \( \hat{R}_k^e \) containing the origin, and coincides with \( S_1^e \). The diffeomorphism

\[ \phi : S_i^e \to S_i^e, \quad (z^1, \ldots, z^{(m+1)b}) \mapsto (z^1, \ldots, z^{(m+1)b}, 0, 0, \ldots, z^i, z^{(m+1)b}, 0, 0 \ldots) \]
carries trajectories of (26) into trajectories of (35), so that the stabilizability of (26) implies the stabilizability of (35). These arguments complete the proof since we have shown how to stabilize the dynamics (34) and preserve noninteraction.

4.3 An example

Consider the nonlinear decoupled system

\[
\begin{align*}
    x_1(k+1) &= x_1(k) + u_1(k) \\
    x_2(k+1) &= x_2(k) + u_2(k) \\
    x_3(k+1) &= 2x_3(k) + a_1(x_1(k)) + a_2(x_2(k)) \\
    y_i(k) &= x_i(k) \quad i = 1, 2
\end{align*}
\]

and assume that for \( i = 1, 2, a_i(0) = 0, \frac{\partial a_i(x_i)}{\partial x_i} \bigg|_{x=0} \neq 0, \) and the two single–input systems

\[
S_1: \begin{cases}
    x_1(k+1) = x_1(k) + u_1(k) \\
    \bar{x}_3(k+1) = 2\bar{x}_3(k) + a_1(x_1(k))
\end{cases}
\quad S_2: \begin{cases}
    x_2(k+1) = x_2(k) + u_2(k) \\
    \bar{x}_3(k+1) = 2\bar{x}_3(k) + a_2(x_2(k))
\end{cases}
\]

are stabilizable respectively with the regular static feedback laws

\[
u_1 = \alpha_1(x_1, \bar{x}_3), \quad \text{and} \quad u_2 = \alpha_2(x_2, \bar{x}_3)
\]

The drift term \( f_0(x) \) and the vector fields \( iG^0_i(\cdot, u), i=(1,2), \) are respectively

\[
f_0(x) = \begin{pmatrix} x_1 \\ x_2 \\ 2x_3 + a_1(x_1) + a_2(x_2) \end{pmatrix}, \quad iG^0_1(\cdot, u) = 1G^0_1(\cdot) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad iG^0_2(\cdot, u) = 2G^0_2(\cdot) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Since \( \frac{\partial a_i(x_i)}{\partial x_i} \bigg|_{x=0} \neq 0 \) for \( i = 1, 2, \) the transport of the vector fields \( iG^0_1 \) and \( 2G^0_1 \) result to be

\[
iG^1_1(\cdot, u) = \begin{pmatrix} 1 \\ 0 \\ \tau_1(x_1, u_1) \end{pmatrix}, \quad iG^1_2(\cdot, u) = \begin{pmatrix} 0 \\ 1 \\ \tau_2(x_2, u_2) \end{pmatrix}, \quad \tau_1(0,0) \neq 0, \quad \tau_2(0,0) \neq 0.
\]

Let \( \tau^0_i(x_i) = \tau_i(x_i, 0), i = 1, 2. \) The distributions \( R^*_1, R^*_2 \) and \( R^* \) are then given by

\[
R^*_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \tau^0_1(x_1) \end{pmatrix} \right\}, \quad R^*_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \tau^0_2(x_2) \end{pmatrix} \right\}, \quad R^* = R^*_1 \cap R^*_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}
\]
Consequently \((R_1^*)^\perp = (dx_2), (R_2^*)^\perp = (dx_1), (R^*)^\perp = (dx_1, dx_2)\), and (37) is not stabilizable via a regular static state feedback preserving noninteraction since \(x_3(k+1) = 2x_3(k)\) is unstable. However \(\Delta_{mix} = 0\) and by assumption the two subsystems in (38) are stabilizable, in particular by the regular static feedback laws (39). Moreover the given dynamics is strongly accessible, has strong relative degree with respect to the output functions and satisfies \(H_2^*\). The conditions of Theorem 4.1 are thus satisfied and we can compute a controller which solves the problem.

According to the procedure illustrated in Section 4, let us first consider the regular dynamic feedback (28) which is given by

\[
\begin{align*}
\lambda_1(k+1) &= x_1(k) + u_1(k) + x_1(k) - \lambda_1(k) + w^{11}(k) \\
\mu_1(k+1) &= 2\mu_1(k) + a_1(x_1(k)) + w^{12}(k) \\
\lambda_2(k+1) &= x_2(k) + u_2(k) + x_2(k) - \lambda_2(k) + w^{21}(k) \\
\mu_2(k+1) &= 2\mu_2(k) + a_2(x_2(k)) + w^{22}(k)
\end{align*}
\]

For the extended system (37–40), set \(w = 0\) and compute the associated distributions \(R_1^{e*}\), \(R_2^{e*}\) and \(R^{e*}\) which are given by

\[
R_1^{e*} = \left\{
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}, \quad R_2^{e*} = \left\{
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}, \quad R^{e*} = \left\{
\begin{pmatrix} 0 \\ 0 \\ \tau_1^0(x_1) \\ 1 \\ 1 \\ \tau_2^0(x_2) \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}\right.\right.\]

Set \((R_1^{e*} + R_2^{e*})^\perp = (d\xi_3)\), \((R_1^{e*})^\perp = (d\xi_2, d\xi_3)\), \((R_2^{e*})^\perp = (d\xi_1, d\xi_3)\) By construction

\[
\begin{align*}
d\xi_3 &= (d\lambda_1 - dx_1, d\lambda_2 - dx_2, dx_3 - d\mu_1 - d\mu_2) \\
d\xi_2 &= (d\lambda_2, d\mu_2) \\
d\xi_1 &= (d\lambda_1, d\mu_1)
\end{align*}
\]

In the new coordinates, one obtains

\[
\xi_{11}(k+1) = \xi_{11}(k) - 2\xi_{31}(k) + u_1(k) + w^{11}(k)
\]
Due to the stability of (42), \( \xi_3 \to 0 \) and the two subsystems \( S3 \) and \( S4 \) in (45) reduce respectively to \( S1 \) and \( S2 \) in (38). As a consequence noninteraction with stability is achieved with the feedback \( u_1 = \alpha_1(\lambda_1, \mu_1) \) and \( u_2 = \alpha_2(\lambda_2, \mu_2) \), obtained from (39) by setting \( x_1 = \lambda_1 \), \( \tilde{x}_3 = \mu_1 \), \( x_2 = \lambda_2 \) and \( \hat{x}_3 = \mu_2 \).

In conclusion, substituting (43) into (40), we get the expression of the dynamic compensator solving the problem which is

\[
\begin{align*}
\lambda_1(k+1) &= \frac{3}{2}x_1(k) - \frac{1}{2}\lambda_1(k) + u_1(k) \\
\mu_1(k+1) &= \frac{1}{2}\mu_1(k) + a_1(x_1(k)) + \frac{3}{2}(x_3(k) - \mu_2(k))
\end{align*}
\]
\[
\lambda_2(k + 1) = \frac{3}{2}x_2(k) - \frac{1}{2}\lambda_2(k) + u_2(k)
\]
\[
\mu_2(k + 1) = 2\mu_2(k) + a_2(x_2(k))
\]
\[
u_1(k) = \alpha_1(\lambda_1, \mu_1), \quad u_2(k) = \alpha_2(\lambda_2, \mu_2)
\]

**Appendix**

**Proof of Corollary 3.1.** As for the involutivity of \(\Delta_{mix}\), let us consider two elements \(\tau_1\) and \(\tau_2\) of \(\Delta_{mix}\). Using the same arguments as those of Proposition 3.1, the Lie bracket \([\tau_1, \tau_2]\) can be written as a sum of elements of the form \(ad_{\eta_1 G_{t_1}^{r_1}} \cdots ad_{\eta_r G_{t_r}^{r_r}}(\tau_2)\). By recalling that \(ad_{\eta G_{t}^{r}}(\tau_2) = Ad_{f_{t}^{r}}\left(ad_{\eta G_{0}^{r}}(Ad_{f_{t}^{r}}\tau_2)\right)\), using the invariance property iteratively one proves that \([\tau_1, \tau_2]\) is an element of \(\Delta_{mix}\).

We must now show that \(\Delta_{mix} \subset R^*\). As stated above, any term of \(\Delta_{mix}\) can be supposed of the form \(\tau = ad_{\eta_1 G_{s_1}^{r_1}} \cdots ad_{\eta_{r-1} G_{s_{r-1}}^{r_{r-1}}} (Id)\), with the sequence \(\eta_{r-1} \cdots \eta_1 \neq I\) and \(k_1 \leq \cdots \leq k_r\). In order to prove the result it is sufficient to show that \(\tau\) is also an element of \(\sum_{j \neq i} R_j^*\) for \(i = (1, \cdots, m)\). Three cases are possible:

a) \(\eta_r = j \bar{\eta}\) with \(j \neq i\); then \(\eta_r G_{s_r r}^{k_r} \in \sum_{j \neq i} R_j^*\) and, due to the invariance property, such is \(\tau\);

b) \(\eta_r = i \bar{\eta}\) with \(\bar{\eta} \neq I\); then, without any loss of generality, we can assume \(\bar{\eta} = j \hat{\eta}\) with \(j \neq i\). In this case, because by construction
\[
\frac{\partial}{\partial u_i} G_0(\cdot, u) = \frac{\partial}{\partial u_j} G_0(\cdot, u) + [j G_0(\cdot, u), \eta G_0(\cdot, u)],
\]
we have in general
\[
ij \hat{\eta} G_{s_r r}^{k_r} = i j \hat{\eta} G_{s_r}^{k_r} + \sum [\eta G_{t_{t_1}}^{k_{t_1}}(\cdot), \eta G_{t_{t_2}}^{k_{t_2}}(\cdot)] \in \sum_{j \neq i} R_j^*.
\]

Again, due to the invariance property, this relation shows that \(\tau \in \sum_{j \neq i} R_j^*\), being the sum of elements in \(\sum_{j \neq i} R_j^*\);

c) \(\eta_r = I\), \(\eta_{r-1} \neq I\) and \(k_{r-1} < k_r\); this case can be reduced to situations a) or b) by noting that, due to the invertibility of the drift term,
\[
\eta G_{s}^{0}(x) = \alpha_1(x) \eta G_{s}^{t+1}(x) + \cdots + \alpha_n(x) \eta G_{s}^{t+n}(x)
\]

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where $t \geq 0$ is an arbitrary integer. Consequently

$$ad_{\eta_{i-1}G_{s_{i-1}}^{k_{i-1}}} ad_{\eta_{i}G_{s_{i}}^{k_{i}}} (Id) = -[\eta_{i} G_{s_{i}}^{k_{i}}, \eta_{i-1} G_{s_{i-1}}^{k_{i-1}}] = -[\eta_{i} G_{s_{i}}^{k_{i}}, \sum_{t=1}^{n} \alpha_{t} \eta_{i-1} G_{s_{i-1}}^{k_{i+t}}]$$

$$= - \sum_{t=1}^{n} \alpha_{t} [\eta_{i} G_{s_{i}}^{k_{i}}, \eta_{i-1} G_{s_{i-1}}^{k_{i+t}}] - \sum_{t=1}^{n} (L_{\eta_{i} G_{s_{i}}^{k_{i}}} \alpha_{t}) \eta_{i-1} G_{s_{i-1}}^{k_{i+t}} \in \sum_{j \neq i} R_{s}^{*}$$

thus proving that $\Delta_{mix} \subset R^{*}$. \hfill \Box

**Proof of Property 3.2.** Let $r_{i}^{e}$ be the relative degree of the output $y^{i}$ with respect to the extended system (17), then by definition $r_{i}^{e} \geq r_{i}$ since

$$h^{i} \circ x_{k+s}^{i} = h^{i} \circ x_{k+s}^{i} = h^{i} \circ f_{0}^{s}(x_{k}), \quad s \leq r_{i} - 1.$$ Moreover since the two systems (1-2) and (17) are decoupled, from Lemma 2.1

$$L_{jG^{p}(u_{k+1} \cdots u_{k+p})(v_{e}, u_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x)) = 0 \quad \forall u \in U^{p+1}, \quad \forall j \neq i, \quad \forall p \geq 0$$

and

$$L_{jG^{p}(v_{k+1} \cdots v_{k+p})(v_{e}, v_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x)) = 0 \quad \forall v \in V^{p+1}, \quad \forall j \neq i, \quad \forall p \geq 0.$$ As a consequence from (18) we get

$$L_{jG^{r_{i}}(x_{k}^{e}, v_{k})}(h^{i})(x) = L_{jG^{0}(x_{k}^{e}, v_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x))$$

$$= \frac{\partial \gamma_{j}^{i}(x_{k}^{e}, v_{k})}{\partial v_{k}^{i}} \bigg|_{x_{k}^{e}=x_{k+s}^{i}} L_{iG^{0}(x_{k}^{e}, v_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x)) = 0.$$ Since by assumption of strong relative degree $r_{i}$, $L_{iG^{0}(x_{k}^{e}, v_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x)) \neq 0$, then (19) must be satisfied for $p = 0$. Iteratively, suppose that it is satisfied for $p = (0, \cdots s - 1)$, then for any $j \neq i$ and $p = s$, we get

$$L_{jG^{s}(u_{k+1} \cdots u_{k+s})(v_{e}, v_{k})}(h^{i} \circ f_{0}^{r_{i}-1}(x)) =$$

$$= \sum_{l=0}^{s} \frac{\partial \gamma_{j}^{i}(x_{k+s+l}^{e}, v_{k+s+l})}{\partial v_{k}^{i}} \bigg|_{x_{k+s+l}^{e}} L_{iG^{0}(u_{k+s+l} \cdots u_{k+s})(v_{k+s+l})}(h^{i} \circ f_{0}^{r_{i}-1}(x))$$

$$= \frac{\partial \gamma_{j}^{i}(x_{k+s}^{e}, v_{k+s})}{\partial v_{k}^{i}} \bigg|_{x_{k+s}^{e}} L_{iG^{0}(x_{k+s}^{e}, v_{k+s})}(h^{i} \circ f_{0}^{r_{i}-1}(x))$$

so that necessarily $\frac{\partial \gamma_{j}^{i}(x_{k+s+p}^{e}, v_{k+s+p})}{\partial v_{k}^{i}} = 0$ for $j \neq i$ which recursively proves (19) for any $p \geq 0$. 25
Moreover, since by assumption the \(i\)-th output \(y^i\) has strong relative degree \(r^e_i\) with respect to the extended system, then from Definition 2.1,

\[
L_{iG^e(v_{k+1}, \ldots, v_{k+s})}(x^e, v_k)(h^i \circ f_0^{-r_i-1}(x)) = 0, \quad \forall x^e, \ \forall s < r^e_i - r_i = \varrho_i
\]

\[
L_{iG^e(v^e_{k+1}, \ldots, v^e_{k+s})}(x^e) \neq 0, \quad \forall x^e \in \mathcal{X}^e_0
\]

which using the same arguments as above iteratively proves that \(\frac{\partial \gamma^i(x^e_{k+s}, v_{k+s})}{\partial v^i_k} = 0\) for \(s < \varrho_i\) and \(\frac{\partial \gamma^i(x^e_{k+s + \rho}, v_{k+s + \rho})}{\partial v^i_k} \bigg|_{v = 0} \neq 0\), i.e. (20) and (21).

**Proof of Lemma 3.2.** In the sequel we will first prove that \(\pi_s \Delta^e_{mix} \subset \Delta^e_{mix} \circ \pi\). To this end let us preliminary note that the generic element \(_i\eta G_s^e(\cdot)\) has the form

\[
_i\eta G_s^e(\cdot) = \left(\sum \alpha_j \tau_j \right) + \frac{Y}{\text{term in ker } \pi_s}, \quad \text{where } \begin{cases} \tau_j \in R^*_i & \text{if } \eta = \mathcal{I} \\ \tau_j \in \Delta^e_{mix} & \text{if } \eta \neq \mathcal{I} \end{cases}
\]

and the coefficients \(\alpha_j\) are appropriate functions of the feedback \(u = \gamma(x^e, v)\) and satisfy the condition \(L_{x^e} \alpha_j = 0\) for any \(\tau^e_{mix} \in \Delta^e_{mix}^3\).

In fact from Property 3.3 the canonical vector field \(_iG^e(v_{k+1}, \ldots, v_{k+p})(\cdot, v_k)\) of the decoupled dynamics (17) has the "decoupled" form (22). It immediately follows that for \(p < \varrho_i\), \(_i\eta G_s^e(\cdot)\) \(\in \ker \pi_s\).

Let us now set in (22) \(p = \varrho_i\) and \(v_{k+1} = \ldots = v_{k+p} = 0\). We have

\[
_iG^0(\cdot, v_k) := Ad_{f_0}^{\varrho_i} G^0(\cdot, v_k) = \left(\frac{\partial \gamma^i(x^e_{k+\varrho}, v_{k+\varrho})}{\partial v^i_k} \bigg|_{x^e_{k+\varrho \eta_i+1}} \right)_{x^e_{k+\varrho \eta_i+1}=0}
\]

where \(\bar{u}_{k+\varrho} = \gamma(f^{e-1}(\cdot, v_{k+\varrho}), v_{k+\varrho})\), and from the definition of strong relative degree

\[
\frac{\partial \gamma^i(x^e_{k+\varrho}, v_{k+\varrho})}{\partial v^i_k} \bigg|_{v = 0} = \gamma_i(0) \neq 0.
\]

As a consequence the generic term \(_i\mathcal{I}_s G_s^e(x^e_k)\), with \(\mathcal{I}_s := i \cdots i\) of length \(s\), can be obtained by considering the successive derivatives of \(_iG^e(\cdot, v_k)\) with respect to \(i\), i.e.

\[
_i\mathcal{I}_s G_s^e(\cdot) := Ad_{f_0}^{\varrho_i} \bigg( \mathcal{I}_s G_s^e(\cdot) = \frac{\partial^{s-1}}{\partial (v^i_k)^{s-1}} \bigg|_{v^i_k=0} \bigg)_{x^e_k=0} \bigg)_{x^e_k=0}
\]

\[\text{This last statement is a direct consequence of Corollary 3.1 and a result in [2] which states that for any}\]

\[\tau \in R^*, \text{ a noninteractive feedback } u = \gamma(\cdot, v) \text{ satisfies the condition } L_{\tau} \gamma(\cdot, v) = 0.\]
According to (49), and by recalling that the feedback $u = \gamma(\cdot, v)$ is noninteractive, so that for any $j \neq i$, \( \frac{\partial}{\partial v_k^j} \left( \frac{\partial \gamma^i(x_{k+p}^j, v_k)}{\partial v_k^j} \right) \bigg|_{x_{k+1}^i} = 0 \) and \( \frac{\partial \gamma^j(f_{e}^{-1}(\cdot, v_{k}^j), v_k)}{\partial v_k^i} = 0 \), we get
\[
\mathcal{I}_s \mathcal{G}_s^0(\tau_k^j) = \left( \sum_{l=1}^{s} \mathcal{I}_s \mathcal{G}_s^0(x_k, \bar{u}_k^0) \alpha_l^0(\tau_k^j) \right) = \left( \sum_{l=1}^{s} \mathcal{I}_s \mathcal{G}_s^0(x_k, \bar{u}_k^0) \alpha_l^0(\tau_k^j) \right) + \text{term in } \pi_s
\]
where \( \bar{u}_k^0 = \gamma(f_{e}^{-1}, 0) \). The coefficients \( \alpha_l^0(\tau_k^j) \) are defined by the \( i \)-th component of the feedback \( u^i = \gamma^i(\cdot, v^i) \), so that \( \frac{\partial \alpha_l^0(\tau_k^p)}{\partial v_k^j} = 0 \) for any \( j \neq i \). As a consequence for any \( \tau_k^e \in \Delta^{e}_{\text{mix}} \), \( L_{\tau_k^e} \alpha_l^0(\cdot) = 0 \).

The same arguments can be used to compute from (22) the generic term \( \mathcal{I}_s \mathcal{G}_s^p \) with \( p \geq g_i \). One gets
\[
\mathcal{I}_s \mathcal{G}_s^p(\cdot) = \left( \sum_{l=1}^{s} \mathcal{I}_s \mathcal{G}^p(x_k, \bar{u}_k^0) \alpha_l^0(\tau_k^j) \right)
\]
where \( \bar{u}_k^0 = \gamma(f_{e}^{-p-1}, 0) \) and the coefficients \( \alpha_l^0 \) satisfy the condition \( L_{\tau_k^e} \alpha_l^0(\cdot) = 0 \), for any \( \tau_k^e \in \Delta^{e}_{\text{mix}} \), which proves (48) in the case \( \eta = \mathcal{I} \).

Let us now consider the case \( \eta \neq \mathcal{I} \). For any \( j \neq i \), \( i_j \mathcal{G}_s^0(\cdot, v_k) = \frac{\partial}{\partial v_k^j} \mathcal{G}_s^0(\cdot, v_k) = \left( \frac{\partial \mathcal{G}_s^0(x_k, \bar{u}_k)}{\partial v_k^j} \bigg|_{(x_{k+1}^i)} \right) \frac{\partial \gamma^j(f_{e}^{-1}(\cdot, v_k))}{\partial v_k^i} \right) \).

In order to compute the expression of the generic element \( i_\eta \mathcal{G}_s^0(\cdot) \) when \( \eta \neq \mathcal{I} \), let us first recall that \( i_\eta \mathcal{G}_s^0(\cdot) = i_\eta \mathcal{G}_s^0(\cdot) \), whenever \( \bar{\eta} \) is obtained by permutation of the indices of \( \eta \). As a consequence we can reorder the sequence \( \eta \) as follows. Rewrite \( i \cdot \eta = i \cdot \eta_0 \cdot 1_{s_1} \cdots m_{s_m} \), where \( i \cdot \eta_0 \) of length \( s_0 \), is the greatest sequence of indices contained in \( \eta \) without repetitions, and \( \mathcal{J}_{s_j} : j \cdots j \) of length \( s_j \).

Due to the structure of \( i_\eta \mathcal{G}_s^0(\cdot, v_k) \) it immediately follows that
\[
i_\eta \mathcal{G}_s^0(\cdot) = \left( \sum_{l=1}^{s} \cdots \sum_{m=1}^{s} \mathcal{G}_s^0(x_k, \gamma_0(F_{e}^{-1})) \alpha_{l_1}^{0} \cdots m_{l_m+1}^{0} \right)
\]
and more generally, for any \( p \geq 0 \),
\[
i_\eta \mathcal{G}_s^p(\cdot) = \left( \sum_{l=1}^{s} \cdots \sum_{m=1}^{s} \mathcal{G}_s^p(x_k, \gamma_0(F_{e}^{-p})) \alpha_{l_1}^{p} \cdots m_{l_m+1}^{p} \right)
\]
As before, the coefficients \( j_\mathcal{G}_s^{p+1} \) for \( j \in [1, m] \), \( p \geq 0 \), depend on the \( j \)-th input so that for any \( \tau_k^e \in \Delta^{e}_{\text{mix}} \), \( L_{\tau_k^e} j_\mathcal{G}_s^{p+1} \) = 0. This proves (48) in the case \( \eta \neq \mathcal{I} \).
Let us now consider a generic element $\tau_{mix}^e \in \Delta_{mix}^e$ which without any loss of generality can be assumed of the form $\tau_{mix}^e : \text{ad}_{\eta_1 G_{e_{k_1}}^e} \circ \cdots \circ \text{ad}_{\eta_r G_{e_{k_r}}^e} (Id)$, with $k_1 \leq \cdots \leq k_r$ and $\eta_{r-1} \cdot \eta_r \neq \mathcal{I}$. We will show that $\tau_{mix}^e$, can be expressed as the sum of two terms, one in ker $\pi_s$, the other of the form $(\tau_{mix}^e, 0)^T$ with $\tau_{mix} \in \Delta_{mix}$. As a consequence the projection of $\Delta_{mix}^e$ is contained in $\Delta_{mix}$. In order to show this result, assume for instance $\eta_r \neq \mathcal{I}$. Then according to (48) and the properties of the coefficient $\alpha_j$, standard computations show that

$$\text{ad}_{\eta_{r-1} G_{e_{k_{r-1}}}^e} \circ \text{ad}_{\eta_r G_{e_{k_r}}^e} (Id) = \left[ \left( \sum \alpha_j \tau_j \right)_0, \eta_r G_{e_{k_r}}^e \right] + \text{terms in ker } \pi_s + \left( \tau_{mix}^e \right)_0$$

and since $[\tau_j, \tau_{mix}^e] \in \Delta_{mix}$ we can immediately conclude that

$$\text{ad}_{\eta_{r-1} G_{e_{k_{r-1}}}^e} \circ \text{ad}_{\eta_r G_{e_{k_r}}^e} (Id) = \left( \tau_{mix}^e \right)_0 + \text{terms in ker } \pi_s$$

Iterating the reasoning one immediately gets that

$$\text{ad}_{\eta_1 G_{e_{k_1}}^e} \circ \cdots \circ \text{ad}_{\eta_r G_{e_{k_r}}^e} (Id) = \left( \tau_{mix}^e \right)_0 + \text{terms in ker } \pi_s$$

Accordingly $\pi_s \Delta_{mix}^e \subset \Delta_{mix} \circ \pi$.

In order to complete the proof we must show that any element of $\Delta_{mix}$ can be obtained by projection of an element of $\Delta_{mix}^e$. This can be proven through technical manipulations of the exponential representation introduced in [12] and by using similar arguments as above. Hereafter we give the general lines.

Let $\tilde{\tau}_i \geq 1$ and $j_i$ be the smallest indices which satisfy the following relation

$$\exists j_i \geq 0 : \forall t < j_i, \left. \frac{\partial^j \gamma^i(x_{k+l}^{e_{k+l}}(t), v_k)}{\partial(v_{k+l})^j} \right|_{v=0} = 0, \quad \forall j \geq 0, \ l = 0, \cdots, \tilde{\tau}_i - 1$$

(50)

For $i = 1, \cdots, m$ the pair $(\tilde{\tau}_i, j_i)$ certainly exist. Let us first assume that $\tilde{\tau}_i \geq 1$. Then according to Property (3.2), $\left. \frac{\partial \gamma_i(x)}{\partial x} \right|_{x=0} G_{i_1}^{e_{i_1}} \neq 0$. Due to the invertibility of the drift we have that there

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exists an index \( s : \bar{\theta}_i \leq s \leq n^e - 1 \) (\( n^e \) being the dimension of the extended system) such that \( iG^e_1, \ldots, iG^e_{s_1} \) are linearly independent and \( iG^e_{s_1+1} \in \text{span}\{iG^e_1, \ldots, iG^e_{s_1}\} \). As a consequence there exists an index \( t \geq 0 \) such that \( iG^e_{s_1}(-t) := Ad_{F_0^{-1}}^* iG^e_1 \in \text{span}\{iG^e_{s_1}, \ldots, iG^e_{s_2}\} \) and \( iG^e_{s_1}(-t) \notin \text{span}\{iG^e_{s_1}, \ldots, iG^e_{s_2-1}\} \) which guarantees that at least \( \frac{\partial \gamma^i(x^{e,-1}_{k+1}, v_k)}{\partial v_k} \bigg|_{v=0} \neq 0 \), i.e. \( \bar{\theta}_i \leq t \).

Instead if \( \bar{\theta}_i = 0 \) then either \( \bar{\theta}_i = 1 \) or we have that since

\[
\frac{\partial \gamma^i(\bar{\theta}_i x^{e,-1}_k)}{\partial v_k} \bigg|_{v=0} = \left( \frac{\partial \gamma^i(\cdot, v_k)}{\partial v_k} \bigg|_{v=0} - \frac{\partial \gamma^i_0}{\partial x} G^e_1(-1) \right) F_0^{-1} = 0,
\]

so that \( \frac{\partial \gamma^i}{\partial x} G^e_1(-1) = \frac{\partial \gamma^i_0}{\partial v_k} \bigg|_{v=0} \neq 0 \), which can be treated as before.

Let us now recall that from (22) that for \( p \geq \bar{\theta}_i \)

\[
iG^e_{s_1} \bigg( x^e_k \bigg) = \left( \sum_{l=0}^{p-\bar{\theta}_i} iG^e_l \bigg( \bar{u}_{k+p-l+1}^0, \ldots, \bar{u}_{k+p}^0 \bigg) \bigg( x^e_k, \bar{v}_{k+p+1}^0 \bigg) \right) \text{span} \{ \alpha^{p-\bar{\theta}_i-1} \}
\]

where \( \bar{u}_{k+j}^0 := \gamma(f_0^{-(p+1-j)}), 0 \) and the coefficients \( i\alpha^l(x^e_k) \) are linked to the \( i \)-th component of the feedback \( \gamma(\cdot, v) \) and satisfy the condition \( L_s G^e_{s_1} \alpha^l = 0, \forall j \neq i \) and \( \forall s \geq 0 \).

We will first show how to generate by projection of terms of \( \Delta^e_{\text{mix}} \), terms of the form \( i_s G^e_{s+1}(\cdot) \), \( s \neq i, \forall j \geq 0 \). To this end let us preliminarily note that, due to the invertibility of the drift, for any fixed \( t, \forall j \geq 0 \), \( i_s G^e_{s+1}(\cdot) \in \text{span}\{i_s G^e_{s}(\cdot), \ldots, i_s G^e_{s+n-1}(\cdot)\} \).

Let now \( (\bar{\theta}_s, j_s) \) be the pair satisfying (50) with reference to the \( s \)-th component of the feedback \( u = \gamma(\cdot, v) \). Then for \( s \neq i \) and \( p = \bar{\theta}_i + \bar{\theta}_s \), we have that

\[
\frac{\partial j_s}{\partial (v_{k+\bar{\theta}_i+\bar{\theta}_s})} iG^e(\bar{v}_i+\bar{v}_s)(v_{k+1}, \ldots, v_{k+\bar{\theta}_i+\bar{\theta}_s})(x^e_k, v_k) \bigg|_{v=0} = \left( i_s G^e_{s}(x_k) \beta_1(\cdot) \bigg|_{v=0} \right) + \left( \sum_{i} \tau_i^{\bar{\theta}_i} \beta_2(\cdot) \bigg|_{v=0} \right) + \left( 0 \bigg|_{Y} \right)
\]

where, the coefficients \( \beta_1(\cdot) \) and \( \beta_2(\cdot) \) satisfy the conditions \( \beta_1(0) \neq 0 \) and \( \beta_2(0) = 0 \).

Consequently the projection of \( \frac{\partial j_s}{\partial (v_{k+\bar{\theta}_i+\bar{\theta}_s})} iG^e(\bar{v}_i+\bar{v}_s)(v_{k+1}, \ldots, v_{k+\bar{\theta}_i+\bar{\theta}_s})(x^e_k, v_k) \bigg|_{v=0} \) which is an element of \( \Delta^e_{\text{mix}} \) generates \( i_s G^e_{s+1}(x_k) \) since

\[
\pi_s \frac{\partial j_s}{\partial (v_{k+\bar{\theta}_i+\bar{\theta}_s})} iG^e(\bar{v}_i+\bar{v}_s)(v_{k+1}, \ldots, v_{k+\bar{\theta}_i+\bar{\theta}_s})(x^e_k, v_k) \bigg|_{v=0} = \left( i_s G^e_{s}(x) \beta_1(0, 0) + \sum_i \beta_2(x, 0) \tau_i^{\bar{\theta}_i} \right)
\]

with \( \beta_1(0, 0) \neq 0 \) and \( \beta_2(0, 0) = 0 \).
Consider now the term \( Ad_{f_0}(\frac{\partial s}{\partial v^i_{k+\theta_1+\theta_2}})G^e(\theta^i+\theta_1)(v_{k+1}, \cdots v_{k+\theta_1+\theta_2})(x^e_k, v_k) \big|_{v=0} \). According to Property 3.2, and equation (51) we have

\[
Ad_{f_0} \left( \frac{\partial s}{\partial v^i_{k+\theta_1+\theta_2}} \right) \mid_{v=\gamma_0(\cdot)} = \left( \frac{\partial f(\cdot, u)}{\partial x^c} \right)_{s=\gamma_0(\cdot)}^{\gamma_0(\cdot)}(v_{k+1}, \cdots v_{k+\theta_1+\theta_2})(x^e_k, v_k) \big|_{v=0}
\]

which generates by projection \( i_sG_2^{\theta_1+1} \). Iteratively we can deduce \( i_sG_2^{\theta_1+1} \) through the projection of \( Ad_{f_0} \left( \frac{\partial s}{\partial v^i_{k+\theta_1+\theta_2}} \right) \mid_{v=\gamma_0(\cdot)} = \left( \frac{\partial f(\cdot, u)}{\partial x^c} \right)_{s=\gamma_0(\cdot)}^{\gamma_0(\cdot)}(v_{k+1}, \cdots v_{k+\theta_1+\theta_2})(x^e_k, v_k) \big|_{v=0} \). According to the discussion above, it is sufficient to consider \( l = 0, \cdots, n - 1 \).

We can now generate \([i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)]\), \( i \neq s \) by projecting \([i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)]\). In fact, according to (22), we have that

\[
[i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)] = \left[ \left( i G^0(x^e_k, u^0_{k+\theta_1}), \alpha^0(x^e_k) \right), \left( s G^0(x^e_k, u^0_{k+\theta_1}), \alpha^0(x^e_k) \right) \right]
\]

Let us now denote

\[
\frac{\partial (s G^0(x^e_k, u^0_{k+\theta_1}), \alpha^0(x^e_k))}{\partial x^e_k} = s \alpha^0(x^e_k) \frac{\partial s G^0(x^e_k, u^0_{k+\theta_1})}{\partial x^e_k} + \frac{\partial (s G^0(x^e_k, u^0_{k+\theta_1}), \alpha^0(x^e_k))}{\partial x^e_k} \frac{\partial s G^0(x^e_k, u^0_{k+\theta_1})}{\partial x^e_k}
\]

where with \( w^i_{\theta_1} \), \( \bar{u}^0_{k+\theta_1} \), we have denoted the \( i \)-th component of \( w^i_{\theta_1} \), \( \bar{u}^0_{k+\theta_1} \), respectively.

Consequently we have that

\[
[i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)]
\]

so that

\[
\pi_*([i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)]) = s \alpha^0(x^e_k, 0) \alpha^0(x^e_k, 0)[i G_1^0(x^e_k), s G_1^0(x^e_k)]
\]

so that

\[
\pi_*([i G_1^{\theta_1}(x^e_k), s G_1^{\theta_1}(x^e_k)]) = s \alpha^0(x^e_k, 0) \alpha^0(x^e_k, 0)[i G_1^0(x^e_k), s G_1^0(x^e_k)]
\]

\[
+ \beta_1(x^e_k, 0) s G_2^0(x^e_k) + \beta_2(x^e_k, 0) i_s G_2^0(x^e_k) + \sum l \beta^l(x^e_k) r^l_{mix}
\]

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with \( \alpha^0(0,0) \neq 0 \), \( s \alpha^0(0,0) \neq 0 \) while \( \beta^i(0) = 0 \).

It is only a matter of computation to verify that the projection of \( [i G_1^{\theta_i} (x_k^e), s G_1^{\theta_s+j} (x_k^e)] \), \( i \neq s \), \( j \geq 1 \), iteratively generates \( [i G_1^0 (x_k), s G_1^j (x_k)] \). In fact we have that

\[
\pi_*([i G_1^{\theta_i} (x_k^e), s G_1^{\theta_s+j} (x_k^e)]) = \sum_{l=0}^{j} \alpha^0(x_k,0) s \alpha^{l-j}(x_k,0) [i G_1^0 (x_k), s G_1^j (x_k)]
\]

\[
+ \sum_{l=0}^{j} \beta_1(x_k,0) s i G_2^j (x_k) \cdot \beta_2(x_k,0) s i G_2^0 (x_k) + \sum_l \beta^l(x_k) \tau^l_{\text{mix}}
\]

which proves the result since \( i \alpha^0(0,0) \neq 0 \), \( s \alpha^0(0,0) \neq 0 \) while \( \beta^i(0) = 0 \).

Similar arguments can be used to generate iteratively terms of the form \( \eta_i G^j_{s+1} \), with \( \eta = i_1 \cdots i_s \neq I_s \) and \( \bar{\eta}_i \geq \cdots \geq \bar{\eta}_s \) by projection of terms of the form

\[
\left. \frac{\partial^{i_1}}{\partial(v_k^{i_1} + \bar{\eta}_1)} \cdots \frac{\partial^{i_s}}{\partial(v_k^{i_s} + \bar{\eta}_s)} G(x^e)_{\eta_1} \right|_{v=0}
\]

as well as terms of the form \( \left[ \eta_1 G^k_{s_1} \cdots \eta_{r-1} G^k_{s_{r-1}} \eta_r G^k_{s_r} \right] \) with \( \eta_{r-1} \eta_r \neq I \) through the repeated Lie brackets of terms of the form (52).

\[
\pi_* (R^* _i + \Delta^e_{\text{mix}}) = (R^* _i + \Delta^e_{\text{mix}}) \circ \pi \text{ can be proved by using the same arguments as above.}
\]

Finally the fact that \( \gamma(x^e,0) = 0 \) for any \( x^e \in S_{\Delta^e_{\text{mix}}} \) is a direct consequence of Property 3.3 and the fact that \( \Delta^e_{\text{mix}} \subset R^* \). \( \square \)

**Lemma A.1** Consider the extended system (27–28) and let \( \bar{R}^*_i \) be the associated distribution when \( w_k = 0 \), \( i = (1, \cdots, m) \). Locally around the origin, \( \bar{R}^*_i \) has the same dimension \( s_i \) as \( \bar{R}^*_i ; \) therefore, it is involutive and invariant under (27–28). Moreover, \( \bar{R}^*_i \subset \bigcap_{j \neq i} \ker \{ dh^j \} \) and \( \bar{R}^*_i \cap \sum_{j \neq i} \bar{R}^*_j = 0 \).

**Proof of Lemma A.1.** Since \( \Delta^e_{\text{mix}} = 0 \) the generic element \( \tau_i(z) \in \bar{R}^*_i \) has the form

\[
\tau_i(z) = \begin{pmatrix}
0 \\
\tau^i(z^i) \\
0 \\
\tau^{(m+1)i}(z)
\end{pmatrix}
\]

\( \left\langle \text{of dimension } \sum_{j=1}^{i-1} \rho_j \right\rangle \left\langle \text{of dimension } \frac{m}{i+1} \rho_i \right\rangle \left\langle \text{of dimension } \sum_{j=i+1}^{m} \rho_j \right\rangle \left\langle \text{of dimension } \rho_{(m+1)i} \right\rangle \).
The generic elements $\tau^e_i(z) \in \hat{R}^e_i$ is defined by $\tau_i(z) \in R^*_i$ given above as follows:

$$\tau^e_i(z) = \begin{pmatrix}
\tau_i(z) \\
0 \\
\tau^e_i(z') \\
\tau^{(m+1)bs}_i(z^i, \mu^i)
\end{pmatrix} \leftarrow \begin{pmatrix}
\tau_i(z) \\
0 \\
\tau^e_i(z') \\
\tau^{(m+1)bs}_i(z^i, \mu^i)
\end{pmatrix}$$

of dimension $\sum_{j=1}^{i-1} \rho_j + (i-1)\rho_{(m+1)b}$

of dimension $\rho_i$

of dimension $\rho_{(m+1)b}$

of dimension $\sum_{j=i+1}^{m} \rho_j + (m-i-1)\rho_{(m+1)b}$

where we have denoted by $\tau^{(m+1)bs}_i(z^i, \mu^i) := \tau^{(m+1)bs}_i(z) |_{z^j=0, j \in [1,m], j \neq i, \mu^i}$.

Moreover $\hat{R}^e_i$ is spanned by $s_i$ vectors $\theta_1 \cdots \theta_{s_i}$ of the form of $\tau_i$, it is involutive and invariant under $\hat{F}_0$ and $\hat{G}_0^0_s$, therefore

$$Ad_{\hat{F}_0} \theta_l = \sum_{l=1}^{s_i} \alpha_l \theta_l, \quad \text{and} \quad [\hat{G}_0^0_s, \theta_l] = \sum_{l=1}^{s_i} \beta_l \theta_l$$

If $\sigma \in \sum_{j \neq i} \hat{R}^e_j$, since $\Delta_{mix} = 0$, it follows that $[\sigma, Ad_{\hat{F}_0} \theta_l]$ and $[\sigma, [\hat{G}_0^0_s, \theta_l]]$ are equal to zero. More precisely

$$[\sigma, Ad_{\hat{F}_0} \theta_l] = [\sigma, \sum_{l=1}^{s_i} \alpha_l \theta_l] = \sum_{l=1}^{s_i} \theta_l \frac{\partial \alpha_l}{\partial z} \sigma = 0, \quad [\sigma, [\hat{G}_0^0_s, \theta_l]] = [\sigma, \sum_{l=1}^{s_i} \beta_l \theta_l] = \sum_{l=1}^{s_i} \theta_l \frac{\partial \beta_l}{\partial z} \sigma = 0$$

Since the vector fields $\theta_l$ are independent, it follows that $\alpha_l$ and $\beta_l$ depend only on $z^i$. Using this property and by considering the form of the vector fields generating $\hat{R}^e_i$ it is easily seen that $\hat{R}^e_i$ has dimension $s_i$, is invariant with respect to $F^e(z^e, u)$, and is involutive. Moreover $\hat{R}^e_i \cap \sum_{j \neq i} \hat{R}^e_j = 0$ and $\hat{R}^e_i \subset \bigcap_{j \neq i} \ker \{dh^j \}$.

References


