Robust Nonlinear Attitude Stabilization of a Spacecraft through Digital Implementation of an Immersion & Invariance Stabilizer

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Abstract: The paper deals with the problem of robust nonlinear attitude stabilization of a rigid spacecraft. In particular, an Immersion and Invariance robust attitude stabilizer is proposed, taking into account actuator dynamics in control design. The proposed continuous-time controller is then implemented under sampling using an approximated single-rate strategy to match, at the sampling instants, the zero-going evolution of the off-the-manifold coordinates. Simulations show the effectiveness of the proposed controller.

Keywords: Robust nonlinear control, Immersion and Invariance, Sampled-data control.

1. INTRODUCTION

In this paper, a nonlinear control strategy to stabilize the attitude of a rigid spacecraft robustly with respect to actuators dynamics is proposed. The control law is applied under sampling using a single-rate digital approach with first order corrector term, which overcomes in performance the direct implementation through zero-order hold, as shown by simulations. A robust attitude stabilizer is necessary for long range communications satellites, especially when a high throughput is involved. The capability of the spacecraft to maintain a fixed orientation despite external disturbances and maintaining uncertainties is crucial when dealing with satellite internet access at high data speeds (Pietrabissa and Fiaschetti (2012)).

The technique of Immersion & Invariance (I&I), first introduced by Astolfi and Ortega (2003), is a tool for the stabilization of nonlinear systems via state-feedback. The existence of a reduced globally asymptotically stable dynamics, called target dynamics, which can be immersed into the system to be controlled, plus the invariance and attractiveness of the corresponding manifold, together with the boundedness of the trajectories of an extended system are sufficient conditions for the GAS of a chosen equilibrium of the controlled system. The method is specifically suitable for systems that admit a fast-slow dynamics decomposition, e.g. singularly perturbed systems, systems in feedback form but also underactuated systems requiring non-standard control solutions (Astolfi et al. (2008)). Applications to spacecraft have been proposed mostly in the adaptive context in presence of flexible dynamics, as in Lee and Singh (2009). In the sampled-data context, a multi-rate implementation of an I&I stabilizer is introduced for a class of feedback systems in Mattei et al. (2014). Nonlinear sampled-data controllers are also developed to solve the attitude tracking problem in Di Gennaro et al. (1999) and in the flexible case in Monaco et al. (1986).

In this work we propose an I&I solution for systems in strict-feedback form which is particularly suited to counteract the degrading effect of unmodeled actuator dynamics on the overall control systems. In fact, I&I can be regarded as a tool to robustify a given nonlinear controller with respect to higher-order dynamics, exploiting at its best the knowledge of such dynamics during the control design phase. Thus, this approach can be considered “robust” nonlinear control. The obtained continuous-time controller is then implemented under sampling using a single-rate control strategy with truncation of series expansions at the second order in the sampling period. Simulations at increasing sampling times show the effectiveness of using a first order corrector term with respect to the simpler implementation through zero-order hold device (emulated control Nesic et al. (1999)). In particular, the maximum allowable sampling period (MASP) is increased, thus the sampled-data controller shows robustness w.r.t. δ, sampling time.

Notations All the functions, maps and vector fields are assumed smooth and the associated dynamics forward complete. Given a vector field \( f \), \( L_f \) denotes the associated Lie derivative operator,

\[
L_f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i},
\]

\( e^f \) denotes the associated Lie series operator,

\[
e^f := 1 + \sum_{i \geq 1} \frac{L_f^i}{i}.
\]

For any smooth real valued function \( h \), the following result holds \( e^f h(x) = e^f h|_x = h(e^f x) \) where \( e^f \) stands for \( e^f/L_f(x) \) with \( L_f \) the identity function on \( \mathbb{R}^n \) and \( (\cdot) \) (or equivalently \( \chi \)) denotes the evaluation at a point \( x \) of a generic map. The evaluation of a function at time \( t = k\delta \), \( \delta \) sampling period, indicated by \( _{t=k\delta} \) is omitted, when it is obvious from the context. The notation \( O(\delta^p) \) indicates that the absolute value of the approximation error in the series expansions is bounded from above by a linear function of \( \delta^p \), for \( \delta \) small enough. With col\((x_1, x_2, \ldots, x_p)\) we denote the column vector of components \( x_1, x_2, \ldots, x_p \). A class \( \mathfrak{C} \) function is a continuous function \( w(\cdot) : [0,a) \rightarrow [0,\infty) \), which is strictly increasing and
such that \( w(0) = 0 \). A class \( \mathcal{K}_\infty \) function is a class \( \mathcal{K} \) function with \( a = \infty \) and such that \( \lim_{r \to \infty} w(r) = \infty \). Throughout the paper, the gradient of a scalar function is considered a row vector.

2. SPACECRAFT DYNAMIC MODELING

Consider a symmetric rigid spacecraft characterized by a diagonal inertia matrix \( J \), namely

\[
J = \begin{pmatrix}
J_x & 0 & 0 \\
0 & J_y & 0 \\
0 & 0 & J_z
\end{pmatrix}.
\]

The kinematic model used is based on the modified Cayley-Rodrigues parameters, which provide a global and non-redundant parametrization of the attitude of a rigid body (Dwyer et al. (1987)). In the following, \( S(\cdot) \) denotes the three-dimensional skew-symmetric matrix, which for a generic vector \( r \in \mathbb{R}^3 \) takes the form

\[
S(r) = \begin{pmatrix}
0 & -r_3 & r_2 \\
r_3 & 0 & -r_1 \\
r_2 & r_1 & 0
\end{pmatrix}.
\]

Defining \( \rho \in \mathbb{R}^3 \) the modified Cayley-Rodrigues parameters vector and \( \omega \in \mathbb{R}^3 \) the angular velocity in a body-fixed frame, the kinematic equations take the form

\[
\dot{\rho} = H(\rho)\omega.
\]

The matrix-valued function \( H : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3} \) denotes the kinematic jacobian matrix of the modified Cayley-Rodrigues parameters, given by

\[
H(\rho) = \frac{1}{2} \left( I - S(\rho) + \rho^T \rho - \frac{1}{2} \rho^T \rho - \frac{1}{2} \right)
\]

where \( I \) denotes the \( 3 \times 3 \) identity matrix. The matrix \( H(\rho) \) satisfies the following identity (Tsiotras (1996))

\[
\rho^TH(\rho)\omega = \frac{1}{2} (1 + \rho^T \rho) \rho^T \omega
\]

for all \( \rho, \omega \in \mathbb{R}^3 \).

According to Euler’s law, the kinematic and dynamic equations can be written as

\[
\dot{\rho} = H(\rho)\omega \\
\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}u.
\]

If first-order actuator dynamics with time-constants \( T_i \) \( (i = 1, 2, 3) \) are considered, equations (5)-(6) are dynamically extended as follows

\[
\dot{\rho} = H(\rho)\omega \\
\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}\tau \\
\tau = Au + u
\]

where \( \tau \in \mathbb{R}^3 \) represents the torque generated by the actuators according to the reference torque \( u \in \mathbb{R}^3 \) and \( A = \text{diag}(-\frac{1}{T_1}, -\frac{1}{T_2}, -\frac{1}{T_3}) \) is a Hurwitz diagonal matrix whose eigenvalues, all negative real, depend on the time constants of the actuators.

3. IMMERSION AND INVARIANCE STABILIZATION

3.1 Recalls

Let us recall the continuous-time I&I main result in the general case (the proof is detailed in Astolfi et al. (2008)).

Theorem 1. Consider the nonlinear system

\[
\dot{x} = f(x) + g(x)u
\]

with state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \) and an equilibrium point \( x^* \in \mathbb{R}^n \) to be stabilized. Suppose that (10) satisfies the following four conditions.

H1c (Target System) - There exist maps \( \alpha(\cdot) : \mathbb{R}^p \to \mathbb{R}^p \) and \( \pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n \) such that the sub-system \( \xi = \alpha(\xi) \) with state \( \xi \in \mathbb{R}^p, p < n \), has a (globally) asymptotically stable equilibrium at \( \xi^* \in \mathbb{R}^p \) and \( x^* = \pi(\xi^*) \).

H2c (Immersion condition) - For all \( \xi \in \mathbb{R}^p \), there exists a map \( c(\cdot) : \mathbb{R}^p \to \mathbb{R}^m \) such that

\[
f(\pi(\xi)) + g(\pi(\xi))c(\xi) = \frac{\partial \pi}{\partial \xi}(\xi)\alpha(\xi)
\]

H3c (Implicit manifold - \( \mathcal{M} \)) - There exists a map \( \psi(\cdot, \cdot) : \mathbb{R}^{n-p} \to \mathbb{R}^p \) such that the identity between sets \( \{ x \in \mathbb{R}^n | \psi(x) = 0 \} = \{ x \in \mathbb{R}^n | x = \pi(\xi) \} \) for \( \xi \in \mathbb{R}^p \) holds.

H4c (Manifold attractivity and trajectory boundedness) - There exists a map \( \psi(\cdot, \cdot, \cdot) : \mathbb{R}^{n-p} \to \mathbb{R}^m \) such that all the trajectories of the system

\[
\dot{\xi} = \frac{\partial \phi}{\partial \eta}(f(x) + g(x)\psi(x, z)) \\
\dot{z} = f(x) + g(x)\psi(x, z)
\]

are bounded and satisfy \( \lim_{t \to \infty} z(t) = 0 \).

Under these four conditions, \( x^* \) is a globally asymptotically stable equilibrium of the closed-loop system

\[
\dot{x} = f(x) + g(x)\psi(x, \phi(x))
\]

The following definition is straightforward.

Definition 3.1. (I&I Stabilizability) A nonlinear system of the form (10) is said to be I&I stabilizable with target dynamics \( \dot{\xi} = \alpha(\xi) \), if it satisfies conditions H1c to H4c of Theorem 1.

Note that the target dynamics is the restriction of the closed-loop system to the manifold \( \mathcal{M} \), implicitly defined in H3c. The control law \( u = \psi(x, z) \) is designed to steer to zero the off-the-manifold coordinate \( z \) and to guarantee the boundedness of system trajectories. On the manifold, the control law is reduced to \( \psi(\pi(\xi), 0) = c(\xi) \), and it renders \( \mathcal{M} \) invariant according to H2c. The complete control law can thus be decomposed in two parts:

\[
u = \psi(x, \phi(x)) = \psi(x, 0) + \psi(x, \phi(x))
\]

with \( \psi(\pi(\xi), 0) = c(\xi) \) on the manifold and \( \psi(x, 0) = 0 \). Note that \( \psi(x, 0) \) can be seen as a nominal control law, designed on the model of the dynamics restricted on the manifold to obtain a GAS target dynamics. In this sense, the term \( \psi(x, \phi(x)) \) is a robustness-improving addendum which takes into account the off-the-manifold behaviors generated, for instance, by higher-order actuator dynamics. The overall control law provides the I&I “robust” nonlinear stabilizer.

3.2 The class of systems under study

In this work, we consider the problem of state-feedback stabilization of the following class of systems in feedback form

\[
\dot{\xi} = f(\xi) + g(\xi)\eta \\
\dot{\eta} = u
\]

where \( \xi \in \mathbb{R}^p, \eta \in \mathbb{R}^{n-p}, u \in \mathbb{R}^m \) (with \( m = n - p \), \( x = \text{col}(\xi, \eta) \)) and \( \xi^* = 0 \) is a globally asymptotically stable equi-
librium of $\dot{\xi} = f(\xi)$. It is also assumed that we know a radially unbounded Lyapunov function $V(\xi)$ such that

$$\frac{\partial V}{\partial \xi} f(\xi) < -w(\|\xi\|)$$ (16)

with $w(\cdot)$ the function. Note that the existence of $V$ is guaranteed by the converse Lyapunov theorems, although in many cases its knowledge or construction could be difficult to achieve. For systems like (15) constructive solutions of the I&I stabilization problem do exist, as shown with more detail in (Astolfi et al., 2008). In fact, the target dynamics condition H1c is trivially satisfied by $\dot{\xi} = f(\xi)$. Moreover, the mappings

$$x = \begin{bmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \quad u = \varphi(\xi, \eta) = \eta$$ (17)

are such that conditions H2c and H3c hold. Since the off-the-manifold component $z = \eta$ is a partial-coordinate, condition H4c is verified if it is possible to find a control law $u = \psi(\xi, \eta)$ such that the trajectories of the closed-loop system

$$\dot{\xi} = f(\xi) + g(\xi) \eta$$ (18)

are bounded and $\lim_{t \to \infty} \eta(t) = 0$ (manifold attractivity). To this end, it is possible to relax the assumption on $V$ to be a weak Lyapunov function, namely such that (16) holds for all $\|\xi\| > M > 0$, for a proper, problem-dependent, choice of $M$ (see (Astolfi et al., 2008) for more details). With this in mind, we can state the following result.

**Theorem 2.** Consider system (18) with all the related properties and assumptions. The system satisfies condition H4c, i.e. manifold attractivity and trajectory boundedness, with the following choice for the control law:

$$\psi(\xi, \eta) = -(\mu(\xi) + k_\eta) \eta$$

$$\mu(\xi) > \left| \frac{\partial V}{\partial \xi} g(\xi) \right| k_\eta > 0.$$ (19)

**Proof.** Substituting (19) into (18) yields

$$\dot{\xi} = f(\xi) + g(\xi) \eta$$

$$\eta = -(\mu(\xi) + k_\eta) \eta.$$ (20)

Consider now the Lyapunov function

$$W(\xi, \eta) = V(\xi) + \frac{1}{2} \eta^T \eta.$$ (21)

The derivative of (21) along the trajectories of (20) takes the form

$$\dot{W} = \frac{\partial V}{\partial \xi} f(\xi) + \frac{\partial V}{\partial \xi} g(\xi) \eta - \mu(\xi) \eta^T \eta - k_\eta \eta^T \eta$$

$$\leq -w(\|\xi\|) + \left| \frac{\partial V}{\partial \xi} g(\xi) \right| \|\eta\|^2 - \mu(\xi) \|\eta\|^2 - k_\eta \|\eta\|^2$$

$$\leq -w(\|\xi\|) + \left( \left| \frac{\partial V}{\partial \xi} g(\xi) \right| - \mu(\xi) \right) \|\eta\|^2 - k_\eta \|\eta\|^2$$

$$\leq -w(\|\xi\|) - k_\eta \|\eta\|^2$$ (22)

which is negative definite for all $\|\xi\| > M > 0$ thanks to the proper choice of $\mu(\xi)$ and $k_\eta$, thus condition H4c is verified, which concludes the proof.

### 3.3 Problem setting

Assume now that the control input $u$ is maintained piecewise constant on intervals of fixed length, namely the sampling period. We seek a possibly sampling-dependent controller, which maintains the stability properties achieved by the continuous-time control law under sampling. In particular, the digital controller should verify the sampled-data versions of conditions H1c-H4c of Theorem 1, achieving manifold attractivity and keeping the boundedness of the trajectories under sampling.

### 4. SAMPLED-DATA CONTROL DESIGN

We seek a sampled-data controller which, following the closed-loop continuous-time evolution of the function $\varphi(\xi, \eta) = \eta$, ensures attractivity of the manifold $\mathcal{M}$ and boundedness of the closed-loop trajectories under sampling. The closed-loop system under digital control should be globally asymptotically stable. To begin with, let us briefly recall the sampled-data equivalent dynamics under single-rate sampling. In order to simplify the problem and to fit it to that of attitude stabilization of the rigid spacecraft, the following two assumptions are straightforward.

**Assumption 4.1.** Consider system (15). With the aim of designing the sampled-data controller, it is assumed that the dimension of the vector $\xi$ is equal to that of $\eta$ and $u$, namely $p = m$, thus $n = 2m$. As a consequence, the matrix $g(\xi)$ is square, $g(x) \in \mathbb{R}^{m \times m}$.

**Assumption 4.2.** Consider system (15). For the sake of simplicity of the sampled-data control design it is assumed that the matrix $g(\xi)$ is diagonal and does not depend on $\xi$, namely $g(\xi) = G$. Moreover, it is assumed that it is non-singular, i.e. $\det G \neq 0$.

Next, let us rewrite system (15) in the following form ($i = 1, \ldots, m$):

$$x_i = f_i(x) + u_i \tilde{g}$$ (23)

where $x_i = col(\xi_i, \eta_i)$. $u_i$ is the $i$th component of the control input, $f_i(x) = col(f_i(\xi_i) + G_i \eta_i, 0)$, $G_i$ is the $i$th row-$i$th column element of $G$ and $\tilde{g} = col(0, 1)$. In this way, the $n$-dimensional multi-input system is decomposed into $m$ single-input systems, each of dimension two.

The single-rate sampled-data equivalent model is the discrete-time dynamics reproducing, at the sampling instants, the solution of (23) when the control variable $u_i(t)$ is kept constant over time periods of length $\delta$, namely $u_i(t + \tau) = u_i(t) = u_{i0}$ for $0 \leq \tau \leq \delta$, $i = k \delta$, $k \geq 0$. It is described by the $\delta$-parametrized map $F^\delta(x, u_{ik})$ (the pair $u, F^\delta$) admitting the Lie exponential series expansion:

$$x_{i(k+1)} = F^\delta(x_{ik}, u_{ik}) = e^{\delta(f_{i}+u_{i} \tilde{g})}x_{ik}$$ (24)

The sampled-data control is designed employing a single-rate strategy (see Monaci and Normand-Cyrot (1997a), Monaci and Normand-Cyrot (1997b)), which achieves manifold attractivity under sampling by matching the controlled continuous-time evolution of $\varphi = \eta$ at the sampling instants. Since the continuous-time conditions are fulfilled by (19), $\phi$ will vanish asymptotically. A digital control law reproducing the $\phi$ behavior is proposed to steer system trajectories on the manifold. The behavior on the manifold is reduced, by construction, to that of the target system under sampling, yielding global asymptotic stability of the digital control system.

**Theorem 3.** Consider the class of feedback systems (15) satisfying assumptions 4.1 and 4.2 with an equilibrium $x^*$ to be stabilized. Moreover, consider the stabilizing continuous-time control law (19), $\psi(\xi, \eta) \in \mathbb{R}^m$. There exist sampled-data control laws of the form ($i = 1, \ldots, m$)
such that $x^*$ is a globally asymptotically stable equilibrium of the closed-loop dynamics ($i = 1, \ldots, m$)

$$x_i(k+1) = F_i^\phi(x_i, \psi_i^\phi(\xi_k, \eta_k))$$

(26)

Proof. Following assumptions H1c to H4c of the continuous-time result, we can define for (15) a sampled-data equivalent target system, with state $\xi \in \mathbb{R}^p$.

$$\dot{\xi}_{k+1} = f_d(\xi_k)$$

(27)

where $f_d(\xi) = \phi(\xi) \xi$ has a globally asymptotically stable equilibrium at $\xi^* \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$. Condition H2c can be reformulated as follows, for $i = 1, \ldots, m$,

$$F^\delta(\pi(\xi_k), 0) = \pi(f_d(\xi_k))$$

(28)

which, exploiting the properties of the exponential representation, yields that $\pi$ is equal to the identity function for the first $p$ components, while it is necessarily equal to zero for the remaining $n - p = m$ components. Thus invariance under sampling is ensured. Keep in mind that, for the class of systems at study, $c(\xi) = 0$. As a consequence, the implicit manifold condition H3c is verified under sampling with the choice $\phi_1(x) = x_{p+1}, \phi_2(x) = x_{p+2}, \ldots, \phi_m(x) = x_n$ or, alternatively, $\phi(x) = \eta$. The existence of a single-rate controller of the form (25) for (15) is ensured provided a controllability-like condition (27) is satisfied. Consequently, the dynamics in $\xi$ are guaranteed to be globally exponentially stable for $i = 1, \ldots, m$.

Consider now the extended system under sampling ($i = 1, \ldots, m$)

$$\dot{z}_i(k+1) = \phi(F_i^\delta(x_i, \psi_i^\phi(\xi_k, \eta_k)))$$

(29)

$$x_i(k+1) = F_i^\delta(x_i, \psi_i^\phi(\xi_k, \eta_k))$$

(30)

Attractivity of $\mathcal{H}$ is ensured by matching the continuous-time trajectories of $\phi(x)$, which for the systems at study reduces to a simple input-state matching, guaranteed by the existing single-rate solution according to the results in Monaco and Normand-Cyrot (2007). As a consequence, $\lim_{k \to \infty} z_i = 0$. Boundedness of the trajectories of (29) is guaranteed by one-step consistency property plus forward completeness of the vector fields of (15), hence the thesis.

5. ATTITUDE STABILIZATION OF THE RIGID SPACECRAFT

We consider the dynamical model introduced in section 2 to design a robust nonlinear control law in continuous-time using the result introduced in 3 and implementing it under sampling using the result provided in 4.

5.1 Continuous-time control design

The basic idea of continuous-time control design lies in the exploitation of identity (4) to show global exponential stability of the kinematic subsystem with a virtual feedback linear in $\rho$, namely $\dot{\omega} = -k_3 \rho$, yielding the closed-loop kinematic subsystem

$$\dot{\rho} = -k_3 H(\rho) \rho$$

(31)

With this aim, consider the Lyapunov function

$$V(\rho) = \rho^T \rho$$

(32)

Using (4), the derivative of $V$ along the trajectories of (31) is given by

$$\dot{V} = -2k_3 \rho^T H(\rho) \rho = -2k_3 \left(1 + \rho^T \rho\right) \rho^T \rho \leq -\frac{k_3}{2} V$$

(33)

which yields global exponential stability with rate of decay $k/2$. With this in mind, in the following it is shown how the dynamical model of the spacecraft (7)-(8)-(9) can be cast into the form (15) using simple transformations. First, we perform the classical backstepping transformation $\xi = \omega + k_3 \rho$, which transforms the dynamics into

$$\dot{\rho} = -k_3 H(\rho) \rho + H(\rho) \xi$$

(34)

$$\dot{\xi} = J^{-1} S(\omega) J \omega + J^{-1} (J^T H(\rho) \rho + k_3 H(\rho) \xi)$$

(35)

$$\zeta = \dot{\alpha} + u$$

(36)

Now, it is possible to “virtually” stabilize sub-system (34)-(35) using the backstepping control law

$$\tau_1(\rho, \omega) = \tau_1(\rho, \omega) + \tau_2(\rho, \omega)$$

(37)

$$\tau_1(\rho, \omega) = -S(\omega) J \omega$$

(38)

As a consequence, the resulting target-dynamics when $\tau = \tau_b$ in $(\rho, \xi)$ coordinates takes the following form

$$\dot{\rho} = -k_3 H(\rho) \rho + H(\rho) \xi$$

(31)

under which the dynamics in $(\rho, \omega)$ coordinates takes the form

$$\dot{\rho} = H(\rho) \omega$$

(40)

$$\dot{\omega} = J^{-1} S(\omega) J \omega + J^{-1} \tau_b$$

(41)

$$\zeta = \dot{\alpha} + u - \tau_b$$

(42)

Setting now $u = -A \tau + \delta \tau + v$, the representation reduces to a strict-feedback form like (15), namely

$$\left(\begin{array}{c}
\dot{\rho} \\
\dot{\omega}
\end{array}\right) = 
\left[
\begin{array}{c}
H(\rho) \omega \\
J^{-1} S(\omega) J \omega + J^{-1} \tau_b(\rho, \omega)
\end{array}\right] + 
\left[
\begin{array}{c}
0_{3 \times 3} \\
J^{-1}
\end{array}\right] z$$

(43)

$$\zeta = v$$

(44)

where the drift term in (43) is exactly the GAS target dynamics. Keeping in mind that in our case a proper Lyapunov function for the target dynamics is known from backstepping design, namely

$$W(\rho, \zeta) = \rho^T \rho + \frac{1}{2} \zeta^T \zeta$$

(45)

we can directly apply the result of theorem 2, obtaining the continuous-time control law:

$$v = -(\mu(\rho, \zeta) + k_3) \zeta$$

(46)

$$\mu(\rho, \zeta) > \|\col(\zeta, J \zeta, J^{-1}, \zeta J^{-1}, J^{-1} \zeta, J^{-1})\|$$

which ensure global asymptotic stabilization of the attitude of the rigid spacecraft.

5.2 Sampled-data control design

Taking as output $z$, the vector relative degree of system (43)-(44) is equal to 3, thus it is possible to design three digital single-rate control laws to bring $z$ to zero, by matching the
continuous-time control law at the sampling instants. In particular, let us consider the controlled sub-system:
\[
\begin{align*}
\dot{\omega} &= J^{-1}S(\omega)J\omega + J^{-1}v_{i}(\rho, \omega) + J^{-1}z, \\
\dot{z} &= -\mu(\rho, \zeta)z - k_{i}z
\end{align*}
\]  
(47) \hspace{1cm} (48)

for which assumption 4.1 and 4.2 are satisfied, since \( \dim(\omega) = \dim(z) \) and \( J^{-1} \in \mathbb{R}^{3 \times 3} \) is squared, diagonal, constant and non-singular. We want to construct a digital version of the linear part of the continuous-time controller, i.e. \( v_{i} = -k_{i}z \), which is exactly the component bringing to zero the off-the-manifold coordinate \( z = \phi \). With this aim, let us express (47)-(48) in the following way, for \( i = 1, 2, 3 \):  
\[
\begin{pmatrix}
\phi_i \\
\zeta_i
\end{pmatrix} = \begin{pmatrix}
\tilde{f}_{i}(\rho, \omega, z_i) \\
-\mu(\rho, \zeta)z_i
\end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_{i} = f_{i}(\rho, \omega, z_i) + g_{i}v_{i}
\] 
(49)

where  
\[
\begin{align*}
\tilde{f}_{i}(\rho, \omega, z_i) &= [J^{-1}S(\omega)J\omega + J^{-1}v_{i}(\rho, \omega) + J^{-1}z], \\
v_{i}(z_i) &= -k_{i}z_i
\end{align*}
\] 
(50)

The matching equations for the \( i \)-th component of the digital control law take the form  
\[
e^{\delta f_{i}(\cdot, \cdot)+v_{i}(\cdot)}|_{\rho, \omega, z} = e^{\delta f_{i}(\cdot, \cdot)+v_{i}(\cdot)\rho}|_{\rho, \omega, z}.
\] 
(51)

The corresponding digital control laws, truncated at the first term in \( O(\delta^{2}) \), are, for \( i = 1, 2, 3 \),  
\[
v_{d_{i}} = v_{d_{0}} + \frac{\delta}{2} v_{d_{1}}
\] 
(52)

or, in more compact form, \( v_{d_{i}} = v_{d_{0}} + (\delta/2)v_{d_{1}} \), with \( v_{d_{0}} = v_{i}(z)|_{z} \) emulated controller, and \( v_{d_{1}} = v_{i}(z)|_{z} \) first order cor-rector term, for which we simply obtain:  
\[
v_{d_{1}} = (k_{i}^{2}z + k_{i}\mu(\rho, \zeta)z)|_{\rho, \zeta, \mu, \zeta}
\] 
(53)

5.3 Simulations

The performance of the proposed sampled-data controller (52)-(53) is tested through numerical simulations in comparison with the emulated controller \( v_{d_{0}} \) at different sampling periods. The rigid spacecraft introduced in section 2 is characterized by the inertia moments listed below in Tab. 1. The following initial conditions in modified Cayley-Rodrigues (CR) parameters, angular velocities and torques are considered:

| \( J_{1} \) | 30.08 kg \cdot m^2 |
| \( J_{2} \) | 30.12 kg \cdot m^2 |
| \( J_{3} \) | 29.89 kg \cdot m^2 |

\( (\rho_{1}(0), \rho_{2}(0), \rho_{3}(0)) = (4, 8, 12) \)

\( (\omega_{1}(0), \omega_{2}(0), \omega_{3}(0)) = (6, 3, 9) \) \( [\text{rad/s}] \)

\( (\tau_{1}(0), \tau_{2}(0), \tau_{3}(0)) = (10, 11, 8.5) \) \( [\text{N} \cdot \text{m}] \).

In the first scenario, the emulated and first-order corrector controllers are tested when the sampling period is \( T = 0.2 \) s. Figures from 1 to 4 illustrate the results: even if both controllers achieve asymptotic stabilization, the improved performances of the proposed controller in the transients of \( \tau(t) \) and \( z(t) \) are evident. Moreover, the result is obtained with a control effort which is smaller even in comparison with the continuous-time controller (Fig. 5). In all the figures below, the closed-loop continuous-time trajectories are depicted in red, those under emulated control in green and those under the improved controller in blue. In the second scenario, the emulated and first-order corrector controllers are tested when the sampling period is \( T = 0.5 \) s. As shown in Fig. 6, 7, 8, 9 and 10.
the emulated controller cannot achieve stabilization, while the proposed controller still behaves well, also in terms of control effort.

Note that the maximum allowable sampling time for the emulated controller is about $T_{\text{MA}S\text{P}} = 0.4 \text{ s}$, a sampling period at which the improved controller still works.

6. CONCLUDING REMARKS

A robust nonlinear stabilizer for a rigid spacecraft has been developed taking into account actuator dynamics in control design. The proposed control law is a special case of I&I stabilizer and it has been implemented under sampling using a single-rate control strategy with first-order corrector term. The effectiveness of the proposed control solution is shown in several simulations at two different sampling periods.

REFERENCES


