Input-state matching under piecewise constant control for systems on matrix Lie groups

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Abstract—The constructive controllability problem under piecewise constant control is discussed for systems evolving on matrix Lie groups. Two approaches are followed: an indirect one, in which a continuous time control is matched by a multi-rate sampled data controller ensuring coincidence at the sampling instants and a direct one, in which a digital point to point maneuver is designed. Simulations are performed to compare the continuous and digital approaches.

I. INTRODUCTION

There is a well documented nonlinear control literature drawing attention to driftless systems with fewer controls than state variables. Motion planning of mechanical articulated structures as wheeled robots with non holonomic constraints or satellites with vibrational appendages, underwater vehicles are well known examples. Provided the Lie algebra rank condition is satisfied, there exists a control law that drives the system to the origin from any initial state. However, such a rank condition is not constructive for the design. Several different approaches dedicated to build controls achieving complete controllability have been developed especially in terms of time-varying solutions [1], [2], [4], [5], [10], [12], [16], [17], [18]. The basic requirement is to move in the directions induced by the Lie brackets of vector fields needed to achieve complete controllability. Following in particular [4], [10], [11], [12], [16], the present paper focuses on driftless, left-invariant systems evolving on matrix Lie groups of the form

\[ \dot{X} = \epsilon X U, \quad U(t) = \sum_{i=1}^{m} A_i^{(i)}u_i(t) \]  

where \( X(t) \) is a curve in a matrix Lie group \( G \) of dimension \( n \), \( U(t) \) is a curve in \( \text{span}\{A_1^1, \ldots, A_m^m\} \) with \( m \leq n \), the control authority or actuator capability of the system, where \( \{u_1^1, \ldots, u_m^m\} \) can be actuated independently; \( \epsilon \) is a small parameter \( (0 < \epsilon < 1) \) such that \( \epsilon u_i(.) \) are interpreted as small amplitude controls. Denoting by \( \mathcal{G} \) the matrix Lie algebra of \( G \), the Lie bracket \([., .]\) on \( \mathcal{G} \) is the matrix commutator \([A, B] = AB - BA\), for \( A, B \in \mathcal{G} \), (see [6] for an introduction to matrix Lie groups and Lie algebras). Averaging in the present context is referred to the secular terms in the asymptotic expansions characterizing the system behavior associated with (1) as in [3] for the Brockett case study, a two-input nilpotent system on \( R^3 \).

Our interest is motivated by work deriving periodic time-varying control solutions from average system behavior. More precisely, algorithms for constructing open loop controls for point-to-point maneuvering are proposed in terms of sinusoidal controls that solve the constructive controllability problem with \( O(\epsilon^p) \) accuracy; i.e. \( p \)-order average solution. For lower average order, \( p = 2, 3 \), control solutions can be constructed as functions of structure constants associated with the control authority.

Our goal is to specialize multi-rate digital solutions proposed in [7], [8], [14] to this context, exploiting the Lie group structure. More precisely, we address the following problem: given initial and final conditions \( X_i, X_f \in G \) and a time \( T_f \), find a piecewise constant control sequence driving \( X_i \) at time \( t = 0 \) to \( X_f \) at time \( T_f \).

This is equivalently to the constructive controllability problem under piecewise constant control. Referring to average behavior of (1), multi-rate digital solutions are designed driving exactly the average behavior of the system thereby driving the exact solution approximately. The proof is constructive.

Let us note that the generality of the representation (1) in the Lie group framework leads to coordinate-free system behavior and thus coordinate-free control algorithms. Furthermore, left-invariance guarantees global results, so that the current position can always be set at the identity of the Lie group.

The paper is organized as follows. Section II describes the problem in its geometric framework and recalls the instrumental tools for characterizing the flows under continuous-time and piecewise constant controls. Section III describes the digital solution in terms of the continuous-time one for average orders equal to 1, 2, 3. Section IV describes a point-to-point digital solution independently on the existence of a continuous-time solution. Some simulations illustrate the results in Section III to allow some comparative analysis.

II. PRELIMINARIES AND PROBLEM STATEMENT

Averaging theory for systems evolving on Lie groups has been derived in different contributions [4], [12], [17] so providing time-varying periodical open loop control solutions for point-to-point maneuvering, the most popular being the sinusoidal ones [3], [10], [16], [18]. Approximate average solutions make reference to the order of accuracy in \( \epsilon \) involved in the computations. It is also related to the order of Lie brackets needed to capture the Lie bracket motion of the dynamics. In this driftless context, two basic representations...
of the flows associated with (1) are used: as a product of exponentials, the Wei-Norman representation, and as a single exponential, the Magnus single exponential representation [13]. Short recalls, given below, are instrumental in the present digital context.

A. The Magnus single exponential

From Theorem III in [13], the solution to (1) at any time \( t = \Delta \) can be expressed as \( X(\Delta) = e^{Z(\Delta)}X(0) \), where \( Z(\Delta) \in \mathcal{G} \), once the infinite series \( Z(\Delta) = \sum_{q \geq 1} \epsilon^q Z^{(q)}(\Delta) \) is introduced, so that

\[
Z(\Delta) = \epsilon \int_0^\Delta U_c(\tau) \, d\tau + \frac{\epsilon^2}{2} \int_0^\Delta [\dot{U}_c(\tau), U_c(\tau)] \, d\tau + \frac{\epsilon^3}{12} \int_0^\Delta \dot{U}_c(\tau), [\dot{U}_c(\tau), U_c(\tau)] \, d\tau + \ldots
\]

with \( \dot{U}_c(t) = \sum_{l=1}^n A_l \dot{u}_l(t) \) and \( \ddot{u}_l(t) = \frac{3}{l^2} u_l(t) \) \( \epsilon \).

B. The multi-rate trajectory

Assuming in (1) the controls constant over time intervals of length \( \delta \), \( U = \sum_{l=1}^n u^l A^l \), the solution at time \( \delta \) reduces to

\[
X_{\delta}(\delta) = e^{Z_{\delta}(\delta)}X(0) = e^{\epsilon \delta U_1} \ldots e^{\epsilon \delta U_p}X(0)
\]

a single exponential form of degree 1 in \( \epsilon \). Composing such exponential forms over \( p \) time intervals of length \( \delta \) for possibly different control values \( (U_1, \ldots, U_p) \), one gets the solution at time \( \Delta = p\delta \) under piecewise constant controls; i.e. the \( p \)-order multi-rate trajectory

\[
X_{mr}(\Delta) = e^{Z_{mr}(\Delta)}X(0) = e^{\epsilon \delta U_1} \ldots e^{\epsilon \delta U_p}X(0)
\]

with

\[
Z_{mr}(\Delta) = \sum_{q \geq 1} \epsilon^q Z^{(q)}(\Delta) = \mathcal{BCH}_p(\epsilon \delta U_1, \ldots, \epsilon \delta U_p)
\]

where \( \mathcal{BCH}_p(\ldots) \) denotes the generalized Baker-Campbell-Hausdorff exponent [13] associated with the product of exponentials (given two formal variables \( X, Y \), \( e^Xe^Y = e^{BCH_2(X,Y)} \) with \( BCH_2(X,Y) = X + Y + 1/2[X,Y] + 1/12([X,[X,Y]] + [Y,[Y,X]] + [Z,[Z,Z]]) + \ldots \) \( \epsilon \).

As before, \( X_{mr}^{(q)}(\Delta) = \epsilon \sum_{j=1}^q e^{Z_{mr}^{(j)}(\Delta)}X(0) \) denotes the \( q \)-order average approximation of \( X_{mr}(\Delta) \). The generalized exponent \( \mathcal{BCH}_p(\ldots) \) in (5) can be rewritten in \( \mathcal{G} \) as follows.

**Proposition 2.1:** Setting \( U_r = \sum_{l=1}^n u^l A^l \) for \( r \in [1,p] \)

\[
\mathcal{BCH}_p(\ldots) = \sum_{r=1}^m \sum_{q \geq 1} \epsilon^q \delta^q \alpha^q_1(u_1, \ldots, u_p) A^r + \sum_{i=m+1}^n \sum_{q \geq 2} \epsilon^q \delta^q \alpha^q_1(u_1, \ldots, u_p) A^i
\]

for suitable real functions \( \alpha^q_i(u_1, \ldots, u_p) \).

As before, average multi-rate solutions can be defined in terms of truncations in \( O(\epsilon^{q+1}) \) of these asymptotic expansions. A major advantage to evolve on matrix Lie groups under piecewise constant controls is that the \( \alpha^q_i(\ldots) \) which express the decomposition in \( span\{A^1, \ldots, A^n\} \) of the successive Lie brackets, are functions of their arguments, the multi-rate controls, with real coefficients.

C. Problem statement

Setting, without loss of generality, \( X(0) = I \), problem (P) can be specialized as follows. Given \( X_f = e^{Z_f} \) a target point in \( \mathcal{G} \) with \( Z_f \in \mathcal{G} \) and time \( T_f \in [0,T] \), find a multi-rate control of order \( p \), \( (U_1, \ldots, U_p) \) so that

\[
e^{T_f \epsilon U_1} \ldots e^{T_f \epsilon U_p}X(0) = e^{Z_f}
\]

or equivalently in \( \mathcal{G} \), so that

\[
\mathcal{BCH}_p(\epsilon \frac{T_f}{p} U_1, \ldots, \epsilon \frac{T_f}{p} U_p) = Z_f
\]

Two situations are discussed in the paper. First, the existence of a continuous-time control solution \( \dot{U}_c(t) = \sum_{l=1}^n A_l \dot{u}_l(t) \) driving exactly \( X(0) = I \) to \( X_f \) at time \( T_f \) is assumed and the multi-rate solution is computed in terms of the continuous-time one. Average digital solutions driving exactly \( X(0) = I \) to the \( q \)th order approximated continuous solution \( X_f^{(q)} \) are detailed for average orders \( q = 1, 2, 3 \). This approach corresponds to an indirect strategy because one is interested in reproducing, at the sampling instants, the continuous-time law and, as a consequence, in matching its performances; it will be referred to as the average input-state matching problem (AISMP) under multi-rate digital control. The second solution proposed is represented by a digital point-to-point maneuver, driving \( X(0) = I \) to \( X_f \) in an approximate way. One has in this case a direct digital control strategy, not based on the continuous time solution. In such a case, the target point is an element of the Lie group, \( X_f = e^{Z_f} \) with \( Z_f \in \mathcal{G} \), \( Z_f = \sum_{i=1}^n z_f A^i \).

D. Some geometric aspects

Denoting by \( \{A^1, \ldots, A^n\} \) a given basis for the Lie algebra \( \mathcal{G} \), a depth-p Lie bracket refers to the number of iterated brackets: a depth-zero Lie bracket is just an element of \( \mathcal{G} \) and a depth-one Lie bracket an element of the form \( [A^1, A^2] \). Arguing so, we can then get the depth-p structure constants. For the first ones, the depth-1 structure constants \( \Gamma^q_{ij} \) verify, for \( i, j \in [1,n] \)

\[
[A^1, A^2] = \sum_{q=1}^n \Gamma^q_{ij} A^q.
\]

The skew symmetry of the Lie bracket implies \( \Gamma^q_{ij} = -\Gamma^q_{ji} \).

Analogously, the depth-two structure constants, \( \theta^q_{ijk} \) verify for \( i, j, k \in [1,n] \)

\[
[[A^1, A^2], A^k] = \sum_{q=1}^n \sum_{l=1}^n \Gamma^q_{ij} \Gamma^l_{jk} A^q = \sum_{q=1}^n \theta^q_{ijk} A^q.
\]

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The Jacobi identity implies $\theta_q^{ijk} + \theta_q^{jki} + \theta_q^{kij} = 0$. These structure constants enter in the description of the $\alpha_q^{(i,j,k)}$ in (6). As an example, for a $2^n d$-order multi-rate and $3^rd$-order average, one gets with an error in $O(\epsilon^4)$

$$BCH_2\{\epsilon T_j/2 U_1, \epsilon T_j/2 U_2\} = \epsilon T_j/2 \sum_{i=1}^{m} (u_i^1 + u_i^2) A^i +$$

$$\epsilon^2 T_j^2/8 \sum_{i<j=1}^{m} (u_i^j u_j^i - u_j^i u_i^j) \sum_{i=1}^{n} \Theta_{ijk} A^i -$$

$$\epsilon^3 T_j^3/24 \sum_{i<j<k=1}^{m} (u_i^k - u_k^i)(u_j^i u_k^j - u_j^j u_k^i) \sum_{i=1}^{n} \Theta_{ijk} A^i.$$

### III. AISMP UNDER MULTI-RATE CONTROL

The $q^{th}$-order AISMP under $p^{th}$-order multi-rate control: given $U_c(t) = \sum_{i=1}^{m} u_i^c(t) A^i$ driving $X(0) = I$ to $X_f^{(q)}$ at time $T_f$ exactly, find a $p^{th}$-order multi-rate control $(U_1, ..., U_p)$ such that

$$e^{T_f/p} U_{p}, ..., e^{T_f/p} U_{1} = e^{T_f} Z_f^{(l)}$$

or equivalently

$$BCH_p\{T_f/p U_1, ..., T_f/p U_p\} = \sum_{l=1}^{q} \epsilon^l Z_f^{(l)} + O(\epsilon^{q+1})$$

that is from (6) for each $l \in [1, q]$ with an error in $O(\epsilon^{q+1})$

$$\sum_{i=1}^{n} \frac{T_f}{p} \alpha^i(u_1, ..., u_p) A^i = Z_f^{(l)}$$

with $Z_f^{(l)}$ described according to (2).

#### A. First order average digital solution

Rewriting (12) for $l = 1$ and $p = 1$ one gets the first order average digital solution;

**Lemma 3.1:** Let system (1) on the Lie group $G$ with Lie algebra $\mathcal{G}$ and a continuous-time controller $U_c(t) = \sum_{i=1}^{m} u_i^c(t) A^i$, then, the $1^{st}$-order average input-state matching is achieved under single digital control $U_d(T_f) = \sum_{i=1}^{m} u^d(T_f) A^i$ iff $m = n$ and $u^i$ given for all $i \in [1, m]$ by

$$u^i(T_f) = \frac{1}{T_f} \int_0^{T_f} u_i^c(t) dt = u_0^i + \sum_{l=1}^{T_f} (l+1)! u_l^i.$$

We note that the so computed digital solution is the average over $T_f$ of $u_i^c(t)$.

For $m < n$, higher order average and higher order multi-rate strategies are needed to capture all the controllability directions associated with the Lie brackets.

#### B. Second order average digital solution

**Theorem 3.1:** (Second order average digital solution) Consider system (1) on the Lie group $G$ with Lie algebra $\mathcal{G}$. Let $U_c(t) = \sum_{i=1}^{m} u_i^c(t) A^i$, a given continuous-time control, then the $2^{nd}$-order average input-state matching is achieved under $p^{th}$-order multi-rate digital control $(U_1, ..., U_p)$ with the $u_i^c$ for $i < j \in [1, m], r, s \in [1, p]$ solutions of

$$\sum_{r=1}^{p} u_r^c = \frac{p}{T_f} \int_0^{T_f} u_i^c(t) dt,$$

$$\sum_{r<s=1}^{p} u_r^c u_s^c - u_s^c u_r^c = \frac{p^2}{T_f} \int_0^{T_f} (u_i^c(\tau) u_j^c(\tau) - u_j^c(\tau) u_i^c(\tau)) d\tau.$$

**Proof:** The proof works out rewriting the equality (12) for $q = 2$ and a generic $p$

$$U_1 + ... + U_p = \frac{p}{T_f} \int_0^{T_f} U_c(t) dt$$

$$\sum_{r,s=1}^{p} \epsilon T_f \int_0^{T_f} [U_r(t) U_s(t)] d\tau$$

from which we deduce the term-by-term equalities (14,16).

Once again, for $r \in [1, p]$, each digital solution $U_r(T_f/p) = \sum_{i=1}^{m} u_r^c(T_f/p) A^i$ is described by its series expansion in $T_f$

$$u_r^c(T_f/p) = u_0^r + \sum_{l=1}^{T_f} \frac{T_f^l}{p(l+1)!} u_r^l.$$

In (16), all the possible Lie brackets of depth one $[A^i, A^j], i, j \in [1, m]$ are taken into account so obtaining $m + C_m^n$ equations to be satisfied with $m \times p$ control variables represented by the $u_r^l$, $i \in [1, m], r \in [1, p]$. However, in practice, as some depth-one Lie brackets do not capture a new controllability direction, the number of equations to satisfy should be reduced to $n = \dim \mathcal{G}$ at most, provided all the depth-one Lie brackets are constructive to achieve complete controllability.

Specializing the digital solution (14-16) to multi-rate of order 2 ($p=2$), we get for $i < j \in [1, m]$

$$u_1^2 + u_2^2 = \frac{2}{T_f} \int_0^{T_f} u_i^c(t) dt$$

$$u_1^i u_2^j - u_2^i u_1^j = \frac{A}{T_f} \int_0^{T_f} (u_i^c(\tau) u_j^c(\tau) - u_j^c(\tau) u_i^c(\tau)) d\tau$$

which corresponds to solve at most $m + C_m^n$ equations with $2m$ degrees of freedom; for $m = 2$ and $m = 3$ the problem is solvable, while for $m > 3$ it depends on the number of new independent directions of the form $[A^i, A^j]$.

#### C. Third order average digital solution

As far as the design of a third order multi-rate control, one has

**Theorem 3.2:** (Third order average digital solution) Consider system (1) on the Lie group $G$ with Lie algebra $\mathcal{G}$. Let $U_c(t) = \sum_{i=1}^{m} u_i^c(t) A^i$, a given continuous-time control, then the $3^{rd}$-average input-state matching is achieved under a $3^{rd}$-order multi-rate digital control $(U_1, U_2, U_3)$ with the
Theorem 4.1: (Direct second order digital solution) Consider system (1) on the Lie group \( G \) with Lie algebra \( \mathcal{G} \) of rank \( n \), then the \( p^\text{th} \)-order multi-rate digital described for \( r \in [1, p] \), \( i \in [1, n] \) by

\[
u_i^r(T_f/p) = u_{r0}^i + \frac{T_f}{2p} u_{r1}^i
\]

with the \( u_{r0}^i, u_{r1}^i \) solutions, for \( l \in [1, m] \), of

\[
\sum_{r=1}^{p} u_{r0}^i + \sum_{i<j=1}^{m} \sum_{r<s=1}^{p} (u_{r0}^i u_{s0}^j - u_{s0}^i u_{r0}^j) \Gamma_{ij}^q = z_f^q
\]

and solutions, for \( q \in [m+1, n] \), with at least one \( \Gamma_{ij}^q \neq 0 \), of

achieves point-to-point maneuvering from \( X(0) = I \) to \( X_f \) at time \( T_f \) with accuracy in \( O(T_f^2) \) provided full controllability is achieved through depth-one Lie bracket.

Remark: For optimal closed-loop control, \( \epsilon \) must be selected such that the discrete-maps \( \Psi_r \) are\( \mathcal{C}^{q} \) at any time \( T_f \).
\[
\frac{T_r^2}{4p^3} \sum_{i,j=1}^m \sum_{i<j=1}^3 \left( u_{r0} u_{s1} + u_{r1} u_{s0} - u_{s0} u_{r1} - u_{s1} u_{r0} \right) \Gamma_{ij}^q \\
\frac{T_r^4}{4p^3} \sum_{i,j,k=1}^m \left( (u_{r0} - u_{s0})(u_{r0} - u_{s0}) \right) + 2u_{r0} \left( u_{20} u_{10} - u_{21} u_{10} \right) + 2u_{r0} \left( u_{20} u_{30} - u_{21} u_{30} \right) \right) \eta_{ijk}^q = \frac{\gamma_j^q}{2} \\
\]

and solutions, for \( q \in [m + 1, n] \), with \( \Gamma_{ij}^q = 0 \) and at least one \( \theta_{ijk} \neq 0 \), of
\[
\frac{T_r^4}{4p^3} \sum_{i,j,k=1}^m \left( (u_{r0} - u_{s0})(u_{r0} - u_{s0}) \right) + 2u_{r0} \left( u_{20} u_{10} - u_{21} u_{10} \right) + 2u_{r0} \left( u_{20} u_{30} - u_{21} u_{30} \right) \right) \eta_{ijk}^q = \frac{\gamma_j^q}{2}
\]
achieves point-to-point maneuvering from \( X(0) = I \) to \( X_f \) at time \( T_f \) with accuracy in \( O(T_f^2) \) (error in \( O(T_f^3) \)), provided full controllability is achieved through depth-two Lie brackets.

**Remark:** Assuming depth \( q \)-controllability, that is Lie brackets of depth \( q \) are needed to achieve full rank of \( G \), then the results above say that the digital controller has to be computed with an order of accuracy in \( T_f \) of order \( q \) at least.

**Remark:** Also in this case, the multi-rate order of the solution cannot be given a priori.

V. Simulation Results

Numerical simulations have been performed in order to show the behavior of the proposed solution for the AISM problem as in section III. Following [11] the case study of a rigid body attitude control is considered. \( G = SO(3) \), the Lie group of rotational matrices; a basis for the associated Lie algebra is given by the three skew-symmetric matrices
\[
A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
where \([A_1, A_2] = A_3, \ (A_1, A_2) = A_1^2\) is the control authority.

In [11], the following 2\(nd\)-order average solutions are used
\[
\varepsilon u_1(t) = \begin{cases}
\frac{g_{d1} \sin(\epsilon t)}{2} & t_0 \leq t \leq t_3 \\
\frac{g_{d1} \sin(\epsilon(t - t_3))}{2} & t_3 \leq t \leq T_f
\end{cases}
\]
\[
\varepsilon u_2(t) = \begin{cases}
\frac{g_{d2} \sin(\epsilon t)}{2} & t_0 \leq t \leq t_1 \\
\frac{g_{d2} \sin(\epsilon(t - t_2))}{2} & t_1 \leq t \leq t_2 \\
0 & t_2 \leq t \leq t_3
\end{cases}
\]
with \( t_0 = 0, t_1 = 1, t_2 = 21, t_3 = 22, T_f = 24, g_{d1} = g_{d2} = 0.05, \epsilon = \frac{\pi}{10} \) and \( \psi = \frac{\pi}{4} \).

The solution proposed in section III to the AISM for a 2\(nd\)-order input-state matching under a \( p\)\(^{th}\) order multirate digital control, according to Theorem 3.1, requires the computation of \( U_1, U_2 \), with \( U_1 = \sum_{k=1}^m u_k A^k, i = 1, 2 \), solving equations (14,16).

\[
\frac{1}{T_f} \int_0^{T_f} u_1(t) dt = \frac{1}{T_f} g_{d1} \left( \int_0^{T_f} \omega \sin(\omega t) dt + \int_0^{T_f} -\frac{1}{2} \omega \sin(\omega(t - t_3)) dt \right) = 0.1
\]

\[
\frac{1}{T_f} \int_0^{T_f} u_2(t) dt = \frac{1}{T_f} g_{d2} \left( \int_0^{T_f} \omega \cos(\omega t) dt + \int_0^{T_f} \frac{1}{4} \omega \cos(\omega(t - t_3)) dt \right) = \frac{\pi}{40}
\]

Setting \( p = 4 \), one gets the equations
\[
\frac{1}{4} (u_1 + u_2 + u_3 + u_4) = 0.1
\]
\[
\frac{1}{4} (u_1^2 + u_2^2 + u_3^2 + u_4^2) = 0.05
\]
\[
\frac{1}{2} \left( (u_1 u_2 - u_1^2 u_2) + (u_1 u_3 - u_1^2 u_3) + (u_1 u_4 - u_1^2 u_4) + (u_2 u_3 - u_2^2 u_3) + (u_2 u_4 - u_2^2 u_4) + (u_3 u_4 - u_3^2 u_4) \right) = \frac{\pi}{40}
\]

A numerical solution is
\[
\begin{align*}
u_1 &= 0.0509 \\
u_2 &= 0.0420 \\
u_3 &= -0.0425 \\
u_4 &= 0.0488 \\
u_5 &= 0.0021 \\
u_6 &= 0.0488 \\
u_7 &= 0.0021 \\
u_8 &= 0.0446
\end{align*}
\]

Figures 1 to 5 illustrate the comparison between the digital solution and the continuous-time one in [11] for \( g_1(t), g_2(t), g_3(t) \) and \( u_1(t), u_2(t) \). A comparison from the energetic point of view is reported in Fig. 6, where the term \( \int_0^{T_f} (u_1^2 + u_2^2) dt \) is plotted. The difference can be imputed to the different way the motion along the \( n - m \) directions \( \left\{ A^{m+1}, \ldots, A^n \right\} \) is achieved: the digital approach allows to compute the constant values, in the control series expansion, corresponding exactly to the amount required to reach the final condition (the coefficient of the corresponding Lie bracket). On the other hand, the continuous approach requires a periodic control whose integral only represents the actual contribution. Since the integral of a periodic signal is periodic while the integral of its square is positive and monotonically increasing, the energy required in the continuous case can be sensibly greater.

VI. Conclusions

In this paper two solutions to the problem of approximated point to point control have been given.

The former considers average approximated solutions computed under continuous control and computes an equivalent multi-rate digital input for getting, exactly, the same approximated final point. The latter is based on a direct digital approach, computing the multi-rate digital control on the basis of the approximated dynamics, obtained by truncation of the exponential series expansion.
A comparison between the behavior of the continuous and the equivalent digital control is discussed, through some numerical simulation results, showing some energetic advantages for the digital solution.

REFERENCES