DIGITAL CONTROLLERS FOR NILPOTENT LIE ALGEBRA OF ORDER 3

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Abstract The paper deals with input-state matching under sampled data control of a given input-affine dynamics under nilpotency assumptions set over its controllability Lie-algebra. Nilpotency at order 3, under the additional assumption that the zero-time controllability Lie-algebra is of maximal dimension 4, ensures the existence of a sampled-data multi-rate controller of order 4 which can be computed at any order of approximation.

1 Introduction

Consider a continuous-time input-affine dynamics

\[ \dot{x} = f(x) + u_c g(x) \]

where \( f \) and \( g \) are analytic vector fields on a manifold \( \mathcal{X} \) and \( u \in U \), a set of admissible controls. It has been conjectured in some recent works by the authors (e.g. [14]) that input-state matching under sampled-data control can be achieved making use of either multirate or generalized sampled controllers of order equal to the dimension of the zero-time controllability Lie-algebra associated to (1). The computation of the sampled-data control law is rather involved and exact solutions have been described for nilpotency at the first orders only.

Following the approach proposed in [19] and on the basis of first results in [15] the problem is here addressed under suitable nilpotency assumptions on the controllability Lie-algebra \( \mathcal{L}(f, g) \), i.e. the Lie-algebra generated by the vector fields \( f \) and \( g \) in (1). It results that the order of the multirate is linked to the characteristics of the nilpotency and an elegant approach for describing the solution can be proposed.
For a formal setting of the problem we refer to the terminology in [19]: $x(t, u_c, x_0)$ will denote the solution of (1) at time $t$ when $x(0) = x_0$; the system is assumed complete, i.e. $x(t, u_c, x_0)$ is well defined and unique for all $x_0 \in X, u \in U, t \in R$; given some subset $V$ of $U$ and a time $T \geq 0$, $R(T, V, x_0) := (x(T, v, x_0) : v \in V)$ will denote the set of states accessible at time $T$ from $x_0$ using controls in $V$. The conditions which guarantee

$R(T, V, x_0) = R(T, U, x_0)$

and the computation of a control in $V$ associated with a given control in $U$ are the problems we will investigate. $\mathcal{MR}^p$, the set of $p$-order multirate sampled-data controls over time intervals of length $T$, i.e. constant controls over time sub-intervals of length $T/p$, will take the place of $V$ in (2) and $\mathcal{MR}^1$ restitutes the set of usual sampled-data controls. Let us recall that when $u_c(t) = u_k$ for $k \geq 0$, the sampled equivalent dynamics to (1) is described by a nonlinear difference equation given in the form of a map as

$$(2) \quad x_{k+1} = F^\delta(x_k, u_k) = x_k \circ e^{\delta(f+u_k g)}$$

where, with reference to the notations in [1], $x_k \circ e^{\delta(f+u_k g)}$ represents the exponential form of the flow associated with (1): the asymptotic expansion in powers of $\delta$ of the map $F^\delta(., u_k)$.

The study is developed under the assumption that $\mathcal{L}(f, g)$ is nilpotent at order three, i.e. all the elements in $\mathcal{L}(f, g)$ resulting from more than three appearances of $f$ and $g$ are identically zero (equivalently the Lie brackets of length greater or equal to 3 vanish), noted $\mathcal{L}^4(f, g) = 0$. Under such an assumption, the case here addressed in details corresponds to assuming dimension of $\mathcal{L}_0$, the Lie ideal of $\mathcal{L}$ generated by $g$, is equal to 4. As a matter of facts it is readily verified that when $\mathcal{L}^4(f, g) = 0$ the dimension of $\mathcal{L}_0$ can be at most 4; it has been shown in [15] that the additional nilpotency at the first order of $\mathcal{L}_0$ implies its dimension at most three and a multirate solution of order 3 has been given. As stressed in the sequel, relaxing the assumption of nilpotency at order 1 of $\mathcal{L}_0$ and assuming the maximal dimension significantly complicates the analysis and gives generality to the investigation.

More precisely, denoting as usual $\mathcal{L}^2 := [\mathcal{L}, \mathcal{L}], \mathcal{L}^3 := [\mathcal{L}, \mathcal{L}^2], \mathcal{L}^4 := [\mathcal{L}, \mathcal{L}^3]$, we assume

$A : \quad \mathcal{L}^4 = 0 \quad \text{and} \quad \dim \mathcal{L}_0 = 4$

The study, as previously noted generalizes the results obtained in [15] for nilpotent Lie algebra at order less than 3 or assuming in addition $[\mathcal{L}_0, \mathcal{L}_0] = 0$. 

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The pursued approach is based on the integration of (1) over time intervals of length $T$, for $u_c(t) \in \mathcal{U}$ described by its Taylor’s type asymptotic expansion in powers of $t$; i.e.

(4) \[ u_c(t) = u_c(0) + \sum_{i \geq 1} \frac{t^i}{i!} u_{ci} \]

with constant values $u_{ci}$ for $i \geq 0$. Arguing so, we replace (1) by a time-varying dynamics

(5) \[ \dot{x} = f_t^t(x) = f_0(x) + \sum_{i \geq 1} \frac{t^i}{i!} f_i(x) \]

where $f_0 := f + u_c(0)g$ and $f_i := u_{ci}g$ for $i \geq 1$. The flow associated with such a time-varying dynamics can be described either by an ordinary exponential form with exponent Lie series in $\mathcal{L}(f, g)$ [13] or as a product of Chen series around the free evolution as proposed by Sussmann [18]. On the other hand, the computation of the solution at time $T$ under multirate sampled-data control of order 4 is computed from (3) through composition of the flows when assuming 4 different constant controls over time intervals of length $T = 4\delta$:

\[ x(T) = x_0 e^{\delta(f + u_0g)} e^{\delta(f + u_1g)} e^{\delta(f + u_2g)} e^{\delta(f + u_3g)}. \]

In the posed context nilpotency conditions over $\mathcal{L}(f, g)$ can significantly simplify the investigation since giving rise to series expansions of finite length more tractable in practice.

The present study makes use of the differential/difference representations of sampled dynamics as an alternative to the representation in the form of a map [11]. Such a representation enables one to describe, as the Chen-Sussmann series does, the reached states as a composition of exponential forms evaluated around the free evolution. The solution to the problem under study is then computed through a term-by-term comparison of the respective series expansions describing the state behaviours at the sampling instants. Truncation at prefixed order $q$ in $\delta$ yields to approximate solutions valid up to errors in $O(\delta^q)$, an infinitely small of prefixed order $q$ in $\delta$.

The paper is organized as follows. Section 2 sets the problem and introduces notations and tools. Section 3 describes the sampled state space representations of multirate dynamics with nilpotent Lie algebra at order 3 as well as the Chen series representation of the continuous-time flow. Section 4 gives the main result for computing the sampled controller achieving input-state matching at the sampling instants.
2 Some instrumental tools

In the sequel we compare the behaviours of (1) under (4) or 4-th order multirate sampled-data controls respectively. For, we refer to the time-varying dynamics (5) or the multirate sampled model equivalent to (1) respectively. The following equivalent formulation of assumption A will be referred to:

\[ A : \quad \mathcal{L}^4 = 0 \quad \text{and} \quad \mathcal{L}_0 = \text{span}(g, \text{ad}_f g, \text{ad}_f^2 g, \text{ad}_g \text{ad}_f g). \]

**Definition 2.1.** For a given \( T \geq 0 \) and \( x_0 \in \mathcal{X} \), input-state matching at time \( T \) under multirate control of order \( M \) is solvable for system (1) if for any given smooth control \( u_c(t) \), there exist \( M \) constant controls \( u_{d1} \ldots, u_{dM} \) := \( u_{d1/M} \), each \( u_{dp} \) constant over \( T/M \) - \( M \)-th order multirate control - such that

\[ (6) \quad x(T, u_c, x_0) = x(T, u_{d1/M}, x_0). \]

The problem is said to admit an approximate solution at the order \( q \) if the equality (6) holds true till the approximation at order \( q \) in \( T \) (error in \( O(T^{q+1}) \)). Exact solvability stands for the equality between the asymptotic expansions of the two members in (6) at any order.

### 2.1 The continuous-time flow

Let us first adapt to the present context a result in [13] which describes the flow associated with (5).

**Theorem 2.1.** [13] Given (5), one has for any \( T \geq 0 \) and \( x_0 \in \mathcal{X} \)

\[ (7) \quad x(T, u_c, x_0) = x_0 e^{T \mathcal{F}(f^T_{u_c})} \]

where the exponent series \( T \mathcal{F}(f^T_{u_c}) \) is described by its asymptotic expansion

\[ (8) \quad T \mathcal{F}(f^T_{u_c}) = \sum_{i \geq 1} \frac{T^i}{i!} B_i(f + u_{c0}g, u_{c1}g, \ldots, u_{ci-1}g). \]

Each \( B_i \), the coefficient of \( T^i \) in the asymptotic expansion (8), can be decomposed as a Lie polynomial of degree \( i \geq 1 \) in the vector fields \( f + u_{c0}g := f_0 \) (said of degree 1) and for \( i \geq 1 \), \( u_{ci}g = f_i \) (said of degree \( i + 1 \)).
Proposition 2.1. Under the assumption $A$, the Lie polynomials $B_i$ simplify as

$$B_1 = f_0 = f + u_c g$$

(9)  $$B_{i+1} = u_{c1} g + \frac{1}{2} \sum_{q=0}^{i-1} C^q_i \text{ad}_{B_{i-q}} f_q + \frac{1}{12} \sum_{q=0}^{i-2} \sum_{j=1}^{i-q-1} C^q_i C^j_{i-q} \text{ad}_{B_{j}} \text{ad}_{B_{i-j}} f_q$$

with $C^q_i := \frac{i!}{q!(i-q)!}$.

For the first terms, taking out from the Lie brackets the $u_{c1}$’s assumed constant, we compute under $A$

$$B_1 = f + u_c g; \quad B_2 = u_{c1} g$$

(10)  $$B_3 = u_{c2} g + \frac{u_{c1}}{2} \text{ad}_f g; \quad B_4 = u_{c3} g + u_{c2} \text{ad}_f g$$

$$B_5 = u_{c4} g + \frac{3u_{c3}}{2} \text{ad}_f g + \left(\frac{u_{c2} u_{c0}}{6} - \frac{u_{c1}^2}{2}\right) \text{ad}_g \text{ad}_f g + \frac{u_{c2}}{6} \text{ad}_f^2 g; \quad B_6 = ...$$

2.2 The multirate sampled flow

Assuming in (1), the control $u_c$ constant over small time intervals of amplitude $T > 0$, say $u_c(t) = u_{dk}$ over $[kT, (k+1)T]$, ZOH device, the evolution at times $t = kT$ is described by the sampled model (3) so that the state reached at time $T \geq 0$ from $x_0$ under $u_d = u_d$ is given by

(11)  $$x(T, u_d, x) = x_0 e^{T(f + u_d g)}.$$

Assuming again in (1), the control $u_c$ piecewise constant over time intervals of length $T$ and characterized by 4 possibly different values, $u_{d1} \cdots u_{d4}$ over $[k + \frac{(p-1)T}{4}, k + \frac{pT}{4}]$ - multirate sampling of order 4 - the sampled state evolution at time $t = kT$ is represented by the composition of 4 exponential forms as in (11), each one being parameterized by the corresponding constant value $u_{dp}$. The state attained at time $T$ from $x_0$ is

(12)  $$x(T, u_{d1/...4}, x_0) = x_0 e^{\frac{T}{4}(f + u_{d1} g) \cdots e^{\frac{T}{4}(f + u_{d4} g)}}$$

To replace (12) by a unique exponential form involves the BCH-exponent series. Let us recall for completeness the Baker-Campbell-Haussdorff formula [10]. Denoting by $(X,Y)$ a pair of non commuting formal variables, one has

$$e^X e^Y = e^{BCH(X,Y)}$$
where $BCH(X,Y)$ indicates the Baker-Campbell-Haussdorff exponent described as an infinite Lie series in the variables $(X,Y)$. Provided all the Lie brackets of length greater or equal to 3 vanish, then the $BCH(X,Y)$ exponent simplifies as

$$e^X e^Y = e^{BCH(X,Y)} = e^{X+Y + \frac{1}{12}[X,[X,Y]] + \frac{1}{120}([X,[X,Y]]+[Y,[Y,X]])}.$$ 

2.3 The solution

The solution to the set problem can be directly described by comparing term-by-term the exponent series associated with the continuous-time flow (7) and the multirate sampled flow (12) respectively. One has from (8)

**Proposition 2.2.** Given (5) and assuming $A$, then the multirate sampled control computed to solve term-by-term the equality of series below

$$\sum_{i \geq 1} T^i i! B_i(f+u_{c0}g, u_{c1}g, ..., u_{ci-1}g) = BCH\left(\frac{T}{4}(f+u_{d1}g), ..., \frac{T}{4}(f+u_{d4}g)\right)$$

achieves input-state matching at time $T$.

The proof works out noting that both members in equality (13) can be expressed by means of terms in span $(g, ad_fg, ad^2_fg, ad_gad_fg)$. Some technical arguments guarantee the existence of a multirate controller of order 4 equal to the dimension of $L_0$. It results that each control solution $u_{dp}$ is given by its asymptotic expansion in powers of $T$; i.e.

$$u_{dp}^{T/4} = u_{dp0} + \sum_{i \geq 1} T^i i! 4^i u_{dpi}$$

where the successive $u_{dpi}$ are computed in terms of the constant values $u_{cj}$ in (4) by comparing in (13) terms of the same power in $T$. Arguing so and because of (9), the first terms can easily be computed so providing an approximate multirate control solution which is currently sufficiently accurate in practice. However, to describe the exact solution remains a difficult task which can be greatly simplified following the approach proposed in the sequel. Finally, we note that input-state matching at time $T$ guarantees input-state matching at sampling instants $kT$ for $k \geq 0$.

3 Characterization of the evolutions around the drift

The approach proposed to solve the problem is based on the characterization around the drift of both the evolutions under sampled or continuous-
time controls. It makes reference to the differential/difference representation of sampled dynamics proposed in [11] on one hand and to the Chen series representation of the continuous-time flow proposed in [18] on the other hand. Both representations are specified in the nilpotent case under investigation.

3.1 Under sampled-data control

In the present context, it is suitable to make use of the *Differential Difference Representation - DDR -* of (3) proposed in [11]. Denoting by \( x^+(u) \) a curve in \( \mathbb{R}^n \) parameterized by \( u \), we get under \( A \), the following DDR of (3)

**Proposition 3.1.** The DDR of the sampled equivalent to (11) under \( A \) takes the form

\[
\begin{align*}
(15) \quad x^+ &= x \circ e^{Tf} \\
(16) \quad \frac{dx^+(u_d)}{du_d} &= G^T(x^+(u_d), u_d); \quad x^+(0) = x^+
\end{align*}
\]

with

\[
(17) \quad G^T(., u_d) := \int_0^T e^{-sa} (u_d) ds = G_0^T + u_d G_1^T
\]

and

\[
(18) \quad G_0^T := T g - \frac{T^2}{2!} \text{ad}_f g + \frac{T^3}{3!} \text{ad}_f^2 g \quad \text{and} \quad G_1^T := \frac{T^3}{3!} \text{ad}_g \text{ad}_f g.
\]

The state attained at time \( T \geq 0 \) from initial condition \( x_0 \) under \( u_d \) is computed through integration of (16) around (15) so getting from \( x^+(0) = x_0 \circ e^{Tf} \)

\[
x(T, u_d, x_0) = x^+(u_d) = x^+(0) + \int_0^{u_d} G^T(x^+(v), v) dv.
\]

Thanks to nilpotency, the integration of (16) greatly simplifies, so getting around the free evolution

\[
(19) \quad x(T, u_d, x_0) = x_0 \circ e^{Tf} e^{u_d G_0^T + \frac{T^2}{2!} G_1^T}
\]

Denoting by \( \tilde{G} \), the transport along \( e^{Tf} \) of any vector field \( G \), its existence is ensured as \( e^{Tf} \) is invertible so that \( \tilde{G} := e^{Tf} G e^{-Tf} = e^{T\text{ad}_f G} \), we get equivalently to (11) or (19)

\[
(20) \quad x(T, u_d, x_0) = x_0 \circ e^{u_d G_0^T + \frac{T^2}{2!} G_1^T} e^{Tf}
\]
Under assumption 12, given (13), under multirate control of order 4, we get from Proposition 3.1

Proposition 3.2. Under assumption A, the DDR representation under 4-th order multirate sampling takes the form

\[
\begin{align*}
  x^+ &= x_0 e^{Tf}; \quad x^+(0) = x^+
  \\
  \frac{dx^+(u_{d1/\ldots,4})}{du_k} &= G_p^{T/4}(x^+(u_{d1/\ldots,4}), u_{d1/\ldots,4}); \quad p = 1/\ldots,4
\end{align*}
\]

with

\[
\begin{align*}
  G_1^{T/4}(\ldots, u_{d1/\ldots,4}) &= e^{-T \text{ad}_f + u_{d2}g_s^e} e^{-T \text{ad}_f + u_{d3}g_s^e} e^{-T \text{ad}_f + u_{d4}g_s} G^{T/4}_1(\ldots, u_{d1}) \\
  G_2^{T/4}(\ldots, u_{d2/\ldots,4}) &= e^{-T \text{ad}_f + u_{d3}g_s^e} e^{-T \text{ad}_f + u_{d4}g_s} G^{T/4}_2(\ldots, u_{d2}) \\
  G_3^{T/4}(\ldots, u_{d3/\ldots,4}) &= e^{-T \text{ad}_f + u_{d4}g_s^e} G^{T/4}_3(\ldots, u_{d3}) \\
  G_4^{T/4}(\ldots, u_{d4}) &= G^{T/4}(\ldots, u_{d4}) \\
  G^{T/4} &= \int_0^T e^{-s \text{ad}_f g_s} ds.
\end{align*}
\]

The state attained at time \( T \geq 0 \) from initial condition \( x_0 \) under the sequence \( u_{d1/\ldots,4} \) is computed through successive integrations with respect to \( u_{dp} \) starting from

\[
x(T, u_{d1/\ldots,4}, x_0) = x^+(u_{d1/\ldots,4}) = x^+(u_{d1/\ldots,3}, 0) + \int_0^{u_{d4}} G^{T/4}_4(x^+(u_{d1/\ldots,3}, v), u_{d4}) dv
\]

up to

\[
x^+(u_{d1}, 0/\ldots, 0) = x^+(0/\ldots, 0) + \int_0^{u_{d1}} G^{T/4}_1(x^+(v, 0/\ldots, 0), u_{d1}, 0/\ldots, 0) dv
\]

with \( x^+(0/\ldots, 0) = x_0 e^{Tf} \). One gets in conclusion

Proposition 3.3. Given (1) under A, then the state reached at time \( T \) from \( x_0 \) under multirate control of order 4 admits the asymptotic expansion (12) or equivalently

\[
(22) \quad x(T, u_{d1/\ldots,4}, x_0) = x_0 e^{Tf} e^{u_{d1} G^{T/4}_1} + \frac{3}{4} G_1^{T/4} e^{u_{d2} G^{T/4}_2} + \frac{4}{3} G_2^{T/4} e^{u_{d3} G^{T/4}_3} + \frac{5}{4} G_3^{T/4} e^{u_{d4} G^{T/4}_4} + \frac{6}{5} G_4^{T/4}.
\]
with

\[
G_{10}^{T/4} := e^{-\frac{3T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{7T^2}{24} \text{ad}_f g + \frac{28T^3}{3144} \text{ad}_f^2 g
\]

\[
G_{20}^{T/4} := e^{-\frac{2T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{5T^2}{24} \text{ad}_f g + \frac{14T^3}{3144} \text{ad}_f^2 g
\]

\[
G_{30}^{T/4} := e^{-\frac{T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{3T^2}{24} \text{ad}_f g + \frac{7T^3}{3144} \text{ad}_f^2 g
\]

\[
G_{40}^{T/4} := \frac{T^4}{4} - \frac{T^2}{24} \text{ad}_f g + \frac{T^3}{3144} \text{ad}_f^2 g
\]

\[
G_{p1}^{T/4} = \frac{T^3}{3144} \text{ad}_g \text{ad}_f g
\]

or equivalently

\[
(23) \quad x(T, u_{d1/...4}, x_0) = x_0 e^{u_{d1}G_{10}^{T/4} + \frac{3T}{4} G_{11}^{T/4} + \ldots + u_{d4}G_{40}^{T/4} + \frac{3T}{4} G_{41}^{T/4}} e^{Tf}
\]

with

\[
\tilde{G}_{10}^{T/4} := e^{-\frac{3T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{7T^2}{24} \text{ad}_f g + \frac{28T^3}{3144} \text{ad}_f^2 g
\]

\[
\tilde{G}_{20}^{T/4} := e^{-\frac{2T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{5T^2}{24} \text{ad}_f g + \frac{14T^3}{3144} \text{ad}_f^2 g
\]

\[
\tilde{G}_{30}^{T/4} := e^{-\frac{T}{4} \text{ad}_f G_0^{T/4}} = T - \frac{3T^2}{24} \text{ad}_f g + \frac{7T^3}{3144} \text{ad}_f^2 g
\]

\[
\tilde{G}_{40}^{T/4} := e^{T \text{ad}_f G_0^{T/4}} = T - \frac{3T^2}{24} \text{ad}_f g + \frac{7T^3}{3144} \text{ad}_f^2 g
\]

\[
\tilde{G}_{p1}^{T/4} = \frac{T^3}{3144} \text{ad}_g \text{ad}_f g.
\]

By applying the BCH-formula to (23), we compute

**Proposition 3.4.** Given (1) under $A$, then the state attained at time $T \geq 0$ from initial condition $x_0$ under the sequence $u_{d1/...4}$ admits the asymptotic expansion

\[
(25) \quad x(T, u_{d1/...4}, x_0) = x_0 e^{\sum_{i=1}^{4} u_{d_i} G_{i0}^{T/4} + \frac{3T}{4} G_{i1}^{T/4} + \frac{7T^2}{24} G_{i2}^{T/4} + \frac{21T^3}{3144} G_{i3}^{T/4} + \frac{35T^4}{15552} G_{i4}^{T/4}} e^{Tf}.
\]

### 3.2 Under continuous-time control

Considering again the input-affine dynamics (1) under continuous-time control $u_c(t)$, let us now specify to the nilpotent case under investigation the representation of the associated flow according to the product of Chen series [18]. One obtains

\[
(26) \quad x(T, u_c, x_0) = x_0 e^{\sum_{i=1}^{4} d_i(T,u_c) \text{ad}_g \text{ad}_f g} e^{d_3(T,u_c) \text{ad}_f^2 g} e^{d_2(T,u_c) \text{ad}_g} e^{d_1(T,u_c) \text{ad}_g \text{ad}_f g} e^{Tf}.
\]
with
\[d_2(T, u_c) = \int_0^T u_c(\tau) d\tau\]
\[d_3(T, u_c) = \int_0^T \tau u_c(\tau) d\tau\]
(27) \[d_4(T, u_c) = \frac{1}{2} \int_0^T \tau^2 u_c(\tau) d\tau\]
\[d_5(T, u_c) = \int_0^T \tau (\int_0^\tau u_c(\sigma) d\sigma) u_c(\tau) d\tau.\]

Assuming \(u_c(t)\) in the form (4), one computes
\[d_2(T, u_c) = \sum_{i \geq 0} \frac{T^{i+1}}{(i+1)!} u_{ci}\]
\[d_3(T, u_c) = \int_0^T \tau u_c(\tau) d\tau = \sum_{i \geq 0} \frac{T^{i+2}}{i! (i+2)} u_{ci}\]
\[d_4(T, u_c) = \frac{1}{2} \int_0^T \tau^2 u_c(\tau) d\tau = \frac{1}{2} \sum_{i \geq 0} \frac{T^{i+3}}{i! (i+3)} u_{ci}\]
\[d_5(T, u_c) = \int_0^T \tau (\int_0^\tau u_c(\sigma) d\sigma) u_c(\tau) d\tau = \sum_{i,j \geq 0} \frac{T^{i+j+3}}{(i+1)! j! (i+j+3)} u_{ci} u_{cj}\]

For the first terms in \(O(T^6)\), we get
\[d_2(T, u_c) = T u_{c0} + \frac{T^2}{2} u_{c1} + \frac{T^3}{3!} u_{c2} + \frac{T^4}{4!} u_{c3} + \frac{T^5}{5!} u_{c4}\]
\[d_3(T, u_c) = \frac{T^2}{2} u_{c0} + \frac{T^3}{3} u_{c1} + \frac{T^4}{4} u_{c2} + \frac{T^5}{5} u_{c3}\]
\[d_4(T, u_c) = \frac{T^3}{6} u_{c0} + \frac{T^4}{8} u_{c1} + \frac{T^5}{20} u_{c2}\]
\[d_5(T, u_c) = \frac{T^3}{3} u_{c0}^2 + \frac{3T^4}{8} u_{c0} u_{c1} + \frac{T^5}{10} u_{c1} + \frac{2T^5}{15} u_{c0} u_{c2}\]

By applying the BCH-formula to (26), we compute

**Proposition 3.5.** Given (1) under \(A\), then the state attained at time \(T \geq 0\) from initial condition \(x_0\) under \(u_c(t)\) as in (4) admits the asymptotic expansion

(28) \[x(T, u_c, x_0) = x_0 e^{\mathcal{F}(T, f, g, \text{ad}_f g, \text{ad}_g, \text{ad}_f g, u_{c1})}\]
with
\[ F(T, f, g, ad_f g, ad^2_f g, ad^3_f g, u_{c_1}) = Tf + d_2 g + (d_3 - \frac{1}{2}d_2 d_1)ad_f g \]
\[ + (d_4 - \frac{1}{2}d_3 d_1 + \frac{1}{12}d_2^2 d_2)ad^2_f g + (d_5 - \frac{1}{2}d_3 d_2 - \frac{1}{12}d_2^2 d_1)ad^3_f g. \]

(29) simplifies when the attained state is described around the drift, one gets

**Proposition 3.6.** Given (t) under A, then the state attained at time \( T \geq 0 \) from initial condition \( x_0 \) under \( u_c(t) \) as in (4) admits the asymptotic expansion
\[ x(T, u_c, x_0) = x_0 e^{(d_5 - \frac{1}{2}d_3 d_2)ad_f g + d_4 ad^2_f g + d_3 ad_f g + d_2 g)} e^{Tf} \]

with \( d_i \) as in (27) for \( i = 2, 3, 4 \) and
\[ d_5 - \frac{1}{2}d_3 d_2 = \sum_{i,j \geq 0} \frac{(j - i + 1)T^{i+j+3}}{2(i+1)!j!(j+2)(i+j+3)} u_{c_i} u_{c_j} \]
\[ = \frac{T^3}{12} u_{c_0}^2 + \frac{T^4}{12} u_{c_0} u_{c_1} + \frac{T^5}{60} u_{c_1}^2 + \frac{T^5}{240} u_{c_0} u_{c_2} + O(T^6). \]

4 **The main result**

On the bases of the arguments so far developed, it is easy to understand that given (5) under A, then for any \( x_0 \in \mathcal{X} \) and any time \( T \) small enough, the \( \mathcal{M}R^4 \)-control \( u_{T/4}^{T/4}_{d_1...d_4} \) computed to satisfy \( x(T, u_c, x_0) = x(T, u_{T/4}^{T/4}_{d_1...d_4}, x_0) \), equivalently to satisfy the equality between the asymptotic expansions (23) and (30), can be computed equating the exponents.

**Theorem 4.1.** Given (5) under A, then for any time \( T \) and \( x_0 \in \mathcal{X} \), the \( \mathcal{M}R^4 \)-control \( u_{T/4}^{T/4}_{d_1...d_4} \) computed to satisfy the equality of series
\[ \sum_{i=1}^{4} (u_{d_i}^{T/4} G_{i0}^{T/4} + \frac{(u_{d_i}^{T/4})^2}{2} G_{i1}^{T/4}) + \frac{1}{2} \sum_{j>i=1}^{4} u_{d_i}^{T/4} u_{d_j}^{T/4} [\bar{G}_{i0}^{T/4}, \bar{G}_{j0}^{T/4}] \]
\[ = (d_5 - \frac{1}{2}d_3 d_2)ad_f g + d_4 ad^2_f g + d_3 ad_f g + d_2 g \]
with the \( \bar{G}_{ij} \) given in (24) and the \( d_i \) in (27) achieve input-state matching at the sampling instants.
The proof follows directly from the problem settlement and Proposition 3.4 and Proposition 3.6. Because of (24), the control computation requires the solvability of the three linear and quadratic equalities below with

\[ u_{d_1}^{T/4} = u_{d_0} + \sum_{j \geq 1} \frac{T^j}{j!4^j} u_{d_j} \]

\[
\begin{pmatrix}
\frac{T}{4} & \frac{T}{4} & \frac{T}{4} & \frac{T}{4} \\
\frac{T}{8} & \frac{T}{8} & \frac{T}{8} & \frac{T}{8} \\
\frac{T}{32} & \frac{T}{32} & \frac{T}{32} & \frac{T}{32} \\
\frac{T}{128} & \frac{T}{128} & \frac{T}{128} & \frac{T}{128}
\end{pmatrix}
\begin{pmatrix}
u_{d_1}^{T/4} \\
u_{d_2}^{T/4} \\
u_{d_3}^{T/4} \\
u_{d_4}^{T/4}
\end{pmatrix} =
\begin{pmatrix}d_2 \\
d_3 \\
d_4 \end{pmatrix} :=
\begin{pmatrix}\sum_{i \geq 0} \frac{T^j}{j!4^j} u_{ci} \\
\sum_{i \geq 0} \frac{T^j}{j!4^j} u_{ci} \\
\frac{1}{2} \sum_{i \geq 0} \frac{T^j}{j!4^j} u_{ci}
\end{pmatrix}
\]

and

\[
\frac{T^3}{2.43} \sum_{i=1}^{4} \left(\frac{u_{d_1}^{T/4}}{6} + \frac{u_{d_2}^{T/4}}{3} + \frac{u_{d_3}^{T/4}}{1} \right)
+ \frac{u_{d_4}^{T/4}}{7} = \frac{T^4}{6} + \frac{T^4}{2} + \frac{T^4}{4} + \frac{T^4}{4} + \frac{T^4}{4}
\]

\[ d_5 - \frac{1}{2} d_3 d_2 = \sum_{i,j \geq 0} \frac{(j-i+1)T^{i+j+3}}{2(i+1)!j!(i+j+3)(j+2)} u_{ci} u_{cj}. \]

These equalities can be transformed into the simpler ones below

(32) \[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 5 & 7 \\
1 & 7 & 19 & 37
\end{pmatrix}
\begin{pmatrix}
u_{d_1}^{T/4} \\
u_{d_2}^{T/4} \\
u_{d_3}^{T/4} \\
u_{d_4}^{T/4}
\end{pmatrix} =
\begin{pmatrix}\sum_{i \geq 0} \frac{T^i}{i!4^i} u_{ci} \\
\sum_{i \geq 0} \frac{T^i}{i!4^i} u_{ci} \\
\sum_{i \geq 0} \frac{T^i}{i!4^i} u_{ci} \end{pmatrix}
\]

and

(33) \[
\begin{pmatrix}
u_{d_1}^{T/4} , \nu_{d_2}^{T/4} , \nu_{d_3}^{T/4} , \nu_{d_4}^{T/4}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{6} & 1 & 2 & 3 \\
0 & 1/6 & 1 & 2 \\
0 & 0 & 1/6 & 1 \\
0 & 0 & 0 & 1/6
\end{pmatrix}
\begin{pmatrix}u_{d_1}^{T/4} \\
u_{d_2}^{T/4} \\
u_{d_3}^{T/4} \\
u_{d_4}^{T/4}
\end{pmatrix}
\]

\[
\sum_{i,j \geq 0} \frac{4^3(j-i+1)T^{i+j}}{(i+1)!j!(i+j+3)(j+2)} u_{ci} u_{cj}.
\]

It is easy to check that the solution \( u_{d_1}^{T/4} = u_{c_0} \) satisfies the equalities above (32 -33) for \( T = 0 \), which imply the existence of a solution in the form

\[ u_{d_1}^{T/4} = u_{c_0} + \sum_{j \geq 1} \frac{T^j}{j!4^j} u_{d_2j}. \]
Computing the Jacobian of the equations (32-33) in the unknown $u_{d1}^{T/4}$ and evaluating the result at $u_{d1}^{T/4} = u_{d0}$ and $T = 0$ puts in light that (32) admits a unique solution in terms of one of the $u_{d1}^{T/4}$ while the solution to (33) is not unique; the Jacobian is of rank 3. Solving (32) in terms of $u_{d4}^{T/4}$, easy manipulations show the existence of a unique exact solution after verifying that the $3 \times 3$ matrix in the left hand side is of rank 3. We get

$$
\begin{pmatrix}
    u_{d1}^{T/4} \\
    u_{d2}^{T/4} \\
    u_{d3}^{T/4}
\end{pmatrix} =
\begin{pmatrix}
    11/6 & -1 & 1/6 \\
    -7/6 & 3/2 & -1/3 \\
    1/3 & -1/2 & 1/6
\end{pmatrix}
\begin{pmatrix}
    4d_2 - u_{d4}^{T/4} \\
    2d_3 - 7u_{d4}^{T/4} \\
    3d_3 - 37u_{d4}^{T/4}
\end{pmatrix}.
$$

Replacing these exact solutions in (33), we get an algebraic equation of degree 2 in $u_{d4}^{T/4}$ only. Easy computations show that the discriminant at $T = 0$ of such a two order algebraic equation is positive and non zero so guaranteeing the existence of two solutions for $u_{d4}^{T/4}$. These solutions specify the exact MR^4-control law which achieves input-state matching at the sampling instants of (1) under $u_c(t)$ as in (4). Some computational aspects are detailed below.

### 4.1 Some computational aspects

Let us illustrate the computation over the first terms $u_{d1}$. Setting $u_{d1}^{T/4} = u_{d0} + T^2 u_{d1} + O(T^2)$ in (34), we express each $u_{d1}$ for $i = 1, 2, 3$ in terms of $u_{d4}$; i.e.

$$
\begin{pmatrix}
    u_{d1} \\
    u_{d2} \\
    u_{d3}
\end{pmatrix} =
\begin{pmatrix}
    11/6 & -1 & 1/6 \\
    -7/6 & 3/2 & -1/3 \\
    1/3 & -1/2 & 1/6
\end{pmatrix}
\begin{pmatrix}
    8u_{c1} - u_{d41} \\
    2d_3 - 7d_4 \\
    3d_3 - 37u_{d41}
\end{pmatrix}
$$

or

\begin{align}
    u_{d1} &= 4u_{c1} - u_{d41} \\
    u_{d2} &= -28/3u_{c1} + 3u_{d41} \\
    u_{d3} &= 40/3u_{c1} - 3u_{d41}.
\end{align}

Replacing each $u_{d1}^{T/4}$ by its expression in (36) in the equality of degree 2 below deduced from (33)

$$
\frac{T^3}{243} \sum_{i=1}^{4} (u_{d1}^{T/4})^2 + u_{d1}^{T/4} (u_{d2}^{T/4} + 2u_{d3}^{T/4} + 3u_{d4}^{T/4}) + u_{d2}^{T/4} (u_{d3}^{T/4} + 2u_{d4}^{T/4}) + u_{d3}^{T/4} u_{d4}^{T/4} =
$$

\begin{align}
    &\frac{T^3}{12} u_{c0}^2 + \frac{T^4}{12} u_{d0} u_{c1} + \frac{T^5}{60} u_{c1}^2
\end{align}

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we verify that the terms in $T^3u_{e0}^2$ and $T^4u_{e0}u_{e1}$ coincide in both members of the equality so reducing the problem to the solvability of the equation of degree 2 in $u_{d41}$ below

$$
\frac{T^5}{2.4^5} \left( \frac{1}{6} (4u_{e1} - u_{d41})^2 + (\frac{-28}{3} u_{e1} + 3u_{d41})^2 + (\frac{40}{3} u_{e1} - 3u_{d41})^2 + u_{d41}^2 \right)
+ (4u_{e1} - u_{d41}) (-\frac{28}{3} u_{e1} + 3u_{d41}) + 2(\frac{40}{3} u_{e1} - 3u_{d41}) + 3u_{d41})
+ (\frac{-28}{3} u_{e1} + 3u_{d41}) (\frac{40}{3} u_{e1} - 3u_{d41}) + 2u_{d41}) + (\frac{40}{3} u_{e1} - 3u_{d41})u_{d41})
= \frac{T^5}{60} u_{e1}^2.
$$

Such a property is a consequence of two facts, a solution exists because the discriminant of such a two order algebraic equation is real positive and the rank 3 of the Jacobian implies nonuniqueness of the solution. This two order algebraic equation can be rewritten as

$$
-\frac{9}{4} u_{d41}^2 + 18u_{e1} u_{d41} - \frac{179}{5} u_{e1}^2 = 0
$$

so getting the two solutions

$$
(38) \quad u_{d41} = (4 + 2\sqrt{\frac{5}{15}})u_{e1} \quad \text{or} \quad u_{d41} = (4 - 2\sqrt{\frac{5}{15}})u_{e1}.
$$

In conclusion the $\mathcal{MR}^4$-control law (36) with $u_{d41}$ as in (38) achieves input-state matching at order 2 in $T$ in the direction of $g$, order 3 in $T$ in the direction of $ad_{fg}$, at order 4 in $T$ in the direction of $ad_{fg}^2$ and $ad_{f}ad_{fg}$ of the continuous-time dynamics (1) under continuous-time control (4).

The successive terms $u_{dij}$ for $j \geq 2$ can be computed according to similar arguments. More precisely, for a given $j \geq 1$, the $u_{dij}$ for $i = 1, 2, 3$ are first expressed in terms of $u_{d4j}$ according to

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 5 & 7 \\
1 & 7 & 19 & 37
\end{pmatrix}
\begin{pmatrix}
u_{d1j} \\
u_{d2j} \\
u_{d3j} \\
u_{d4j}
\end{pmatrix}
= \begin{pmatrix}
\frac{4j+1}{2} u_{e1} \\
\frac{4j+2}{3} u_{e1} \\
\frac{4j+3}{5} u_{e1}
\end{pmatrix}
$$

so getting

$$
\begin{pmatrix}
u_{d1j} \\
u_{d2j} \\
u_{d3j}
\end{pmatrix}
= \begin{pmatrix}
11/6 & -1 & 1/6 \\
-7/6 & 3/2 & -1/3 \\
1/3 & -1/2 & 1/6
\end{pmatrix}
\begin{pmatrix}
\frac{4j+1}{2} u_{e1} - u_{d4j} \\
\frac{4j+2}{3} u_{e1} - 7u_{d4j} \\
\frac{4j+3}{5} u_{e1} - 37u_{d4j}
\end{pmatrix}.
$$

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Replacing in (33) each $u_{di}$ for $i = 1, 2, 3$ by its expression above parameterized by $u_{di}$, we get again a two order algebraic equation

\[
\frac{1}{6} \sum_{i=1}^{4} u_{di}^2 + u_{d11}(u_{d21} + 2u_{d31} + 3u_{d41}) + u_{d21}(u_{d31} + 2u_{d41}) + u_{d31}u_{d41} = 2.4^{j+3}u_{cj}^2 \frac{(j+1)(2j+3)}{j+1}
\]

which corresponds to equate terms in $T^{2j+3}$ and which can be solved in $u_{d4j}$. This possibility is again due to the fact that all the previous terms in $T^{i+l+3}_{u_{ci}u_{cl}}$ for $i, l \leq j$ and $i + l < 2j$ are equal in both members of the equality (33) due to singularity of the Jacobian and existence of a solution; i.e. for $i, l \leq j$ and $i + l < 2j$

\[
= \frac{4^{i+1}(l-i+1)u_{ci}u_{cl}}{(i+1)(i+l+3)(l+2)}.
\]

5 Concluding remarks

The problem of matching the input-state evolutions of a given input-affine dynamics under piecewise constant controls has been investigated under nilpotency at order three of the Lie algebra of controllability. The approach proposed is based on multirate sampled-data schemes and computational expansions introduced by the authors in the analysis and design of sampled-data control systems and the Chen-Sussmann representation of continuous-time flows.

References


