Exponential representations of Volterra-like expansions: an explicit computation of the exponent series

S. Monaco, D. Normand-Cyrot, and C. Califano

Dipartimento di Informatica e Sistemistica ‘‘Antonio Ruberti’’
Università di Roma ‘‘La Sapienza’’, Via Eudossiana 18, 00184 Rome, Italy.
Laboratoire des Signaux et Systèmes, CNRS-ESL,
Plateau de Moulon, 91190 Gif-sur-Yvette, France.

Abstract

The paper concerns the exponential form representation of the flow associated with the solution of time varying nonlinear differential equations. Starting from the known fact that the exponent is a Lie series, an iterative computation of the successive Lie polynomials which define this exponent is proposed. The result is specified in the case of nonlinear difference equations.

1 Introduction

There exists a vast literature oriented to characterize the Volterra series representations associated with nonlinear differential equations (see for example [2], [8], [7], [14], [15], [17]). These works essentially concern state affine equations which are linear in the control variables. The object of the present work is to generalize these results to more intrinsically case represented by time varying differential equations and discrete-time ones. This extension is done making use of chronological series and combinatorial tools ([8], [1], [5], [16]). In the discrete-time case the study is based on a preliminary representation of difference equations as differential ones [12]. The main contribution is to show that the Volterra-like series expansions which characterize the input-to-state behaviors of these dynamics exhibit exponential form representations, whose exponent series can be explicitly computed in terms of Lie polynomials. The proof is constructive. While the first two parts of this result were already known as existence results [11], the last part is constructive and constitutes the main originality of the present work. This result further specifies the use of Lie algebras and Lie group techniques in the context of nonlinear equations and the analogies between time-varying differential equations and difference equations.

The paper is organized as follows. Section 2 introduces some tools and results issued from chronological calculus. Then the case of time-varying equations is discussed and the results are proposed in a Theorem which specifies the three aspects. In Section 3, after recalling how difference equations can be seen as differential ones, the results of Section 2 are specified in this context.

2 Time-varying differential equations

Consider at first the time invariant nonlinear differential equation

\[ \dot{x}(t) = f(x(t)) \] (1)

with \( x \in \mathbb{R}^n \) and \( f \) an analytic vector field on \( \mathbb{R}^n \) - an infinitely differentiable function admitting a convergent Taylor series - and assume it complete - every solution of (1) exists for \( t \in \mathbb{R} \). Choosing at time \( t_0 = 0 \), a state \( x_0 \), for small values of \( t \), the solution \( x(t) \) is given by the flow

\[ x(t) = \Phi(t, 0, x_0), \]

which admits an exponential form representation. This is an immediate consequence of a Taylor type series expansion of the solution; i.e.

\[ x(t) = x_0 + \sum_{m \geq 1} \frac{t^m}{m!} x^{(m)}(t) \] (2)

where \( x^{(m)}(t) := \frac{d^m x(t)}{dt^m} |_{t=0} \). In this simple case, according to the differential equation (1), we have \( x^{(m)}(t) = L_f^m(I_{dx}) |_{x_0} \), where \( I_{dx} \) represents the identity function on \( \mathbb{R}^n \), \( L_f(t) = \frac{d}{dt} f \), and \( L_f^m := L_f \circ L_f^{m-1} \), so that we get

\[ x(t) = \Phi(t, 0, x_0) = e^{t L_f} (I_{dx}) |_{x_0} \] (3)

where by definition, \( e^{t L_f} := I + \sum_{m \geq 1} \frac{t^m}{m!} L_f^m \), with \( I \) the identity formal operator.

Consider now a time-varying differential equation of the form

\[ \frac{d}{dt} x(t) = f(x(t), t) \] (4)

Authorized licensed use limited to: Universita degli Studi di Roma La Sapienza. Downloaded on April 23,2010 at 13:36:44 UTC from IEEE Xplore. Restrictions apply.
and assume the vector field \( f(\cdot, \cdot) \), analytic and complete on \( \mathbb{R}^n \times \mathbb{R} \). Let the expansion with respect to \( t \) around \( t = 0 \) of \( f(\cdot, t) \) be

\[
f(\cdot, t) := f_0(\cdot) + \sum_{i \geq 1} \frac{t^i}{i!} f_i(\cdot)
\]

(5)

with \( (f_i(\cdot))_{i \geq 0} \), analytic vector fields on \( \mathbb{R}^n \).

A non-autonomous complete vector field \( f(\cdot, t) \) defines the associated flow as the unique solution of the operator equation

\[
\frac{d}{dt} \Phi(t, 0, x_0) = f(\Phi(t, 0, x_0), t).
\]

so that

\[
x(t) := \Phi(t, 0, x_0). \tag{6}
\]

Generalizing the previous arguments, this flow can be either characterized by a series of multiple integrals, called the right chronological exponential, or through Taylor series type expansions. The right chronological exponential in \( f(\cdot, t) \) is defined in the literature (see [1], [5], [16]) by its asymptotic expansion below

\[
\exp \int_0^t L_{f(\cdot, r)} dr := 1 + \sum_{m \geq 1} \int_0^t \cdots \int_0^t L_{f(\cdot, r_m)} \cdots L_{f(\cdot, r_1)} dr_m \cdots dr_1.
\]

(7)

The object of the present work is to further characterize the expansion of (7) with respect to \( t \). To do so, we note that the solution to (4) can be written as

\[
x(t) = x_0 + \int_0^t f(x(\tau), \tau) d\tau \tag{8}
\]

and that, for any analytic function \( \lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), we have

\[
\lambda(x(t), t) = \lambda(x_0, t) + \int_0^t \frac{d(\lambda(x(\tau), t))}{dr} d\tau
\]

(9)

Substituting \( \lambda(\cdot, t) \) with \( f(\cdot, r) \) into (9), we get

\[
f(x(\tau), \tau) = f(x_0, \tau) + \int_0^\tau L_{f(\cdot, r_2)} f(x(\tau_2), \tau_2) dr_2
\]

which can be substituted into (8). Iterating the procedure we get the solution as the series of iterated integrals below

\[
x(t) = x_0 + \int_0^t f(x_0, \tau_1) d\tau_1 + \sum_{m \geq 2} \int_0^t \cdots \int_0^t L_{f(\cdot, r_m)} \cdots L_{f(\cdot, r_1)} (I_{\mathbb{R}^n})|_{x_0} | d\tau_m \cdots d\tau_1
\]

thus recovering the asymptotic expansion of the chronological series, i.e.

\[
x(t) = \Phi(t, 0, x_0) = \exp \int_0^t L_{f(\cdot, r)} (I_{\mathbb{R}^n})|_{x_0} | ds. \tag{10}
\]

Let us now generalize the Taylor type series expansion to this time varying case by introducing the "extended differential operator" \( \mathcal{D}_s \), defined on \( \mathbb{R}^{n+1} \), as

\[
\mathcal{D}_s := L_{f(s, \cdot)} + \frac{\partial}{\partial s} \tag{11}
\]

According to this definition, the solution \( x(t) \) of (4), still satisfies (2), with presently, \( x^{(m)}(s) := \frac{d^m x(s)}{ds^m} \big|_{s=0} = \mathcal{D}_s^m (I_{\mathbb{R}^n})|_{x_0, 0} \), thus getting

\[
x(t) = e^{\mathcal{D}_s (I_{\mathbb{R}^n})|_{x_0, 0} \cdot t} := (I + t L_{f_0} + \sum_{m \geq 2} \frac{t^m}{m!} \mathcal{D}_s^m (I_{\mathbb{R}^n})|_{x_0, 0}) \tag{12}
\]

By equating (10) and (12), we get the solution

\[
x(t) = \Phi(t, 0, x_0) = \exp \int_0^t L_{f(s, \cdot)} (I_{\mathbb{R}^n})|_{x_0} | ds
\]

(13)

To recover the expansion with respect to \( t \) of the flow, the following computations are performed

\[
\mathcal{D}_s(I_{\mathbb{R}^n})|_{x_0, 0} = L_{f_0} (I_{\mathbb{R}^n})|_{x_0}
\]

\[
\mathcal{D}_s^2(I_{\mathbb{R}^n})|_{x_0, 0} = (\frac{\partial}{\partial s} L_{f(s, \cdot)} + L_{f(s, \cdot)} L_{f(s, \cdot)} (I_{\mathbb{R}^n})|_{x_0, 0})
\]

\[
= (L_{f_1} + L_{f_2}^2 (I_{\mathbb{R}^n})|_{x_0})
\]

\[
\mathcal{D}_s^3(I_{\mathbb{R}^n})|_{x_0, 0} = (L_{f_2} + L_{f_1} L_{f_0} + 2 L_{f_0} L_{f_1} + L_{f_3}^3 (I_{\mathbb{R}^n})|_{x_0})
\]

(14)

Further performing these computations, it becomes possible to regroup, according to the successive powers of \( t \), the expansion of \( x(t) \) thus getting

\[
x(t) = x_0 + \sum_{m \geq 1} \sum_{\iota=0}^{m} \frac{1}{m!} \sum_{\iota \geq 0} \cdots \sum_{\iota \geq 0} c(i_1, \ldots, i_m) L_{f_{i_1}} \cdots L_{f_{i_m}} (I_{\mathbb{R}^n})|_{x_0}
\]

where the \( c(i_1, \ldots, i_m) \)'s are real coefficients satisfying

\[
\mathcal{D}_s^m (I_{\mathbb{R}^n})|_{x_0, 0} = \sum_{\iota \geq 0} \cdots \sum_{\iota \geq 0} c(i_1, \ldots, i_m) L_{f_{i_1}} \cdots L_{f_{i_m}} (I_{\mathbb{R}^n})|_{x_0}
\]

(15)

Let us now state the complete result.
Theorem 2.1 [Exponential representation of time-varying dynamics] The asymptotic behaviour of a time-varying dynamics of the form (4) admits the exponential representation

\[ z(t) = e^{tF(\cdot,t)}(f_{0\alpha}) \bigg|_{x_0}, \]

where \( tF(\cdot,t) : R^n \to R^n \), a smooth vector field parameterized by \( t \), is a Lie element in \( \{ f_i(\cdot) : i \geq 0 \} \)s given by the expansion

\[ tF(\cdot,t) = \sum_{i \geq 1} t^i B_i(f_0, \ldots, f_{i-1}). \]  

(16)

\( B_i \) stands for a homogeneous Lie polynomial of degree \( i \) in its arguments for \( f_i(\cdot) \) said, by convention, to be of degree \( i + 1 \).

• The expansion (16) satisfies

\[ \frac{d}{dt} tF(\cdot,t) = \frac{\text{ad}_x tF(\cdot,t)}{1 - e^{-\text{ad}_x tF(\cdot,t)}}, \]

\[ := Z(\text{ad}_x tF(\cdot,t))f(\cdot,t). \]

(17)

where the function \( Z(\cdot) \) is defined by its Taylor expansion

\[ Z(\psi) = \sum_{i \geq 0} (-1)^i b_i \psi^i \frac{i!}{i!}. \]

The coefficients \( b_i \) are the Bernoulli numbers. For the first, one has: \( b_0 = 0, b_1 = -1/2, b_2 = 1/6, b_{2k+1} = 0 \) for \( k > 0, b_k = -1/30, b_0 = 1/42, b_2 = -1/30 \).

• The decomposition of \( B_i \), for \( i \geq 1 \) can be iteratively computed from \( B_1 = f_0 \), according to

\[ B_{i+1} = \frac{f_i}{(i+1)!} \]

(18)

+ \( \sum_{0 \leq k < i} \sum_{j \geq \max(0, i-k)} \sum_{\pi \in S_{i+1}} (-1)^j b_j \text{ad}_B_{i-k} \ldots \text{ad}_B_{i-j}, \)

with, for any \( j \geq 1 \), \( \sum_{p=1}^{j} t_p + k = i \).

Proof: The proof of the first item is immediately deduced from Lemma 2.1 below and a result in [13] which claims that the formal logarithmic expansion of a series of the form (14), with coefficients satisfying the shuffle relations, is a Lie element.

Lemma 2.1 The real coefficients \( c(i_1, \ldots, i_m) \) defined in (15) verify the shuffle relations

\[ c(i_1)c(i_2) = c(i_1 \cup i_2) := c(i_1, i_2) + c(i_2, i_1), \]

\[ c(i_1)c(i_2, i_3) = c(i_1 \cup i_2, i_3) := c(i_1, i_2, i_3) + c(i_2, i_1, i_3) + c(i_2, i_3, i_1), \]

\[ \vdots \]

\[ c(i_1, \ldots, i_m) := c(i_1 \cup i_2 \cup \ldots \cup i_m). \]

the shuffle product, denoted by "\( \cup \)" , is defined in a recursively way as

\[ 1 \cup i_1 = i_1 \cup \ldots \cup i_1 = \ldots \]

\[ i_1 \cup i_2 = i_2 \cup i_1 = \ldots = i_1 \ldots i_m \cup \ldots \cup i_p = \ldots = \]

\[ i_1(i_2 \ldots i_m \cup \ldots \cup i_p) + \ldots. \]

By reversing the logarithmic expansion we get the result expressed in the first item accordingly to the notation

\[ \log(1 + \sum_{m=1}^{\infty} \mathcal{D}^m_t |_{s=0} := tF(\cdot,t). \]

The second item enlightens the decomposition of the \( (B_i)'s \) as Lie polynomials in the \( (f_i)'s \). The proof is a direct application of the next equality which characterizes the derivative of the formal logarithmic expansion of the flow - or of the chronological series expansion - associated with a differential equation. More precisely

Lemma 2.2 Setting \( \psi = \text{ad}_x \log(\Phi(\cdot,t)) \), one has

\[ \frac{d}{dt} \log(\Phi(\cdot,t)) = \frac{\psi}{1 - e^{-\psi}} f(\cdot,t) = Z(\psi)f(\cdot,t). \]

(19)

when

\[ \Phi(\cdot,t) := \exp \int f(\cdot,s)ds := e^{tF(\cdot,t)}. \]

The formal equality (19), or related versions, are proposed in the literature about chronological calculus (see for example [1], [16]), a combinatoric proof is reported in the appendix.

Accordingly to the present context, equality (19) can be rewritten as

\[ \frac{d}{dt} tF(\cdot,t) = \frac{\text{ad}_x tF(\cdot,t)}{1 - e^{-\text{ad}_x tF(\cdot,t)}}, \]

\[ := Z(\text{ad}_x tF(\cdot,t))f(\cdot,t). \]

(20)

The last step consists in substituting the function \( \frac{\psi}{1 - e^{-\psi}} \) with its Taylor expansion involving the Bernoulli numbers, see [6]; i.e.

\[ \frac{\psi}{1 - e^{-\psi}} = \sum_{i \geq 0} (-1)^i b_i \psi^i \frac{i!}{i!}. \]

The last item is constructive with respect to the decompositions of \( B_i \). To show it, let us rewrite the equality (20) in terms of the expansions of \( tF(\cdot,t) \) and \( f(\cdot,t) \) thus getting

\[ B_1 + \sum_{i \geq 2} (i+1)t^i B_{i+1} = \]

\[ (1 + \sum_{i \geq 1} (-1)^i b_i \text{ad}_B_{i+1}f_0 + \sum_{i \geq 1} t^i f_i). \]

(21)
with
\[ a_{i_1, i_2, \ldots, i_n} = \sum_{i_{p+1} > i_{p}} d^{(i_p)} a_{B_{i_1}, \ldots, a_{B_{i_n}}}, \]
so that we rewrite the last equality as
\[
B_1 + \sum_{i \geq 1} (i + 1) t^i B_{i+1} = f_0 + \sum_{i \geq 1} \frac{t^i}{i!} f_i \\
+ \sum_{i \geq 1} \sum_{j \geq 1} \sum_{k \geq 1} (-1)^k \frac{b_k}{j!} a_{B_{i_1}, \ldots, a_{B_{i_n}}} f_0 \\
+ \sum_{i \geq 1} \sum_{j \geq 1} \sum_{k \geq 1} (-1)^k \frac{b_k}{j!} a_{B_{i_1}, \ldots, a_{B_{i_n}}} \left( \frac{t^j}{j!} \right).
\]

By equating the terms of the same power in \( t \), we prove the last item of Theorem 2.1. For the first terms, we have
\[
t F(.; t) = t B_1(f_0) + t^2 B_2(f_0, f_1) + t^3 B_3(f_0, f_1, f_2) + \\
+ t^4 B_4(f_0, f_1, f_2, f_3) + O(t^5)
\]
\[
B_1 = f_0, \quad 2B_2 = b_0 f_1, \\
3B_3 = \frac{f_3}{3!} - b_1 a_{B_2, f_0} - b_0 a_{B_1, f_1}, \\
4B_4 = \frac{f_4}{4!} - \frac{b_1}{2} a_{B_3, f_1} - b_0 a_{B_2, f_0} + \\
+ b_2 a_{B_2, B_1, f_1} + b_1 a_{B_1, B_0, f_0}
\]
or equivalently
\[
B_1 = f_0, \quad B_2 = \frac{1}{2!} f_1, \\
B_3 = \frac{1}{3!} (f_3 + \frac{1}{2} f_0 f_1), \quad B_4 = \frac{1}{4!} (f_4 + [f_0, f_2]),
\]
where \( a_{F, G} := [F, G] := L_F G - L_G F \), indicates the Lie bracket of two vector fields.

### 3 Discrete-time Volterra expansions

Let us apply the results of Theorem 2.1 to nonlinear difference equations. To do so, we first need to recall a representation of difference dynamics as differential ones proposed in [12]. Let the nonlinear first order difference equation be of the general form
\[
x_{k+1} = F(x_k, u_k)
\]
where the mapping \( F(x, u) \) is assumed to be analytic on its domain of definition. For \( u \in U_0 \), let the expansion of \( F(x, u) \) with respect to \( u \), around \( u = 0 \), be
\[
F(.; u) = F_0(.; u) + \sum_{i \geq 1} \frac{u^i}{i!} F_i(.; u),
\]
where, \( F_0(x) := F(x, 0) \) and the \( \langle F_i(x) \rangle_{i \geq 0} \) are analytic on \( \mathbb{R}^n \).

The following assumption is at the basis of the definition below.

**H1:** There exists an analytic function \( G(x, u) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) satisfying
\[
G(F(x, u), u) = \frac{\partial}{\partial u} (F(x, u)).
\]

Let, for \( u \in U_0 \), the expansion of \( G(x, u) \) be
\[
G(x, u) = G_1(x) + \sum_{i \geq 1} \frac{u^i}{i!} G_{i+1}(x).
\]
Denoting \( F_i(.; u) \) by \( x^+(u) \) to point out that we will consider \( F \) as a function of \( u \), the next definition has been proposed in [12].

**Definition 3.1** [Analytically parameterized single-input discrete-time dynamics - APDTD]

Let \( (x, u) \in \mathbb{R}^n \times U_0 \), where \( U_0 \) is a neighbourhood of \( 0 \) in \( \mathbb{R} \) and assume the maps analytic in their arguments. Given \( F_0(.) \) on \( \mathbb{R}^n \) and a parameterized vector field \( G_i(u) \), on \( \mathbb{R}^n \), an analytically parameterized single-input discrete-time dynamics is defined by the two equations
\[
x_{k+1} = F_0(x)
\]
\[
\frac{d}{du}(x^+(u)) = G^0(x^+(u), u); \quad x^+(0) = x_0^+.
\]

**Some comments**

- As a matter of fact, a nonlinear difference equation of the form (22) can be recovered by integrating (27) between 0 and \( u_k \) with the initial condition \( x_0^+ \) computed from (26) setting \( x_{k+1} := F(x_k) \); i.e.,
\[
x(k + 1) := x^+(u_k) = x_0^+ + \int_0^{u_k} G(x^+(v), v) dv.
\]
The so computed \( F(.; u) \) is analytic on \( \mathbb{R}^n \) and such that \( F(.; 0) = F_0(.) \).

- This definition reinforces a formal link between continuous-time and discrete-time equations. Equations (26) and (27) may represent a wide class of dynamics as (27) is control dependent. In fact, it can be shown, see [11], that the case \( G_i(u) := G_i, \) or equivalently \( G_i = 0 \), for \( i \geq 2 \) turns out to be comparable, with respect to the difficulties encountered for its study and the reached results, to input-affine differential dynamics extensively studied in the continuous-time case.
3.1 Chronological series and exponential representations

Let an - APDTD given by (26, 27) and assume $G_i(\cdot,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ analytic and complete - every solution of (27) exists for $u \in \mathbb{R}$.

Accordingly to Section 2, let $\Phi(u,0,\cdot)$ be the flow associated with $G_i(\cdot,u)$, defined as the unique solution of the operator equation

$$\frac{d}{du} \Phi(u,0,\cdot) = G_i(\Phi(u,0,\cdot),u)$$

and let the right chronological exponential in $G_i(\cdot,u)$ be

$$\exp \sum_{\alpha=1}^{u} L_{G_i(\alpha,u)} dv = I + \sum_{n \geq 1} \sum_{0 \leq \alpha \leq n} \frac{1}{n!} L_{G_i(\alpha,u)}^n$$

The following proposition is immediate.

Proposition 3.1 Given a real analytic vector field $F_i(\cdot,u)$ on $\mathbb{R}^n$ and assuming $H_1$ then, for each $x \in \mathbb{R}^n$ there exists an open neighbourhood $U_0$ of $0 \in \mathbb{R}^n$, depending on $x$, and a smooth map $\Psi_i(\cdot,u) : U_0 \times \mathbb{R} \to G_i(x,\cdot)$ such that the solution of (27), initialized at (26) takes the form

$$x_t^+(u) := \Phi(u,0,x_0^+).$$

Let us now state the main result.

Theorem 3.1 Under $H_1$, the asymptotic behaviour of a discrete-time dynamics of the form (22) admits the exponential representation

- $x_{i+1} = e_1 G_i(x_i,u)$,

where $e_1 G_i(\cdot,u) := \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field parameterised by $u$; it is a Lie element in the $(G_i(\cdot) : i \geq 1)$'s given by its expansion

$$u G_i(\cdot,u) := \sum_{1 \leq i \leq n} u_i B_i(G_1,\ldots,G_i).$$

where $B_i$ stands for a homogeneous Lie polynomial of degree $i$ in its arguments for $G_i(\cdot)$ said, by convention, of degree $i$.

- The expansion (30) satisfies

$$\frac{d}{du} u G_i(\cdot,u) = Z(-\alpha u G_i(\cdot,u))$$

- The decomposition of $B_i$, for $i \geq 1$ can be iteratively computed from $B_i = G_1$, according to

$$B_{i+1} = \frac{G_{i+1}}{(i+1)!} + \sum_{i+1}^{i+1} \sum_{1 \leq j \leq i+1} \sum_{k=1}^{j} \frac{(-1)^{j-k} B_k G_{i+1-k}}{j!}$$

with, for any $f \geq 1$, $\sum_{i=0}^{k} f_i = k = i$.

The proof is the same as for Theorem 2.1 after substituting $f_i(\cdot,u)$ with $G_i(\cdot,u)$ or equivalently for $i \geq 0$ each $f_i(\cdot)$ with $G_{i+1}(\cdot)$, and $\Phi(u,0,\cdot)$ with $e^{\nu u G_i(\cdot)}$.

For the first terms of the expansion (30), we find

$$u G_i(\cdot,u) = u B_1 G_1 + u^2 B_2 G_2 + u^3 B_3 G_3 + u^4 B_4 G_4 + \cdots + u^i B_i G_i$$

with

$$B_1 = G_1, B_2 = \frac{1}{2!} G_2.$$  
$$B_3 = \frac{1}{3!}(G_3 + \frac{1}{2!}[G_1,G_2]), B_4 = \frac{1}{4!}(G_4 + [G_1,G_2] + [G_2,G_3]).$$

The existence of a formal exponential representation of the flow associated with the solution to (27) has already been proved in [11] - first item of Theorem 3.1. The main contribution of the present paper is the explicit computation of the expansion with respect to $u$ of the Lie series exponent - second item of Theorem 3.1. The proof is constructive - third item of Theorem 3.1.

3.2 Some examples linear in $x$

Consider a first-order input-linear discrete-time dynamics,

$$\frac{d}{da} x^+_a = A_0 x_a$$

$$\frac{d}{da} (x^+_a(u)) = A_1 x^+_a(u) \quad x^+_a(0) = x^+_a$$

where $A_0$ and $A_1$ are real square matrices of order $n$.

It is easily verified that the associated integrated form admits a closed form which is nonlinear with respect to $u$ and simply reduces to the exponential of a matrix. One has

$$x_{a+1} := x^+_a(u) = \Phi(u_a,0,x_0^+)$$

$$= A_0 x_a + \int_0^{x_a} A_1 x^+_a(u) du = v_1 A_1 A_0 x_a.$$

Consider a $p$-order input-polynomial discrete-time dynamics defined by the two equations,

$$x^+_a = A_0 x_a$$

$$\frac{d}{du} (x^+_a(u)) = \sum_{i=1}^{p} \frac{u^i}{i!} A_{i} x^+_a(u) \quad x^+_a(0) = x^+_a$$

for some square matrices, $A_0, A_1, \ldots, A_p$ of order $n$. The associated integrated form is nonlinear with respect to $u$, one has

$$x_{a+1} := x^+_a(u) = \Phi(u_a,0,x_0^+)$$

$$= e^{v_1 A_1 A_0 x_a}.$$
with \( u A(u)x = \sum_{i \geq 1} u^i B_i(A_1, \ldots, A_i) \) and
\[
B_1 = A_1 x, \quad B_2 = \frac{1}{2!} A_2 x, \quad B_3 = \frac{1}{3!} (A_3 + \frac{1}{2} [A_2, A_1]) x, \\
B_4 = \frac{1}{4!} (A_4 + [A_3, A_1]) x,
\]
where \([, , ]\) indicates the Lie bracket of matrices.

Appendix

Let \( \Lambda \) be the formal algebra of all formal power series in the non commutative indeterminates \( X \) and \( Y \). Write
\[
v := Y + \frac{YX + XY}{2} + \ldots + \frac{YX^{p-1} + XYX^{p-2} + \ldots + XP^{-1}Y}{p!} + \ldots
\]
and \( Q := \frac{1 - e^{-ad_x}}{ad_x} \). Let \( P_X \) be the linear operator defined by \( P_X Y = XY \), then \( Q \) and \( P_X \) commute and direct computations give
\[
(ad_X)(Q(e^{pX})Y) = (e^{pX} - e^{pX - ad_x})Y = e^X Y - Y e^X = ad_X v.
\]
Since \( \ker(ad_X) \) consists in all the formal series in \( X \) only, it follows that
\[
v = Q(e^{pX})Y = Q e^X Y = e^X QY
\]
or equivalently
\[
Y = Q^{-1} e^{-X} v = \frac{ad_X}{1 - e^{-ad_x}} e^{-X} v.
\]

Setting \( X = t F(. , t), \ Y = \frac{dt(F(. , t))}{dt}, \ v = \frac{d(e^{tF(.)})}{dt} \) we get the result expressed in Lemma 2.2 because
\[
e^{-X} v := f(. , t).
\]

4 Conclusions

It has been shown that the asymptotic expansion of time varying differential equations admits an exponential representation whose Lie series exponent can be recursively computed thanks to chronological calculus tools. This has been specified to the case of nonlinear difference equations preliminarily rewritten as differential equations with respect to the input variable. The single input case has been considered. The generalization of these manipulations to the multi-input case keeps valid thanks to quite heavy notations. According to the same lines it is also possible to deal with parameter dependent or singularly perturbed dynamics.

References


