Quadratic Stabilization of Systems with
Period Doubling Bifurcation

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1 Introduction

To set the study of dynamical systems in terms of specific normal forms is a powerful tool. Such an approach, developed in both cases of dynamical systems defined by vector fields (differential dynamical systems) or maps (discrete-time systems), provides new coordinates under which the systems are transformed into their “simplest” forms [4].

The normal form approach was generalized to control systems with controllable linearization in [14]. This is of peculiar interest since the normal form is then used for investigating stabilizability and for computing a stabilizing controller. This approach can be extended to systems with uncontrollable linearization. In this case, a bifurcation occurs, and the formalism of bifurcation theory can be adopted to conclude about stabilizability or not. The case of differential dynamics has been studied in [15, 16, 10, 11] while difference dynamics are considered in [7, 8, 12, 13, 17, 11]. The analysis and control of unparameterized discrete-time systems with one uncontrollable mode was done in [9], while the parameterized case was done in [7, 8]. Quadratic and cubic normal forms for the general case in discrete-time, without parameters, were derived in [17]. Systems exhibiting a Naimark-Sacker bifurcation were treated in [12, 13, 11] and the present paper deals with systems exhibiting a period doubling bifurcation. A complete discussion about discrete-time systems with one real uncontrollable mode is proposed in the thesis [11].

Period-doubling bifurcations appear, in several applications (see for example [5, 19]), and have been treated in [3, 20] in the discrete-time case where the authors used Taylor series expansions and stability theorems to characterize the stability of the bifurcation. This approach results in a rather tedious computations since all the terms involved in the Taylor expansion of the vector field associated to the system are used. In our paper, thanks to the normal form approach, we only use the terms (quadratic invariant terms) which are involved to solve the stabilization problem. The general stabilization procedure combines centre manifold techniques, normal forms and second-order static state feedbacks. In short, the linear part of the feedback is designed to stabilize the controllable subsystem while its quadratic part is designed for modifying the manifold over which the closed loop reduced dynamics evolves. Classical stability theorems for systems with bifurcations are then used to prove the stability of such a dynamics, under suitable conditions.

Section 2 is devoted to the computation of the normal form and its characterization in terms of invariants, a set of numbers which entirely specifies the normal form. In section 3, we find the centre manifold and then solve the stabilization problem related to the period doubling bifurcation by means of a quadratic controller.

2 Normal Forms and Invariants

Let us consider a single input, parametrized, nonlinear discrete-time dynamics on $\mathbb{R}^n$, described by the nonlinear difference equation

$$\xi^{k+1} = f(\xi^k, \mu, u)$$

where $\xi \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $\mu \in \mathbb{R}$ is a constant parameter, $f(\xi, \mu, u) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is analytic in its arguments. We adopt for any integer $k \geq 0$, the notation $\xi^k := \xi(k+1) := f(\xi(k), \mu, u(k))$. We assume that 0 is an equilibrium point associated to $\mu = 0$, $u = 0$ (i.e. $f(0, 0, 0) = 0$) and that the linear approximation around $(0, 0, 0)$, given by $A := \frac{df}{du}(0, 0, 0)$ and $B := \frac{df}{d\mu}(0, 0, 0)$ is such that

Assumption 1

$$\text{rank}([B \ AB \ A^2B \ \ldots \ A^{n-1}B]) = n - 1,$$  

with the uncontrollable part characterized by one real mode, $\lambda = -1$.

2.1 Linear Normal Form

We first compute the linear normal form of (1) as it is essential to characterize the stability of the whole dynamics.
Lemma 2.1 There exist a linear coordinates change and a feedback under which the system (1), satisfying Assumption 1, takes the form
\[
\begin{align*}
\dot{z}^+ &= -\hat{z} + f_1^2(\hat{z}, \mu) + g_1^1(\hat{z}, \mu) u + O(\hat{z}, \mu, v)^3 \\
\dot{\hat{z}}^+ &= A_2 \hat{z} + B_2 u + f_1^2(\hat{z}, \mu) + g_1^1(\hat{z}, \mu) v + O(\hat{z}, \mu, v)^3 \\
\end{align*}
\]
where \( \hat{z} = [\hat{z}^T, \hat{z}^T]^T, \hat{z} \in \mathbb{R}, \hat{z} \in \mathbb{R}^{n-1} \), the matrices \((A_2, B_2)\) are in the Brunovsky form.
\[
A_2 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & 0 & 1 \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
\vdots \\
0 \\
0 \\
1
\end{bmatrix}.
\]

(4)

\(A_2 \in \mathbb{R}^{(n-1) \times (n-1)}, B_2 \in \mathbb{R}^{(n-1) \times 1}, f_1^2, f_2^3\) stand for the homogeneous polynomials of degree 2 (resp. 1).

In (3), \(O(\hat{z}, \mu)^3\) contains the remaining terms of order \(\geq 3\) in the expansion. In the sequel, for any analytic function \(\Lambda\), for any \(i \geq 0\), \(\Lambda[i]\) denotes the homogeneous \(i\)-th order polynomial of its expansion.

Proof: From linear control theory, we know that there exist a linear coordinates change and a linear feedback, independent on \(\mu\), transforming (1) into
\[
\dot{z}^+ = -\hat{z} + d_{11} \mu + O(\hat{z}, \mu, v)^2 \\
\dot{\hat{z}}^+ = A_2 \hat{z} + B_2 u + D_2 \mu + O(\hat{z}, \mu, v)^2
\]
with \(D_2 = [d_{21}, d_{22}, \ldots, d_{2n-1}]^T\). Then, we easily recover (3) by choosing in addition
\[
\hat{z} = \hat{z} + \frac{d_{21}}{2} \mu, \quad \hat{z} = \hat{z} - \begin{bmatrix}
0 \\
d_{21} \\
\vdots \\
d_{2n-1}
\end{bmatrix} \mu, \quad v = v - \sum_{k=1}^{n-1} d_{2k} \mu.
\]
In the following, \(\Sigma_{PD}\) will denote the dynamics (3).

2.2 The Quadratic Normal Form

Quadratic transformations are defined by the coupled action of a quadratic coordinates change and a quadratic static state feedback of the form,
\[
\begin{align*}
z &= \tilde{z} + \varphi^3(\tilde{z}, \mu) \\
v &= u + \alpha^3(\tilde{z}, \mu) + \beta^3(\tilde{z}, \mu) u
\end{align*}
\]
where \(\varphi^3, \alpha^3, \beta^3\) denote homogeneous polynomials in their arguments. Such a transformation is employed to simplify the quadratic part of (3) while leaving the linear part unchanged. Finally, the quadratic normal form is characterized in the next theorem.

Theorem 2.1 (Quadratic Normal Form) There exists quadratic transformations (5) under which (3) takes the form
\[
\begin{align*}
z^+ &= -z + \sum_{i=1}^{n-1} \delta_{i} z_i^2 + \beta \mu + (\gamma_1 \mu + \delta_{21} u + \gamma_{1z} x_1 + O(\mu)^3 \\
x^+ &= A_2 x + B_2 u + \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} e_{ik} e_{2i} x_k^2 + \delta_{21} u x_1 + O(\mu)^3
\end{align*}
\]
where \(\delta_{i}, \beta, \gamma_1, \delta_{21}, \gamma_{1z}, e_{1, k}, e_{2i}\) are real constants and \(e_{21}\) is the \(i\)-th unit vector in the \(z\)-space.

The proof of Theorem 2.1 is given in [11], it is presently omitted for space limitations.

2.3 The Quadratic Invariants

We show in this section that the quadratic normal form (6) can be characterized by the quadratic invariants, a set of numbers which remain unchanged under quadratic transformations. Moreover, the coefficients of the quadratic normal form can be uniquely determined in terms of these invariants. situation, for the

Theorem 2.2 Consider system \(\Sigma_{PD}\). The coefficients of the normal form of the \(z\)-part, are given by
\[
\begin{align*}
\gamma_1 &= \frac{\partial^2 f_1}{\partial z \partial x_1}(0, 0, 0), \\
\beta &= \frac{\partial^2 f_1}{\partial z \partial \mu}(0, 0, 0), \\
\gamma_1 &= \frac{\partial^2 f_2}{\partial \mu \partial x_1}(0, 0, 0), \\
\delta_{21} &= \frac{\partial^2 g_2}{\partial u \partial x_1}(0, 0, 0), \\
\delta_i &= \frac{\partial^2 f_1}{\partial z_i \partial x_1}(0, 0, 0); \quad 1 \leq i \leq n - 1
\end{align*}
\]
with \(f_1, g_1, f_2\) given by (9). These coefficients are invariant under change of coordinates and feedback.

The following theorem specifies the properties of these numbers as invariants.

Theorem 2.3 Consider the dynamics (9), then

i. The list of numbers (7) do not change under quadratic transformations of the form (5).

ii. The list of numbers (7) are uniquely associated with the coefficients of the quadratic terms in the normal form (6).

iii. Given two dynamics of the form (9), with the same linearization (same \(\omega\), they are quadratically equivalent if and only if the lists of numbers (7) are equal.

The numbers (7) will be denoted as quadratic invariants.
3 Control design

The general procedure combines centre manifold techniques, normal form and the use of second-order static state feedback to achieve stabilizability conditions. Briefly speaking, the linear part of the feedback is designed to stabilize the controllable $x$-part. Then, the basic idea is to suitably modify through the quadratic part of the feedback the manifold equation so as to ensure stability of the closed-loop reduced dynamics. Classical stability theorems of systems with bifurcations can be used to conclude. It has to be noted that the control law is designed by solving a matrix inequality depending on the quadratic invariants.

Let the overall quadratic stabilizing feedback be

$$u(z, x, \mu) = R_1 z + R_2 x + R_3 \mu + [z^T \mu]^T Q_{f,b} [z^T \mu]^T + O(\cdot)^3,$$

where $R_1$, $R_2 := [r_1^T, \ldots, r_{n-1}^T]$, $R_3 := r_3^T$, are row vectors with constant entries ($r_i^2 \neq 0$), and $Q$ is a square matrix of order 2.

First, the coefficients $r_i^2$ stabilize the $x$-part while the stabilization of the $z$-dynamics is achieved through centre manifold techniques. More precisely, let $R_2$ be such that the matrix $A_2 + B_2 R_2$ has eigenvalues inside the unit disk. Due to the structure of $A_2$ and $B_2$, $A_2 + B_2 R_2$ exhibits the canonical form,

$$A_2 + B_2 R_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
r_1^2 & r_2^2 & r_3^2 & \cdots & r_{n-1}^2 
\end{bmatrix},$$

its characteristic polynomial is $P(\lambda) := \lambda^{n-1} - \sum_{i=1}^{n-1} r_i^2 \lambda^{n-i} - 1$.

3.1 Computation of the centre manifold

Let us determine the equations of the centre manifold $z = \Pi(z, \mu)$. We first compute in Lemma 3.1 the linear part of the center manifold.

Lemma 3.1 Consider system $\Sigma_{PD}$ in its normal form (6). The linear part of the center manifold is given by

$$x_i = \Pi_{1,i}^1 z + \Pi_{2,i}^1 \mu + O(z, \mu)^2,$$  \hspace{1cm} (9a)

$$x_i = (-1)^{i-1} \Pi_{1,i}^1 z + \Pi_{2,i}^1 \mu + O(z, \mu)^2, 1 \leq i \leq n - 1.$$  \hspace{1cm} (9b)

and

$$\begin{align*}
\Pi_{1,1}^1 &= \frac{R_1}{P(-1)} \\
\Pi_{2,1}^1 &= \frac{R_3}{P(1)}
\end{align*}$$  \hspace{1cm} (10)

Remark. $P(-1)$ and $P(-1)$ are different from zero, since $A_2$ is Hurwitz and so $P(\mu) = 0$ only for $\mu$ such that $|\mu| < 1$.

Proof: The linear part of the center manifold has the form:

$$x_i = \Pi_{1,i}^1 [z^T \mu]^T + O(z, \mu)^2, \quad i = 1, \ldots, n - 1.$$  \hspace{1cm} (11)

with $\Pi_{1,i}^1 = [\Pi_{1,i}^1 \Pi_{2,i}^1]$. Hence

$$x_i^+ = \Pi_{1,i}^1 [z^T \mu]^T + O(z, \mu)^2 = \Pi_{1,i}^1 [-z^T \mu]^T + O(z, \mu)^2$$  \hspace{1cm} (12)

Moreover, the $x$ part has the form

$$\begin{align*}
x_{i+1} &= x_i + O(z, \mu)^2, \quad 1 \leq i \leq n - 2, \\
x_{i+1}^+ &= R_1 x_i + R_3 \mu + \sum_{i=1}^{n-2} r_i^2 x_i + O(z, \mu)^2, \quad i = n - 1.
\end{align*}$$  \hspace{1cm} (13)

Since equations (12) and (13) are equivalent, and using (11), we get

- For $1 \leq i \leq n - 2$, we have

$$\begin{align*}
\Pi_{1,i+1}^1 &= -\Pi_{1,i}^1 = (-1)^i \Pi_{1,i}^1 \\
\Pi_{2,i+1}^1 &= \Pi_{2,i}^1
\end{align*}$$  \hspace{1cm} (14)

- For $i = n - 1$

$$\begin{align*}
R_1 + \sum_{i=1}^{n-2} r_i^2 \Pi_{1,i}^1 &= -\Pi_{1,n-1}^1 \\
R_1 + \sum_{i=1}^{n-2} r_i^2 \Pi_{2,i}^1 &= \Pi_{2,n-1}^1
\end{align*}
$$

using (14), we deduce

$$\begin{align*}
\Pi_{1,i}^1 &= \frac{R_1}{(-1)^{n-1} - \sum_{i=1}^{n-1} (-1)^{i-1} r_i^2} = \frac{R_1}{P(-1)} \\
\Pi_{2,i}^1 &= \frac{R_3}{1 - \sum_{i=1}^{n-1} r_i^2} = \frac{R_3}{P(1)}
\end{align*}$$

The linear part of the center manifold can be cancelled choosing $R_1$ and $R_3$ in an appropriate fashion.

Proposition 3.1 Given the linear closed-loop dynamics

$$z^+ = -z$$

$$x^+ = A_2 x + B_2 (R_1 z + R_2 x + R_3 \mu)$$

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there exists a linear coordinate change under which the $(z,\mu)$-terms are removed from the $x$-dynamics thus getting

$$
\dot{z}^+ = -\hat{z}
$$
$$
\dot{\hat{z}}^+ = A_2\hat{z} + B_2 R_2 \hat{z}
$$

(15)

\textbf{Proof:} Just set $\tilde{z} = x - z$, $\tilde{x}_i = z_i - (1)^{i-1} \Pi_{1,1}^{j} z_i - \Pi_{2,1}^{j} \mu$ for $i = 1, \ldots, n - 1$ and use (10).

Applying now the feedback (8) to the normal form (6), we write the centre manifold equation which turns out to depend only on the quadratic part of the feedback. Because of Proposition 3.1, we will assume below $R_1 = 0$ and $R_2 = 0$ in (8).

The next theorem gives the explicit expression of the centre manifold, which don’t contain a linear part using proposition 3.1.

**Theorem 3.1** Given the quadratic feedback (8) then, in the coordinates defined in Proposition 3.1, the centre manifold results to be quadratic with the $\hat{z}$-component given by

$$
\Pi_{1,2}^{j}(z,\mu) = [z \quad \mu] Q_i [z \quad \mu],
$$

(16)

with $Q_i$ directly linked to $Q_i$ (through equations (17)-(18) below).

\textbf{Proof:} Consider the normal form (6) and the feedback (8) and assume the centre manifold to be, $x = \Pi(z,\mu)$. According to proposition 3.1, there exists a linear change of coordinates which permits to cancel the linear part of $\pi_i$ and so the linear part part of the center manifold. The $i$-th component of the quadratic part of the center manifold is given by:

$$
\pi_i = [z \quad \mu] Q_i [z \quad \mu],
$$

$i = 1, \ldots, n - 1$

$Q_i$ are real $2 \times 2$ symmetric matrices.

From the centre manifold equation $\pi^\prime = \Pi(\pi^\prime, \mu)$, we get the equations:

$$
Q_{i+1} = \begin{pmatrix} (-1)^i & 0 \\ 0 & 1 \end{pmatrix} Q_i \begin{pmatrix} (-1)^i & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 \leq i \leq n - 2
$$

(17)

and

$$
Q_{11} = \frac{Q_{10}^{j/2}}{Q_{10}^{j/2}},
$$

$$
Q_{12} = \frac{Q_{11}}{Q_{10}},
$$

$$
Q_{12} = \frac{Q_{11}^{j/2}}{Q_{10}} (Q_{10}^{j/2} - \frac{p^{(j)} - \sum_{j=1}^{j=2} \frac{p(1) - \sum_{j=1}^{j=2} \frac{1}{p(-2)}}{p(-2)}}{Q_{12}^{j/2}})
$$

(18)

Theorem 3.1 implies that, for any given matrix $Q_i$, there always exists $Q_{f_2}$, linked to it through (18), so that the associated feedback (8) yields a centre manifold satisfying (16).

**3.2 Stabilization of $\Sigma_P D$**

In this section we will give sufficient conditions for the stabilization of $\Sigma_P D$ and we will design a controller. By stabilization we mean the transformation of the period doubling bifurcation to a supercritical one (see theorem 3.2). Thanks to the theorem 3.2, hereafter, it will be proved that the stability of the period-doubling bifurcation is determined by $Q_i$ and the invariants.

**Theorem 3.2** [6] Consider the dynamics in $\mathbb{R}$. Suppose that for $\mu = 0$, the point $x = 0$, is a non hyperbolic equilibrium point of $z^\prime = \Phi(x,\mu)$, with $(x,\mu) \in \mathbb{R}^2$, and $\Phi_x(0,0) = -1$. If

$$
\hat{\Phi} = 2\Phi_{x\mu} + \Phi_{\mu} \Phi_{xx} \neq 0
$$

and

$$
\hat{\Phi} = \frac{1}{2} \Phi_{xx}^2 + \frac{1}{3} \Phi_{xxx} \neq 0
$$

(the derivatives are evaluated at $(0,0)$).

A period two limit cycle bifurcates from $(0,0)$ for $\mu > 0$ if $\Phi < 0$ or $\mu < 0$. The equilibrium point from which these solutions bifurcates is stable for $\mu > 0$ (resp. $\mu < 0$) and unstable for $\mu < 0$ (resp. $\mu > 0$), for $\dot{\Phi} > 0$ (resp. $\dot{\Phi} < 0$). The bifurcating limit cycle of period two is stable if it coexists with an unstable equilibrium point and nice-verse. The bifurcation is supercritical if the bifurcating solution is stable; it is subcritical if it is unstable.

Substituting (16) into the $z^\prime$ dynamics of the normal form (6), one computes the critical coefficients $\hat{\Phi}$ and gets the next theorem.

**Theorem 3.3** Given a discrete-time dynamics in the normal form (6) with $\beta 1 \neq 0$. If

$$
\beta \left( -\gamma \frac{Q_{11}}{F_1} + \frac{p^{(j)} f_1^{(3)}(z,\mu)}{z^3} \right)_{z=0,\mu=0} < 0
$$

with $f_1^{(3)}$ the cubic part of the $z$-part of the normal form. Then, there exists a stabilizing control law (8) such that the period doubling bifurcation is supercritical.

In conclusion, in the coordinate system given by Proposition 3.1, the stabilizing controller is given by

$$
u(\tilde{x},z,\mu) = R_2 \tilde{x} + \begin{pmatrix} z \quad \mu \end{pmatrix} Q_{f_2} \begin{pmatrix} z \quad \mu \end{pmatrix} + O(\tilde{x},z,\mu)^3,
$$

(19)

with $A_2 + B_2 R_2$ Hurwitz, and $Q_{f_2}$ given by (18).

4 Conclusion

The stabilization of a discrete-time controlled dynamical systems with period doubling bifurcation is studied using quadratic normal forms, centre manifold techniques and quadratic feedbacks. A procedure for designing a quadratic controller is proposed.
References


