Quadratic dynamic feedback linearization with observer in discrete time

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I. INTRODUCTION

The design of nonlinear discrete-time control schemes is quite difficult to handle. This is in part due to the nonlinear action of the control generated by the composition of functions. It results that input-state and input-output behaviours are not easily characterized and that the conditions for feedback linearization under static feedback are quite restrictive as an example. In order to relax some of these conditions and inspired from the literature in the continuous-time case [11], [10], [14], [8], discrete-time feedback linearization in an approximated meaning can be studied. A first step in this direction is to consider quadratic approximations. The interest of such approximation at the degree two has been recently stated in [1] where it is shown that any linearly controllable nonlinear discrete-time dynamics can be linearized up to an error of order three thanks to a dynamic state feedback. This result is recalled hereafter. In order to enlarge the applicability of the control scheme it is completed with an observer scheme. Let the quadratically approximated system \( \Sigma \):

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) + Nz(k)w(k) + Mu(k)^2 \\
&+ Pz(k) \otimes x(k) + O(z(k),w(k),u(k))^2
\end{align*}
\]

\[
y(k) = Cx(k) + Dz(k) \otimes x(k) + O(z(k),w(k),u(k))^2
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), \( y \in \mathbb{R} \), and \( A, B, M, N, P, C, D \) are matrices of appropriate dimensions and where \((A,B,C)\) is assumed controllable and observable with \( A \) invertible. According to the notion of state reconstruction (see [4], [5], [13] for nonlinear discrete-time systems), one proposes to add a dead beat part to a bilinear observer [12] so as to keep the linear filtering action. Coupling this observer with the linearizing quadratic feedback, closed loop stability is not guaranteed a priori. In order to ensure the stability of the closed loop system up to an error of order three in all the variables, one designs, as in [7], the dynamic feedback on the basis of the observer dynamics. The proof omitted for space constraints are detailed in a full version of the paper.

II. OBSERVER DESIGN

Let us define a quadratic observer with dead beat part \( \hat{\Sigma}_b \):

\[
\begin{align*}
\dot{\hat{x}}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) + \hat{M}u(k)^2 + \hat{O}(y(k),\hat{L}(k)u(k)) \\
&+ \hat{\theta}\hat{L}(k) \otimes \hat{L}(k)
\end{align*}
\]

\( \hat{A} \) is chosen stable and \( \hat{L}(k) \), a computable vector, is equal to \((l_0(k),...,l_{n-1}(k))^T\) with

\[
\begin{align*}
l_0(k) &= y(k) = Cx(k) + O(z(k),w(k),u(k))^2 \\
l_1(k) &= y(k-1) + CA^{-1}Bu(k-1) = CA^{-1}z(k) + O(z(k),w(k),u(k))^2 \\
l_2(k) &= y(k-2) + CA^{-2}Bu(k-2) = CA^{-2}z(k) + O(z(k),w(k),u(k))^2
\end{align*}
\]

\[
\hat{U} \in \mathbb{R}^{nxn} \text{ and } \hat{\Theta} \in \mathbb{R}^{nxn^2}.
\]

Combining (1) and (2), one obtains the error dynamics

\[
e(k+1) = \hat{A}e(k) + (A - \hat{A})Cz(k) + (N - \hat{N})Qz(k)w(k)u(k) + (P - \hat{P}Qz(k)Qz(k) + O(z(k),w(k),u(k))^2
\]

where \( Q_0 \) is equal to \((C^T,...,(CA^{-n+1})^T)\).

It can be easily seen that under the conditions

\[
\begin{align*}
(i) & A - \hat{A}G = 0 \\
(ii) & N - \hat{N}Q_0 = 0 \\
(iii) & P - \hat{P}Q_0Q_0 - \hat{Q}D = 0
\end{align*}
\]

one obtains

\[
e(k+1) = \hat{A}e(k) + O(z(k),w(k),u(k))^2
\]

As the pair \((A,C)\) observable, conditions \(i)\) to \(iii)\) are always solvable so that one can state

PROPOSITION 1: Given the linearly observable system \( \Sigma \) there exists a quadratic observer with a dead beat part \( \hat{\Sigma}_b \) such that the observer error dynamic is equal to

\[
e(k+1) = \hat{A}e(k) + O(z(k),w(k),u(k))^2
\]

III. DYNAMIC QUADRATIC FEEDBACK

In this section, one recalls from [1] how any quadratic dynamics can be linearized under dynamic feedback if its linear approximation is controllable (see [10] in the continuous-time case). Let us define a discrete-time quadratic dynamic feedback as

\[
\begin{align*}
w(k+1) &= a^{[0]}(z(k),w(k),u(k)) + \delta^{[1]}(z(k),w(k),u(k)) \\
w(k) &= a^{[0]}(z(k),w(k),u(k)) + \delta^{[1]}(z(k),w(k),u(k))
\end{align*}
\]

where \( w \in \mathbb{R}^r \), \( u \in \mathbb{R} \) \((\mu \text{ the external control})\) and \( a^{[0]}, \delta^{[1]} \) are vector functions of appropriate dimensions, \( a^{[0]}, \delta^{[1]} \) are scalar functions and \( \delta^{[1]}(\cdot) \) denotes a term of order 1 with
respect to its arguments. One recalls the result of [1].

**Theorem 1:** Any linearly controllable discrete-time quadratic dynamics (1) can be transformed under diffeomorphism and dynamic state-feedback into the following extended dynamics

\[
x(k+1) = A_1x(k) + B_1u(k) + O(x,u)^3
\]

with the pair \((A_1, B_1)\) controllable.

In the next section, the observer is slightly modified in order to ensure the closed loop stability when it is coupled with the dynamic quadratic state feedback. Stability is proved up to the order three with respect to \(x, e\) and \(w\).

**IV. Closed Loop Stability up to the Order Three**

Let us consider again the system (1) assumed linearly observable and controllable and satisfying the assumption

**Assumption A.1** The "linear" relative degree of system (1) is equal to \(n_i\) i.e. \(CA^jB = 0\ \forall \ j < n_i\) and \(CA^{n_i-1}B \neq 0\).

Now, let us consider a quadratic observer with dead beat part \(\mathcal{O}_d\), which is similar to (2) but with a vector \(L(k)\) instead of \(L(k)\) and \(L_n(k)\) instead of \(y\) for causality arguments.

\[
\dot{\hat{y}}(k+1) = \hat{\dot{A}}(k) + \hat{B}(k) + M\hat{u}(k)^2 + \hat{O}(L_n(k) + \hat{L}(L) \otimes L(k))
\]

with \(L(k) = (i_n(k), \ldots, i_{2n}(k))^T\) and \(L_n(k) = CA^{n_i-1}x(k) + O(x,u)^2\)

where \(Q_n = ((CA^{n_i-1})_{k=0}^n)^T\). Combining (1) and (8), one gets the error dynamics

\[
e(k+1) = \hat{\dot{A}}(k) + (A - \hat{A} - \hat{\dot{A}}CA^{n_i-1})e(k) + (N - \hat{N}CA^{n_i-1})e(k)u(k)
\]

choosing \(\hat{A}\) stable,

\[
A - \hat{A} - \hat{\dot{A}}CA^{n_i-1} = 0
\]

\[
N - \hat{N}CA^{n_i-1} = 0
\]

the conditions are solvable because of the observability of the pair \((A, C)\). Thus, one obtains

\[
e(k+1) = \hat{\dot{A}}e(k) + O^2(x,u,e)
\]

Replacing now \(x\) by \(\hat{e}\) in (6), one gets

\[
u(k+1) = \hat{d}[\hat{\dot{e}}(k), w(k), u(k), \mu(k)] + \hat{\beta}[\hat{\dot{e}}(k), w(k), u(k), \mu(k)]
\]

In order to be able to prove the closed loop stability, one designs the control scheme on the basis of \(\mathcal{O}_d\) as proposed in [7]. Arguing so, one can state the main result.

**Theorem 2:** Under the assumption A.1, the extended closed loop system composed with the quadratic system \(\Sigma_d\), the observer \(\mathcal{O}_d\) and the dynamic feedback (12), can be transformed under diffeomorphism into

\[
e(k+1) = \hat{A}e(k) + O^2(x,u)
\]

\[
\dot{\chi}(k) = A_2\chi(k) + B_2\mu + E_2\sigma(k) + O^2(x,u)
\]

where \(e \in \mathbb{R}^n, \chi = \phi(z, w) \in \mathbb{R}^{2n}, (A_2, B_2)\) is a controllable pair and \(\hat{A}\) is chosen stable.

By an adequate choice of \(\mu_i\), one immediately verifies

**Corollary 1:** Under the assumption A.1, the closed loop system composed of the quadratic system \(\Sigma\), the observer \(\mathcal{O}_d\) and the dynamic feedback (12), can be locally stabilized.

In this paper a quadratic observer coupled with a quadratic dynamics feedback linearization proposal is proposed. Quadratic stability is ensured as well as linear error filtering. An important open question in this context is the necessity or the opportunity to impose a linear filtering in a nonlinear context? Concerning higher-order extensions of this approach, work is progressing in terms of \(K Nil\) [9]. In order to fulfill reference trajectory tracking, it has been proposed to extend feedback linearization conditions around an equilibrium manifold [14], [6], [2].

**References**


