AN APPROACH TO NONLINEAR DISCRETE-TIME
H$_\infty$-CONTROL*

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Abstract

Following recent works proposed in the continuous-time context concerning the nonlinear equivalent of the H$_\infty$-control problem and its connection with game theory and Isaacs equation, the present paper sets and studies the same problem in the discrete-time context. A digital static feedback law achieving closed-loop stability and disturbance attenuation is firstly designed under full information assumption and then by making use of an observer.

1 Introduction

H$_\infty$-control theory gives an answer to a major control problem, which is to conceive controllers not designed for a single plant under known inputs, but for a class of plants under unknown inputs which can be disturbances, for instance.

H$_\infty$-control, initiated by Zames [22], only arose in the beginning of the eighties.

Initially designed in the frequency-domain, the H$_\infty$-controller goal was to minimize the maximal norm, i.e. the H$_\infty$-norm of an input-output operator linking, for example, an error to an unknown disturbance; the maximum being to be taken over the whole class of disturbances. The first solutions were thus elaborated in the frequency-domain [7], [8]. Later works on a characterization of H$_\infty$-controllers in the time domain showed that a certain Riccati equation was playing an essential role in the resolution of the linear problem [6], [9], [11].

In parallel, in [1], [16], [17], [21], setting the H$_\infty$-control problem as an optimization one, a natural link with differential linear quadratic game theory is proposed.

In particular, in this context, new developments in the nonlinear continuous time case are made possible [10], [12], [14]. In these recent works the existence of a controller solving the problem is shown to be related to the existence of a solution of a particular type of Hamilton-Jacobi equation, known as Isaacs equation. Moreover a solution to such an equation exists under certain assumptions [14], [19].

The present paper deals with the nonlinear discrete-time H$_\infty$-problem. We prove that the existence of a controller providing a solution is related to the existence of a solution to a discrete-time version of Isaacs equation. The solution proposed verifies some extra assumptions which are strictly related to the discrete-time context.

The classic “full-information” case is firstly solved. On this basis, sufficient conditions for the existence of a solution to the H$_\infty$-control problem via measurement feedback, using the same classic type of observer that was used in the continuous-time case [13], are proposed.

The paper is organized as follows. The problem is formulated in a discrete-time context in section 2. Section 3 deals with the “full-information” problem and we will give a solution via measurement feedback in Section 4.

An extended study of this problem where complete proofs are given can be found in [10].

2 Problem formulation

Consider a discrete-time system described by equations of the form:

\[
\begin{align*}
\dot{x}_{k+1} &= f(x_k) + g_1(x_k)u_k + g_2(x_k)w_k \\
z_k &= h_1(x_k) + k_11(x_k)w_k + k_12(x_k)u_k \\
y_k &= h_2(x_k) + k_21(x_k)w_k + k_22(x_k)u_k
\end{align*}
\]

(2.1)

The input variables are denoted by $w \in \mathbb{R}^{m_1}$ (exogeneous input) and $u \in \mathbb{R}^{m_2}$ (control input). The output variables are denoted by $z \in \mathbb{R}^p$ (tracking error) and $y \in \mathbb{R}^q$ (measured variables). The mappings $f(x)$, $g_1(x)$, $g_2(x)$, $h_1(x)$, $h_2(x)$, $k_11(x)$, $k_12(x)$ and $k_21(x)$ **Research supported in part by grants from MURST in Italy and MEN in France**
and \( k_{21}(x) \) are smooth mappings defined in a neighbourhood of the origin in \( \mathbb{R}^n \).

We assume the existence of an equilibrium \( x_0 = 0 \), i.e. \( f(0) = 0 \), and providing a suitable change of coordinates, we assume that \( h_1(0) = 0 \) and \( h_2(0) = 0 \).

Consider a controller described by equations of the form:

\[
\begin{align*}
\theta_{k+1} &= p(\theta_k, y_k) \\
u_k &= q(\theta_k)
\end{align*}
\]

where we assume that \( \theta \) is defined in a neighbourhood of the origin in \( \mathbb{R}^m \), \( p(\theta, y) \) and \( q(\theta) \) are \( C^k \) functions (for some \( k \geq 1 \)).

The purpose of this controller is to provide:

- Local asymptotic stability of the equilibrium \((x, \theta) = (0,0)\).
- Disturbance attenuation in the sense of satisfying the inequality:

\[
\sum_{k=0}^{N} z_k^T z_k \leq \gamma^2 \sum_{k=0}^{N} w_k^T w_k \quad \forall N \in \mathbb{N}
\]

for every sequence \( w = (w_0, \ldots, w_N) \) such that the resulting trajectory remains in a neighbourhood of \( x = 0 \).

The approach here chosen, has been motivated by several developments made in nonlinear continuous-time and linear discrete-time contexts, relying on game theory (see [1], [12], [21]).

The idea is to associate with the \( H_\infty \) discrete-time problem a two-players, zero-sum, difference game:

\[
x_{k+1} = f(x_k) + g_1(x_k)u_k + g_2(x_k)w_k
\]

of fixed duration \( k = 0, \ldots, N \), with value functional

\[
J_N(u, w) = \sum_{k=0}^{N} z^T (x_k, u_k)z(x_k, u_k) - \gamma^2 w_k^T w_k
\]

where \( u \) and \( w \) denote the sequences \((u_k)_{k=0}^{N}\) and \((w_k)_{k=0}^{N} \). Moreover we note \( z(x_k, u_k) = h_1(x_k) + k_{12}(x_k)u_k \).

In our setting exogeneous sequence \( w \) stands for the maximizing player, while control sequence \( u \) stands for the minimizing player whose goal is to achieve (2.3). This kind of situation is studied in [2], p. 254, where the following solution is given:

For a two-players, zero-sum, discrete-time dynamic game of fixed duration \( k = 0, \ldots, N \), the set of strategies \((w^*_k(x), u^*_k(x))\) provide a feedback saddle point solution (i.e., \( J_N(u^*, w^*) \leq J_N(u, w^*) \leq J_N(u^*, w^*) \)) if and only if there exists a function \( V_N(.) : \mathbb{R}^n \rightarrow \mathbb{R} \) such that the following recursive equation is satisfied:

\[
V_N(x) = \min_{w} \max_{u} \left[ V_{N+1}(f(x) + g_1(x)w + g_2(x)u) + \gamma^2 w^T w \right]
\]

Equation (2.4) is a discrete-time equivalent of Isaacs equation.

Since we are interested in finding a time invariant control law, we consider only one function \( V(.) : \mathbb{R}^n \rightarrow \mathbb{R} \). Thus, Isaacs equation can be written in the form:

\[
V(x) = \min_{w} \max_{u} \left[ V(f(x) + g_1(x)w + g_2(x)u) + \gamma^2 w^T w \right]
\]

Moreover, we will not discuss Isaacs condition in the sequel (i.e. interchangeability of operations max and min), because we do not need to ensure that the couple \((w^*(x), u^*(x))\) provides a saddle point solution.

In order to simplify the forthcoming developments, we assume that the mappings characterizing plant (2.1) satisfy the classic assumptions (see [14] for example):

\[
k_{11}(x) = 0 \quad k_{12}^T(x)k_{12}(x) = I \quad k_{12}^T(x)h_1(x) = 0
\]

We will first deal with the "full information" problem, where both state \((x_k)_{k \in \mathbb{N}}\) and exogeneous input \((u_k)_{k \in \mathbb{N}}\) are available for measurement.

### 3 "Full information" problem

As a result of the considered hypotheses, system (2.1) is now described by the following equations:

\[
\begin{align*}
x_{k+1} &= f(x_k) + G(x_k)u_k \\
\dot{x}_k &= h_1(x_k) + k_{12}(x_k)u_k \\
y_k &= [x_k]
\end{align*}
\]

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where \( G(x) = [g_1(x)g_2(x)] \).

Theorem 1 provides a solution to this particular case, in a function \( V(.) \) which is supposed to verify Isaacs equation (2.5), which will turn into a Hamilton-Jacobi type equation by computation of \( (w^*(x), w^*(x)) \).

Let us first give a discrete-time equivalent of a basic definition about detectability (see [3], [14] in the continuous-time case), which will be used in the sequel.

Definition: Suppose \( f(0) = 0 \), \( h(0) = 0 \). The pair \( \{h, f\} \) is said to be detectable if there exists a neighbourhood \( U \) of \( 0 \) such that if \( x_0 \in U \), any trajectory of \( x_{k+1} = f(x_k) \) is such that \( h(x_k) \) is defined for all \( k \in \mathbb{N} \) and verifies:

\[
h(x_k) = 0 \quad \forall k \in \mathbb{N} \quad \Rightarrow \quad \lim_{k \to \infty} (x_k) = 0
\]

Theorem 1. Suppose:

(i) \( h_1(x), f(x) \) locally detectable around \( x = 0 \).

(ii) There exists a smooth positive definite function \( V(.) \), with \( V(0) = 0 \), defined in a neighbourhood of \( x = 0 \) of \( \mathbb{R}^n \), satisfying:

\[
* \quad V(f(x) + G(x)) \text{ is quadratic in } x
\]

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\]

* Hamilton-Jacobi type equation:

\[
V(x) = h_1^2(x)h_1(x) + V(f(x)) - \frac{1}{4} \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x)} G(x).
\]

\[
(R + \frac{1}{2} G^T(x) \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x)} G(x))^{-1} G(x) \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x)} T(x)
\]

(3.2)

for \( R \triangleq \left[ \begin{array}{cc} 0 & - \gamma I \end{array} \right] \).

* There exists a non singular matrix of smooth functions, defined in a neighbourhood of \( x = 0 \),

\[
T(x) \triangleq \left[ \begin{array}{cc} T_{11}(x) & 0 \\ T_{21}(x) & T_{22}(x) \end{array} \right],
\]

such that:

\[
R + \frac{1}{2} G^T(x) \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x)} G(x) = T^T(x) J T(x)
\]

(3.3)

with \( J \triangleq \left[ \begin{array}{cc} 0 & 0 \\ 0 & \gamma \end{array} \right] \).

Then, if we initialize system (3.1) in \( x_0 = 0 \), it is possible to find a controller resolving the \( H_{\infty} \)-control "full-information" problem for this system. This controller is given by:

\[
\hat{u}(x_k, w_k) = u^*(x_k) - (T_{22}(x_k))^{-1} T_{21}(x_k)(w_k - w^*(x_k))
\]

(3.3)

where

\[
\begin{bmatrix}
  w^*(x) \\
  u^*(x)
\end{bmatrix} = \frac{1}{2} \left( R + \frac{1}{2} G^T(x) \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x)} G(x) \right)^{-1}
\]

\[
\frac{\partial V(X)}{\partial X} \Bigg|_{X=f(x)}^T
\]

(3.6)

Sketch of proof:
The basic fact consists in finding a positive definite function \( V(.) \), with \( V(0) = 0 \), such that:

\[
V(x_{k+1}) + \frac{1}{2} \gamma^2 w_k^T w_k - \gamma^2 w_k^T w_k \leq 0 \quad \forall k \in \mathbb{N}
\]

(3.7)

Summing (3.7) from \( k = 0 \) to \( k = N \), choosing \( x_0 = 0 \) and recalling the positivity of \( V(.) \), this all immediately leads to the disturbance attenuation (2.8).

To this end consider:

\[
H(x_k, w_k, u_k) = \frac{1}{2} \gamma^2 w_k^T w_k + V(x_{k+1})
\]

(3.8)

According to (2.8) and (3.2), and applying to \( V \) the Taylor expansion formula, we obtain:

\[
H(x_k, w_k, u_k) = h_1^2(x_k)h_1(x_k) + V(f(x_k))
\]

\[
+ \frac{\partial V(X)}{\partial X} \bigg|_{X=f(x_k)} G(x_k) \left[ w_k \right]
\]

\[
+ \frac{1}{2} \gamma^2 w_k^T \left( R + \frac{1}{2} G^T(x) \frac{\partial^2 V(X)}{\partial X^2} \Bigg|_{X=f(x_k)} G(x_k) \right) \left[ w_k \right]
\]

(3.9)

If it now easy to see that a couple \( [w^*(x), u^*(x)] \) given by (3.6) is such that:

\[
\frac{\partial H(x_k, w_k, u_k)}{\partial u_k} \bigg|_{(x_k, w_k, u_k)} = 0
\]

(3.10)

(The reversibility required by the computation of such a couple is guaranteed by assumption (3.4)).

Then, applying to \( H \) the Taylor expansion formula and remembering (3.4), we can immediately check that:

\[
H(x_k, w_k, u_k) = H(x_k, w^*(x_k), u^*(x_k))
\]

\[
- \frac{1}{2} \gamma^2 w_k^T w_k - \gamma^2 w_k^T w_k \leq 0
\]

(3.11)

Choosing now \( u_0 \) as in (3.5) leads to:

\[
H(x_k, w_k, u_k(x_k)) - H(x_k, w^*(x_k), u^*(x_k)) \leq 0
\]

(3.12)

Using the fact that \( V \) verifies Hamilton-Jacobi equation (3.3), \( H(x_k, w^*(x_k), u^*(x_k)) = V(x_k) \), which is Isaacs equation (2.5), it clearly implies (3.7).

To prove stability, note that inequality (3.7) with \( w_k = 0 \forall k \in \mathbb{N} \), and definite positivity of \( V(.) \) clearly prove that \( V(.) \) is a Lyapunov function for the system.

Finally, asymptotic stability follows from local detectability of \( \{h_1(x), f(x)\} \). The reasoning involving La Salle's invariance principle is rather classic (see [12]).
Remark 1: Choosing $V(z) = z^T X z$ in (3.3) and (3.4) in a linear setting, leads to the Riccati equation and the $M_1, M_2$ factorization of condition (a) in Theorem 3.1 in [11]. So, in a nonlinear discrete-time context, a factorization of the form (3.4) is needed, contrary to the nonlinear continuous-time case. In fact, less restrictive assumptions than (2.6) would require also a factorization in the continuous-time problem, but not involving the function $V(.)$. (See [15]).

Remark 2: What essentially differs from the linear discrete-time case is condition (3.2), which disappears in the linear problem by choosing a quadratic function $V(z) = z^T X z$ which gives the expected results. It would be interesting to examine whether or not this quadratic condition is necessary for the resolution of the $H_\infty$ nonlinear discrete-time problem. Note that, if $V(.)$ is a function verifying Isaacs equation (2.5), we obtain:

$$V(f(z)) + G(z)[$$: $$V((x)) = V(z) + \gamma^2 w^T(z) w(z) - u^T(z) u(z) - h^T(z) h(z)$$

This equation shows that $V(f(z)) + G(z)[$$: $$V((x)) is quadratic in $w(z)$ for every $x$ in a neighbourhood of $z = 0$ in $\mathbb{R}^n$.

Remark 3: If we do not make any of the assumptions (2.6), considering:

$$H(z_k, w(z_k), u(z_k)) = h^T(z_k) h(z_k) + V(f(z_k))$$

$$(2 h^T(z_k) x_k h(z_k) + \frac{\partial V(X)}{\partial X} |_{X = f(z_k)} G(z_k)[w(z_k)]$$

$$+ [w^T(z_k) R(z_k) + \frac{1}{2} G^T(z_k) \frac{\partial V(X)}{\partial X} |_{X = f(z_k)} G(z_k)[w(z_k)]$$

where

$$R(x) \triangleq \left[ \begin{array}{c}
-2 \gamma^2 I + \frac{G^T(z_k)}{2} k_{11}(x) \\
\frac{G^T(z_k)}{2} k_{12}(z) \\
\frac{G^T(z_k)}{2} k_{21}(x) \\
\frac{G^T(z_k)}{2} k_{22}(z)
\end{array} \right]$$

The same arguments of proof lead to a similar result.

Remark 4: It is possible, as it has been done in the continuous time in [14], to present these results in a discrete-time dissipative setting, since (3.7) is a dissipative inequality with supply rate $\gamma^2 w^T w_k - c^T z_k$ and storage function $V(.)$ (see [20] for concepts of dissipativity). Then, the $H_\infty$-control problem can be solved by trying to render the closed-loop system dissipative, with storage function $V(.)$. In order to do so, instead of verifying Isaacs equation, $V(.)$ would suppose to satisfy an inequality of the type:

$$H(z_k, w^*(z_k), u^*(z_k)) \leq V(z_k)$$

In our setting, this inequality would become a Hamilton-Jacobi type inequality:

$$h^T(z_k) h(z_k) + V(f(z_k)) - V(z_k) - \frac{\partial V(X)}{4} |_{X = f(z_k)} G(z_k)$$

$$(R + \frac{1}{2} G^T(z_k) \frac{\partial V(X)}{\partial X} |_{X = f(z_k)} G(z_k)[w(z_k)]$$

$$\leq 0$$

4 Disturbance attenuation via measurement feedback

Assuming $(x_k)_{k \in \mathbb{N}}$ and the exogeneous input $(w_k)_{k \in \mathbb{N}}$ are no longer available for measurement, we consider the following system:

$$x_{k+1} = f(x_k) + G(x_k)[w_k]$$
$$z_k = h_1(x_k) + h_2(x_k) u_k$$
$$y_k = h_3(x_k) + k_21(x_k) w_k$$

In order to use our last result, we introduce an observer (see, for instance, [13]) which is an exact copy of the dynamics with a term proportional to the error introduced by such a choice. This observer is therefore described by equations of the form:

$$\theta_{k+1} = f(\theta_k) + g_1(\theta_k) u(\theta_k) + g_2(\theta_k) w(\theta_k)$$
$$M(\theta_k) [y_k - h_3(\theta_k) - k_21(\theta_k) w(\theta_k)]$$
$$u(\theta_k) = \bar{u}(\theta_k, w(\theta_k))$$

With regards to $w(\theta_k)$, which still has to be chosen, it seems reasonable, as noticed in [13], to take the worst perturbation, namely $w(\theta_k) = w^*(\theta_k)$ according to (3.8).

We are now able to state our main result, for which we will give a sketch of the proof using observer (4.2).

Theorem 2: Suppose:

(i) $(h_1(z), f(z))$ locally detectable.
(ii) There exists a smooth positive definite function $V(.)$, with $V(0) = 0$, defined in a neighbourhood of $x = 0$ in $\mathbb{R}^n$, satisfying the same assumptions as in Theorem 1.
(iii) There exists a $n \times p_2$ matrix of smooth functions $M(\theta)$ defined in a neighbourhood of $\theta = 0$ which locally renders the equilibrium $\theta = 0$ of the following system asymptotically stable:

$$\theta_{k+1} = f(\theta_k) + g_1(\theta_k) u(\theta_k) + g_2(\theta_k) w^*(\theta_k)$$
$$M(\theta_k) [y_k - h_3(\theta_k) - k_21(\theta_k) w^*(\theta_k)]$$

(iv) There exists a smooth positive semi-definite function $W(\cdot, \cdot)$ defined in a neighbourhood of $(x, \theta) = (0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, satisfying:

$$\forall W(0, \theta) > 0 \ \forall \theta \neq 0$$
$$W(f_{obs}(x, \theta) + g_{obs}(x, \theta)) is quadratic$$
$$W(x, \theta) = \bar{w}_T(x, \theta) \bar{u}(x, \theta) + f_{obs}(x, \theta)$$
$$r^T(x, \theta) R_{obs}(x, \theta)^* (x, \theta)$$
$$R_{obs}(x, \theta) < 0$$

where

$$f_{obs}(x, \theta) = \left[ \begin{array}{c}
(f(\theta_k) + g_1(\theta_k) u(\theta_k) + g_2(\theta_k) w^*(\theta_k))
\end{array} \right]$$

$$M(\theta_k) [y_k - h_3(\theta_k) + k_21(\theta_k) w^*(\theta_k)]$$

$$-k_21(\theta) w^*(\theta)$$
\[
\begin{align*}
g_{obs}(x, \theta) &= \left[ g_1(x) \right] M f_{23}(x) \\
R_{obs}(x, \theta) &= \left( T_{11}^{-1}(x) T_{21}(x) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial R_{obs}(x, \theta)}{\partial x} \right) T_{11}(x) - I \\
\frac{\partial^2 W(X)}{\partial X^2} \bigg|_{X=x_{fobs}(x, \theta)} g_{obs}(x, \theta) T_{11}^{-1}(x) - I \\
r^*(x, \theta) &= -\frac{1}{2} R_{obs}(x, \theta) T_{11}^{-1}(x) \left[ 2T_{11}(x) v(x, \theta) + \sum_{i=1}^{n} \frac{\partial R_{obs}(x, \theta)}{\partial x} T_{11}^{-1}(x) v(x, \theta) \right] \\
v(x, \theta) &= T_{22}(x) (u^*(x) - w^*(x)) \\
&= v(x, \theta), T_{11}(x) \text{ and } T_{22}(x) \text{ are defined as in Theorem 1.} \\
\text{Then, if we initialize the closed-loop system in } (x_0, \theta_0) = (0, 0), \text{ the controller resolving the } H_{\infty} \text{-control problem for system (4.1) is given by:} \\
\theta_{k+1} &= f(\theta_k) + g_1(\theta_k) w^*(\theta_k) + g_2(\theta_k) u(\theta_k) + M(\theta_k) \left[ y_0 - h_2(\theta_k) - k_2(\theta_k) w^*(\theta_k) \right] \\
u(\theta_k) &= u(\theta_k, w^*(\theta_k)) \\
&\text{Sketch of proof:} \\
&\text{It is immediate to verify that after applying observer (4.2) to system (4.1), the dynamics is described by:} \\
&\begin{bmatrix} x_{k+1} \\ \theta_{k+1} \end{bmatrix} = f_{obs}(x_k, \theta_k) + g_{obs}(x_k, \theta_k) (w_k - w^*(x_k)) \\
&\text{Now, considering the function:} \\
&H_{obs}(x_k, \theta_k, r(x_k, w_k)) = v^T(x_k, \theta_k, w_k) (x_k, \theta_k, w_k) \\
&-r^T(x_k, w_k) r(x_k, w_k) + W(x_{k+1}, \theta_{k+1}) \\
&\text{where} \\
&r(x_k, w_k) = T_{11}(x_k) (w_k - w^*(x_k)) \\
v(x_k, \theta_k, w_k) = v(x_k, \theta_k) + T_{21}(x_k) T_{11}^{-1}(x_k) r(x_k, w_k) \\
&\text{According to (4.5), } H_{obs} \text{ is quadratic in } r(x_k, w_k). \text{ It is easy to verify that } r^*(x, \theta) \text{ given by (4.8) is such that:} \\
&\frac{\partial H_{obs}(x_k, \theta_k, r^*(x_k, \theta_k))}{\partial r} = 0 \\
&\text{Then, applying to } H_{obs} \text{ the Taylor expansion formula, we obtain:} \\
&H_{obs}(x_k, \theta_k, r(x_k, \theta_k)) = H_{obs}(x_k, \theta_k, r^*(x_k, \theta_k)) + (r(x_k, w_k) - r^*(x_k, \theta_k))^T R_{obs}(x_k, \theta_k) (r(x_k, w_k) - r^*(x_k, \theta_k)) \\
&\text{Define a function } U(., .) \text{ from } \mathbb{R}^n \times \mathbb{R}^n \text{ in } \mathbb{R}, \text{ as:} \\
&U(x, \theta) = V(x) + W(x, \theta) \\
&\text{Hypotheses (ii) and (iii)(4.4) of Theorem 2 show that } U(x, \theta) > 0. \text{ Recalling (3.8), (4.9) and according to the fact that } W \text{ verifies:} \\
&W(x_k, \theta_k) = H_{obs}(x_k, \theta_k, r^*(x_k, \theta_k)) \\
&\text{which is nothing but 4.6, we can easily compute, taking } w_k = 0 \text{ for all } k \text{ in the formula:} \\
&U(x_{k+1}, \theta_{k+1}) - U(x_k, \theta_k) = -||h_1(x_k)||^2 - ||w_k, w^*(\theta_k)||^2 \\
&+ (r(x_k, w_k) - r^*(x_k, \theta_k))^T R_{obs}(x_k, \theta_k) (r(x_k, w_k) - r^*(x_k, \theta_k)) \\
&\text{Remembering (4.7), this clearly shows that } U \text{ is a Lyapunov function for the system. For asymptotic stability, we proceed as in the proof of Theorem 1, using local detectability of } (h_1(x), f(x)), \text{ (4.3), a well-known property of cascade systems (see [18]) and La Salle’s invariance principle.} \\
&\text{Finally, in order to prove disturbance attenuation, note that if we do not take } w_k = 0 \text{ for all } k, (4.10) \text{ becomes:} \\
&U(x_{k+1}, \theta_{k+1}) - U(x_k, \theta_k) = -||h_1(x_k)||^2 \\
&-||w_k, w^*(\theta_k)||^2 \\
&+ (r(x_k, w_k) - r^*(x_k, \theta_k))^T R_{obs}(x_k, \theta_k) (r(x_k, w_k) - r^*(x_k, \theta_k)) \\
&\text{Summing this last inequality from } k = 0 \text{ to } k = N, \text{ using (4.7), the positiveness of } U(., .) \text{ and taking } (x_0, \theta_0) = (0, 0) \text{ immediately leads to the desired disturbance attenuation (2.3).} \\
&\triangle \\
\end{align*}
\]

5 Conclusions

In this preliminary paper, we have presented sufficient conditions for the existence of a static control law which solves the $H_{\infty}$-control problem for a nonlinear discrete-time affine system satisfying some classic assumptions. Some of the results concerning nonlinear $H_{\infty}$-control set in the continuous-time case are proposed for discrete-time systems and difference game theory. The analogies but also the difficulties with respect to the continuous-time situation or the linear case are briefly discussed. Moreover, pursuing the study, one can also easily set the nonlinear discrete-time $H_{\infty}$-control problem in a dissipative context.

Before sending the final version of this paper, the authors became aware of a work from C. Byrnes and W. Lin (see [4]) where nonlinear $H_{\infty}$-control problem in discrete-time via state and full information feedback is solved without quadraticity requirement.

References


