1 Introduction

It is known that discrete-time singularly perturbed (DSP) dynamics do not exhibit a unique canonical representation. A study in the linear context is developed in [19, 17, 13], with respect to discrete-time and sampled dynamics showing that at least three different “Normal Forms” can be assumed for representing a linear discrete time singularly perturbed system (linear DSPS). Starting from the generalization of these concepts to a nonlinear context, the present work studies the properties for the existence of a “nonlinear Column-Form” representation of a nonlinear DSPS. A parallel study is in [2] for a “nonlinear row-Form”.

Let us recall, that a set of nonlinear difference equations depending on a small real positive parameter $\varepsilon$ of the form

$$ y(k+1) = f(y(k), u(k), \varepsilon) $$

(1.1)

describes a nonlinear DSPS if, setting $\varepsilon = 0$, the system degenerates into a system of a lower dimension. Three types of representations, satisfying such a property can generalize the linear situation ([17]).

Formally, if $(x, z, u)$ belongs to $\Omega_x \times \Omega_z \times \Omega_u$, a manifold or a compact and connected set of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$, one sets:

Definition 1: (Nonlinear Row-Form)
The set of difference equations

$$ x(k+1) = f_2(x(k), u(k), \varepsilon) $$

(1.2)

satisfying $\text{Rank}(\frac{\partial f_2}{\partial x}(x, u, \varepsilon)) = m$ for all $(x, z, u) \in \Omega_x \times \Omega_z \times \Omega_u$ will be said a “Row-Form representation” of (1.1).

Definition 2: (Nonlinear Column-Form)
The set of difference equations

$$ x(k+1) = f(x(k), u(k), \varepsilon) $$

(1.3)

satisfying $\text{Rank}(\frac{\partial f}{\partial x}(x, u, \varepsilon)) = m$ for all $(x, z, u) \in \Omega_x \times \Omega_z \times \Omega_u$ will be said a “Column-Form representation” of (1.1).

Definition 3 (Nonlinear Differential-Form)
The set of difference equations

$$ x(k+1) = f(x(k), u(k), \varepsilon) $$

(1.4)

satisfying $\text{Rank}(\frac{\partial f}{\partial x}(x, u, \varepsilon)) = m$ for all $(x, z, u) \in \Omega_x \times \Omega_z \times \Omega_u$ will be said a “Differential-Form representation” of (1.1).

To each of these three sets of difference equations a two time scale representations can be associated. On these bases it is possible to use for the $R$-form and the $C$-form cases, the concept of slow invariant manifold, on which, after a short time the evolution of the system can be forced. The design of the control law will be simplified if this invariant manifold results to be attractive. The control law is composed by fast and slow actions for increasing the domain of attractiveness of the invariant manifold and ensuring the achievement of the control objectives.

Usually, a nonlinear DSPS (or a LDSP system) is not directly written in one of these three forms and it is interesting to study how to recover one of these forms under state space representations, not depending on $\varepsilon$. The purpose of this paper is to give necessary and sufficient conditions in order to transform system (1.1) into the $C$-form (1.3). The paper is organized as follows: The nonlinear discrete time C-form and its associated slow manifold, slow and fast reduced dynamics are introduced in section 2. Necessary and sufficient conditions for transforming (1.1) into a C-form are discussed in section 3; this consists in a straightforward application of the rank theorem [3]. In section 4, two examples are treated. The paper ends with some comments ans conclusions.

2 Nonlinear Discrete Time C-Form

Let us first recall the notion of Discrete Time Invariant Manifold ([4, 15, 7, 11, 20, 5]). Given a manifold $M$, defined on $R^n$, and considering for the sake of simplicity a nonlinear discrete time dynamics without forced term

$$ y(k+1) = f(y(k)) $$

(2.1)

with $y \in R^Q (Q \geq P)$, one recalls the definition of discrete-time invariant manifold as follows:

Definition 2.1 A manifold $M$ is said to be invariant under the discrete-time dynamics (2.1) if, starting from any state on $M$, the state evolution associated to (2.1) remains on $M$.

From this definition, the following characterization of invariance under discrete-time dynamics is straightforward. Let $M$ be defined as $M = \{y, \phi(y) = 0\}$ with $\phi$ a vector function of dimension $Q - P$; then, $M$ is an invariant manifold under (2.1) if and only if

$$ \forall y \in M \quad \phi(f(y)) = 0 $$

(2.2)

In the sequel, we will discuss these concepts of invariance with respect to the C-form representation.
2.1 Invariant slow manifold

Let us consider the set of equations (1.3) with

\[ \text{Rank } \left( \frac{\partial f_1}{\partial x} \right)_{x=\alpha} = m \]

Setting \( \varepsilon = 0 \) in (1.3), one obtains the degenerate \( m \)-dimensional dynamics

\[ x(k+1) = f_1^\varepsilon(x(k), u(k), 0) = f_1^{0,\varepsilon}(x(k), u(k)) \]  \hspace{1cm} (2.3)

and the relation \( x(k+1) = f_2^\varepsilon(x(k), u(k)) \). From (1.3), it becomes intuitive to introduce the slow manifold associated to the C-form, denoted by \( I_{Mc} \), as the manifold defined by

\[ I_{Mc} \triangleq \{ x(k) / z(k) - \Phi(k) = \phi(k) = 0 \} \]  \hspace{1cm} (2.4)

where \( \Phi(k) \) stands for \( \Phi(x(k), u(k-1), \varepsilon_w(k-2), ..., \varepsilon) \) and satisfying, because of the second equation of (1.3) and also (2.2) the invariant manifold condition

\[ \Phi(k+1) = f_2^\varepsilon(x(k), u(k), \varepsilon) + \varepsilon \lambda_2^\varepsilon(x(k), \Phi(k), u(k), \varepsilon) \]  \hspace{1cm} (2.5)

for \( \Phi(u(k) \in \Omega_u \) and \( z(k) \in \Omega_z \).

Remark: Note that \( \Phi(k) \) depends on the previous control \( u(k-1) \), \( i > 1 \) as the slow continuous time manifold depends on the successive derivatives of the control. Such dependency is finite when limited order approximations with respect to \( \varepsilon \) are considered. A difficulty in the previous condition (2.5) is that \( \Phi(k+1) \) is expressed in terms of \( x(k) \) rather than \( x(k+1) \) so that the first equation of (1.3) has to be reversed. For one states

Lemma 2.2: Let

\[ x(k+1) = f_i^\varepsilon(x(k), u(k)) + \varepsilon \lambda_i^\varepsilon(x(k), x(k), u(k), \varepsilon) \]

with \( \text{Rank} \left( \frac{\partial f_i^\varepsilon}{\partial x} \right)_{x=\alpha} = m \) on \( \Omega_x \times \Omega_z \times \Omega_u \), then, there exists \( \varepsilon_0 \) and a smooth mapping \( f : \Omega_x \times \Omega_z \times \Omega_u \times [0, \varepsilon_0] \rightarrow \Omega_x \) such that

\[ x(k) = \tilde{f}(x(k+1), z(k), u(k), \varepsilon) \]

with all the \( \tilde{f}_i \)'s smooth in their arguments.

Proof of Lemma 2.2: As \( f_i^\varepsilon \) is an analytic and because of the rank condition for \( \varepsilon = 0 \), there exists \( \varepsilon_0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), one has

\[ \text{Rank} \left( \frac{\partial f_i^\varepsilon}{\partial x} \right)_{x=\alpha} = m \]

Consequently, by the implicit function theorem, for \( (x(k+1), z(k), u(k), \varepsilon) \in \Omega_x \times \Omega_z \times \Omega_u \times [0, \varepsilon_0] \) there exists a unique reverse function \( \tilde{f} : \Omega_u \times \Omega_z \times \Omega_x \times [0, \varepsilon_0] \rightarrow \Omega_x \), satisfying

\[ x(k) = \tilde{f}(x(k+1), z(k), u(k), \varepsilon) \]

admitting a series expansion with respect to \( \varepsilon \) namely

\[ f_0 + \varepsilon f_1 + \cdots \]

with \( \tilde{f}_0(u, \cdot) : \Omega_u \times \Omega_z \rightarrow \Omega_x \) the reverse of \( f_0(u, \cdot) \) for \( u \in \Omega_u \).

Remark: In practice, the reverse function can be computed up to a certain order of approximation, according to the combinatorial Gröbner formula ([11]).

From the lemma 2.2 and the manifold condition (2.5), one obtains:

Proposition 2.3: The invariant manifold \( I_{Mc} \), associated to the C-form, satisfies the series expansion

\[ \Phi(k) = \Phi_0(k) + \sum_{i \geq 1} \Phi_i(k) \varepsilon^i \]  \hspace{1cm} (2.7)

where each \( \Phi_i(k) \) is an analytic function of \( x(k) \) and \( u(k-j) \) (\( i \geq j \geq 1 \)), which can be iteratively computed.

Proof of the Proposition: Substituting (2.6) into (2.5), taking into account of (2.7) and regrouping and equating terms of the same power in \( \varepsilon \), one iteratively computes the \( \Phi_i(k) \)'s functions. Precisely one has for \( \varepsilon^i \)

\[ \Phi_0(k) = f_2^\varepsilon(x(k), u(k-1), u(k-1)) \]

so that, one notes \( \Phi_0(k) = \Phi_0(x(k), u(k-1)) \)

for \( \varepsilon^1 \)

\[ \Phi_1(k) = \frac{\partial f_2^\varepsilon(x(k), u(k-1))}{\partial x(k)} \]

\[ \times f_0(x(k), u(k-1), \Phi_0(k)) \]

\[ + \varepsilon \lambda_2^\varepsilon(x(k), u(k-1), \Phi_0(k), u(k-1), \varepsilon) \]

so that, one notes

\[ \Phi_1(k) = \Phi_1(x(k), u(k-1), u(k-2)) \]

Iteratively, assuming that \( \Phi_i(k) \) is analytic for \( i \in \{ 0, ..., j \} \) then for \( \varepsilon^{j+1} \) one has

\[ \Phi_{j+1}(k) = \frac{\partial f_2^\varepsilon(x(k), u(k-1))}{\partial x(k)} \]

\[ + \varepsilon \lambda_2^\varepsilon(x(k), u(k-1), \Phi_j(k), u(k-1), \varepsilon) \]

where \( x(k-1) \) is given by (2.6) and

\[ \Phi_{j+1}(k) = \Phi_{j+1}(x(k-1) + \varepsilon \Phi_j(k-1) + \cdots + \varepsilon^j \Phi_{j}(k-1)) \]

and the proof is complete. \( \triangle \)

2.2 Attractiveness of the manifold

After the usual change of variable

\[ \eta(k) = x(k) - \Phi(k) \]

the system (1.3) takes the form

\[ \eta(k+1) = f_1^\varepsilon(x(k), u(k), \varepsilon) + \varepsilon \lambda_1^\varepsilon(x(k), \eta(k) + \Phi(k), u(k), \varepsilon) \]  \hspace{1cm} (2.9)

\[ \eta(k+1) = f_2^\varepsilon(x(k), u(k), \varepsilon) + \varepsilon \lambda_2^\varepsilon(x(k), \eta(k) + \Phi(k), u(k), \varepsilon) \]

where the second equation clearly represents the fast dynamics associated to the C-form.

As previously noted, for \( \varepsilon = 0 \), the slow degenerated dynamics after only one step evolves on \( \Omega_x \times \Omega_u \) independent of \( \eta \) since \( \eta(k+1) = 0 \).

For \( 0 < \varepsilon \leq \varepsilon_0 \), it can easily be noted that the fast dynamics is locally stable around \( \eta = 0 \). This enables to conclude to the local attractiveness of \( I_{Mc} \). One has

Proposition 2.4: The invariant discrete-time manifold \( I_{M} \) is locally attractive for (1.3), namely;

\[ \text{im}_{k \rightarrow \infty} \eta(k) = 0 \]

Proof: For \( \eta(k) = 0 \), one obtains \( \eta(k+1) = 0 \) and in the neighbourhood of \( \eta = 0 \), the approximated dynamics takes the form

\[ \eta(k+1) = \varepsilon \lambda_2^\varepsilon(x(k), \eta(k)) + O(\varepsilon) \]

with the previous condition for \( \varepsilon \geq 0 \) satisfied. This enables to conclude.

\[ \text{im}_{k \rightarrow \infty} \eta(k) = 0 \]
So, for all $1 > K > 0$ there exist $\eta_0 > 0$, $x_0 > 0$, $u_0 > 0$ and $\epsilon_0 > 0$, such that for $|x| < \epsilon_0$, $|\eta(0)| < \eta_0$, $|\eta(x)| < x_0$ and $|u(j)| < u_0$ (for $j \in \{0, \ldots, k\}$) one has

$$|\eta(x)| < K^{\epsilon_0}$$

which ensures that $\eta(k) \to 0$ as $k$ goes to infinity. \triangle

2.3 Slow dynamics

The attractivity property of $IM_a$ greatly simplifies the design of a control procedure which can be set on the basis of the reduced slow dynamics obtaining the slow component of the control.

Definition 2.5 The reduced dynamics

$$x(k+1) = f(\eta(k), u(k), \epsilon) + \epsilon h(x(k), \Phi(k), u(k), \epsilon) \quad (2.11)$$

with $\Phi(k)$ defined by (2.7), is said to be the slow discrete-time dynamics associated to (1.5).

On the basis of (2.10), a slow control law $u$ can be designed. Moreover, in order to accelerate the convergence or extend the attractiveness domain it can be necessary to add a corrective term, namely the fast control component $y$, designed on the fast dynamics (2.9). The same situation occurs when R-form are considered (3.2).

3 Necessary and sufficient conditions for obtaining a C-form

The object of this section is to determine the conditions under which it is possible to transform into the C-form the following general DPS systems

$$y(k+1) = f(y(k), u(k), \epsilon) \quad (3.1)$$

with

$$f_0(y(k), u(k), \epsilon) = f(y(k), u(k), 0)$$

where $y \in M$ (a manifold or an open connected set of dimension $n + m$) and $u \in \Pi_a \subset R^r$ (an open connected set of allowed inputs), and without loss of generality we assume that the equilibrium point is equal to zero

$$0 = f(0,0,0) = f_0(0,0)$$

with $0 \in M$ and $0 \in \Pi_a$. For, let us consider the C-form previously defined in (1.3), and its associated degenerated system $S_0$, for $\epsilon = 0$

$$x(k+1) = f_1(x(k), u(k)) \quad (3.2)$$

Since after one step $S_0$ evolve on $S_0 = \{(x, z) | x = \Phi_0(x, u)\}$, a in assumption C-1 below is straightforward. Moreover, one asks to transform the dynamics (3.1) into the C-form under a diffeomorphism which is not function of $u$, this leads to $b$-condition of R-1. Condition $c$- of R-1 is linked to the regularity of $f_1$.

Assumption C-1: There exist an open set $V \subset M$ (With $0 \in M$) and an open set $W \subset \Pi_a$

a-For all $y \in V$ and $u \in W$

$$\text{Rank}(\frac{\partial f_0(y,u)}{\partial y}) = m$$

b-For all $y \in V$ and $u_1, u_2 \in W$

$$\text{Rank}(\frac{\partial f_0(y,u_1)}{\partial y} - \frac{\partial f_0(y,u_2)}{\partial y}) = m$$

c-For all $y(k) \in V$ and $u(k), w(k+1) \in W$ we have:

$$\text{Rank}(\frac{\partial f_0(y(k+1))}{\partial y} - \frac{\partial f_0(y(k))}{\partial y}) = m$$

Remarks:

i-(C-1-a) is strictly the rank condition, which gives the dimension of the slow state $x$. In a linear context, this implies that the rank of the drift matrix is equal to $m$ for $\epsilon = 0$.

ii-(C-1-b) is a kind of parallelism condition which ensures that the state decomposition is not function of $u$. This condition is always verified in the linear case.

iii-(C-1-c) is an invariance property which generalizes the linear condition certifying that the drift matrix associated to the slow state is “regular” [21].

Theorem 3.1: Consider the control system (3.1) where $f$ is a smooth function defined from $M \times \Omega_a \times [-\epsilon_0, \epsilon_0]$ to $R^{m+n}$ where $M \subset R^{m+n}$ is a manifold or an open and connected subset (with $0 \in M$), $\Omega_a \subset R^r$ is an admissible open and connected control set. Then, there exist an open set $V_0 \in \Pi_a \subset V \subset M$, an open set $W_0$ (with $0 \in W_0 \subset W \subset \Pi_a$) and a diffeomorphism $\psi$ independent on $\epsilon$ such that for any $y \in V_0, u \in W_0$ under the diffeomorphism $\psi$ the control system (3.1) takes the C-form

$$x(k+1) = f_1(x(k), u(k)) + \epsilon h_e(x(k), x(k), u(k), \epsilon)$$

$$x(k+1) = f_2(x(k), u(k)) + \epsilon h_e^{2}(x(k), x(k), u(k), \epsilon)$$

with

$$\text{Rank}(\frac{\partial f_1}{\partial x}) = m$$

and smooth functions $f_1$'s and $h_1$'s, if and only if, C-1 holds.

Proof: According to [3], a constructive sufficient proof is:

sufficiency: As $\text{Rank}(\frac{\partial f_0}{\partial y}) = m$ equal to $m$ on $M \times \Omega_a$ from (C-1-a), it is possible to find a smooth function $\psi_0$ of $\phi_0 \to R^n$ with $\psi_0(z, x) = y$, and $\text{dim}(z) = n$ such that, one has

$$\text{Rank}(\frac{\partial f_0}{\partial y}) = m$$

Furthermore

$$\text{Rank}(\frac{\partial f_0}{\partial y}) = m$$

Consequently, the function $f_0(., u) = \psi^{-1}(f_0(\psi_0, u))$ verifies $\text{Rank}(\frac{\partial f_0}{\partial y}) = m$.

From C-1-b $\phi_0$ does not depend on $u$ and will be denoted by $\psi$. So that, applying the transformation $\psi$ to $f_0$ one has

$$f_0(\psi(u), \psi(u), \epsilon = 0)$$

and from (3.4) and (3.3) one obtains, in the neighbourhood of $\psi^{-1}(s) \in \psi^{-1}(u)$ with $s \in V$ one has

$$\text{Rank}(\frac{\partial f_0}{\partial y}) = m$$

As $\psi_0$ and $W_0$ are open sets in the small region the equilibrium point the diffeomorphism is defined for any $y$ and $y = f_0(y, u)$ (with $y \in \psi_0$ and $u \in W_0$). Consequently from (C-1-c), one has

$$\text{Rank}(\frac{\partial f_0}{\partial y}) = m$$
with $X = (x^T, z^T)^T$ or again

$$
\text{Rank}\left( \begin{bmatrix}
\frac{\partial f_0(z, u(k))}{\partial z} & 1 \\
\frac{\partial f_0(z, u(k))}{\partial u} & 0
\end{bmatrix} \right) = m
$$

which implies:

$$
\text{Rank}\left( \begin{bmatrix}
\frac{\partial f_0(z, u(k))}{\partial z} \\
\frac{\partial f_0(z, u(k))}{\partial u}
\end{bmatrix} \right) = m
$$

Now, since $f$ in (3.1) is a smooth function of $y, u$ and $\epsilon$ one can expand

$$
f(y, u, \epsilon) = \sum_{i=0}^{\infty} f_i(y, u) \epsilon^i
$$

Moreover, as $\psi$ is also a smooth function which does not depend of $\epsilon$ and such that

$$
\psi^{-1}(y) = \left( \psi_1^{-1}(y), \psi_2^{-1}(y) \right)
$$

so one has

$$
f_\epsilon(x, z, u, \epsilon) = \left( \psi_1^{-1}(\sum_{i=0}^{\infty} f_i(y, u) \epsilon^i), \psi_2^{-1}(\sum_{i=0}^{\infty} f_i(y, u) \epsilon^i) \right)
$$

and after Taylor's expansion in $\epsilon$ and regrouping term in $\epsilon^i$, one gets

$$
f_\epsilon(x, z, u, \epsilon) = \left( f_1(x, u) + \epsilon h_1(x, z, u, \epsilon), f_2(x, z, u, \epsilon) \right)
$$

**Necessity:** Let us suppose there exists a diffeomorphism $\psi$, independent of $\epsilon$ and $u$, such that the nonlinear DSDS (1.1) is in the C-form. Then for all $u \in W_0$

$$
\frac{\partial \psi}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} = 0
$$

and

$$
\text{Rank}\left( \frac{\partial \psi}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} \right) = m
$$

which implies, for all $u \in W_0$

$$
\text{Rank}\left( \frac{\partial f_0(z, u)}{\partial y} \right) = m
$$

This proves that condition C-1-a is necessary. Moreover, as $\psi$ is not function of $u$; for all $u_1$ and $u_2$ in $W_0$, one has

$$
\frac{\partial \psi}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u_1)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} \bigg|_{\psi} + \frac{\partial f_0(z, u_2)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} \bigg|_{\psi} = 0
$$

and

$$
\text{Rank}\left( \frac{\partial \psi}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u_1)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} \bigg|_{\psi} + \frac{\partial f_0(z, u_2)}{\partial y} \times \frac{\partial \psi}{\partial \epsilon} \bigg|_{\psi} \bigg) = m
$$

so, one has

$$
\text{Rank}\left( \frac{\partial f_0(z, u_1)}{\partial y}, \frac{\partial f_0(z, u_2)}{\partial y} \right) = m
$$

This proves that condition C-1-b is also necessary.

Finally, $\text{Rank}\left( \frac{\partial f_0(z, u)}{\partial u} \bigg|_{\psi} \right) = m$ gives

$$
\text{Rank}\left( \frac{\partial f_0(z, u(k+1))}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u(k))}{\partial y} \bigg|_{\psi} \right) = m
$$

this implies

$$
\text{Rank}\left( \frac{\partial f_0(z, u(k+1))}{\partial y}, \frac{\partial f_0(z, u(k))}{\partial y} \right) \geq m
$$

and as $\text{Rank}\left( \frac{\partial f_0(z, u(k))}{\partial y} \bigg|_{\psi} \times \frac{\partial f_0(z, u(k+1))}{\partial y} \bigg|_{\psi} \right) = m$

for all $u(k), u(k+1) \in W_0$, i.e. the necessity of (C-1-c). \(\triangle\)

In order to obtain a linear analytic C-form (defined below), one has to request in theorem 3.1 the existence of a function $\psi$ satisfying the following extra conditions:

The diffeomorphism $\psi$, such that, $\psi^{-1}(y) = x$ and $\psi^{-1}(y) = x$ with $\text{dim}(x) = n$, which verifies

$$
\frac{\partial f_0(z, u)}{\partial y} \times \frac{\partial \psi}{\partial y} = 0
$$

and

$$
\text{Rank}\left( \frac{\partial f_0(z, u(k))}{\partial y} \times \frac{\partial \psi}{\partial y} \right) = m
$$

end also, for all $y \in V_0$ and $u \in W_0$

$$
\frac{\partial \psi^{-1}(f_0)}{\partial u} = 0
$$

so, one states:

**Corollary 3.2:** Consider the control system (3.1) where $f$ is a smooth function defined from $M \times \Omega_x \times [-\epsilon_0, \epsilon_0]$ to $R^{\text{dim}(x)}$ where $M \subset R^{\text{dim}(x)}$ is a manifold or an open and connected subset (with $0 \in M$), $\Omega_x \subset R^x$ is an admissible open and connected control set. Then, there exist an open set $V_0 \subset M \subset \Omega_x \subset V \subset M$, an open set $W_0 \subset W_0 \subset \Omega_x \subset M$ and a diffeomorphism $\psi$ independent on $\epsilon$ such that for all $y \in V_0$, $u \in W_0$ under the diffeomorphism $\psi$ the control system (3.1) takes the linear analytic C-form

$$
x(k+1) = f_0(z(k)) + f_1(z(k)) + f_2(z(k)) + f_3(z(k)), y(k), u(k), e(k)
$$

where $f_0 + f_1 u : \Omega_x \times \omega_0 \times [-\epsilon_0, \epsilon_0] \rightarrow R^m$, is a smooth function, and

$$
\text{Rank}\left( \frac{\partial f_0}{\partial y} \bigg|_{\psi} \times \frac{\partial \psi}{\partial y} \right) = m
$$

with smooth functions $f_1$’s and $h_1$’s, if only if, R-1 holds and $\psi$ verifies $\text{Rank}\left( \frac{\partial \psi^{-1}(f_0)}{\partial y} \right) = 0$.

**Proof:** The proof is the same as for the theorem 3.1, with the extra condition $\text{Rank}\left( \frac{\partial \psi^{-1}(f_0)}{\partial y} \right) = 0$, which ensures that

$$
\frac{\partial \psi^{-1}(f_0)}{\partial y} = 0
$$

for all $y \in W_0$.

Consequently the extra condition is necessary and sufficient in order to obtain a linear control dependence of the “stiff” system ($\epsilon = 0$).

\(\triangle\)

4 Examples

Let us briefly discuss the following two simple examples.

**Example 4.1:** Let us study the following linear system

$$
y(k+1) = A(e)y(k) + Bu(k)
$$

where

$$
A(e) = \begin{pmatrix}
3 + e & 2e - 3 \\
1 & -1
\end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
$$

The reduced system is equal to

$$
y(k+1) = A(0)y(k) + Bu(k)
$$
where
\[ A(0) = \frac{1}{2} \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix} \]

As \( \frac{\partial f_0}{\partial y} = A(0) \) and \( \text{Rank}(A(0)) = 1 \) for all \( y \) and \( u \), no condition \( C-1-a \) and \( C-1-b \) are verified. Moreover, \( \text{Rank}(A(0)^2) = 1 \), so the condition \( C-1-b \) holds too. Consequently, choosing \( \psi \) such that
\[ 1.5 \frac{\partial \psi_1}{\partial x} - 1.5 \frac{\partial \psi_2}{\partial x} = 0 \]
\[ 0.5 \frac{\partial \psi_1}{\partial x} - 0.5 \frac{\partial \psi_2}{\partial x} = 0 \]
one gets
\[ \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} = 1 \]
choosing for example
\[ \psi_1 = x + x \]
\[ \psi_2 = x \]
one obtains under this diffeomorphism the C-form system
\[ x(k+1) = (1 + \frac{3}{2}x(k)) + \frac{3}{2}u(k) \]
\[ x(k+1) = \frac{3}{2}u(k) \]

Example 4.2: Let us study the following nonlinear system
\[ v_1(k+1) = \frac{1}{2} ((v_1(k) + v_2(k))^2 + \frac{1}{2} (v_1(k) + y(k)) + \frac{1}{2} (v_1(k) - y(k))^2)] + 2u(k) \]
\[ v_2(k+1) = \frac{1}{2} ((v_1(k) + v_2(k))^2 - \frac{1}{2} (v_1(k) + y(k)) - \frac{1}{2} (v_1(k) - y(k))^2)] + 2u(k) \]
The reduced system is given by
\[ v_1(k+1) = \frac{1}{2} ((v_1(k) + v_2(k))^2 + \frac{1}{2} (v_1(k) + y(k)) + \frac{1}{2} (v_1(k) + y(k))^2)] + 2u(k) \]
\[ v_2(k+1) = \frac{1}{2} ((v_1(k) + v_2(k))^2 - \frac{1}{2} (v_1(k) + y(k)) - \frac{1}{2} (v_1(k) - y(k))^2)] + 2u(k) \]
The Rank of \( \frac{\partial f_0}{\partial y} \) being equal to
\[ \text{Rank} \left( \begin{pmatrix} 0.5(v_1 + v_2 + 1) \\ 0.5(v_1 + v_2 - 1) \end{pmatrix} \right) = 1 \]
for all \( v_1, v_2 \) and \( u \), the conditions \( C-1-a \) and \( C-1-b \) hold. Moreover,
\[ \text{Rank} \left( \frac{\partial f_0}{\partial y} \right) \times \frac{\partial f_0}{\partial y} = m \]
so that condition \( C-1-c \) holds too. Choosing \( \psi \) such that
\[ 0.5(v_1 + v_2 + 1) \frac{\partial \psi_1}{\partial x} + 0.5(v_1 + v_2 + 1) \frac{\partial \psi_2}{\partial x} = 0 \]
\[ 0.5(v_1 + v_2 - 1) \frac{\partial \psi_1}{\partial x} + 0.5(v_1 + v_2 - 1) \frac{\partial \psi_2}{\partial x} = 0 \]
one gets
\[ \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} = 1 \]
One takes, for example
\[ \psi_1 = x + x \]
\[ \psi_2 = x - x \]
and one obtains the C-form system:
\[ x(k+1) = x(k) + x^2(k) + u(k) \]
\[ x(k+1) = x(k) + u(k) \]

5 Some Comments:
Three normal forms can be naturally associated to nonlinear discrete-time singularly perturbed dynamics; one of these forms, the C-form, has been precisely discussed. In particular, the notion of invariant manifold and the corresponding reduced slow system have been defined. The attractiveness of the manifold is locally proved so that control schemes can be defined on the basis of this reduced system. Necessary and sufficient conditions for obtaining such a form under coordinates change are given.

The particular interest in such a form is linked to the fact that under slow sampling \( \delta >> \epsilon \) a continuous-time singularly perturbed system is a C-form discrete-time system. The same kind of analysis can be performed for the R-form [2] and it is interesting to verify that for both classes (C and R form) the local attractiveness of the invariant manifold can be proved. This simplifies the control procedure in the sense that a fast part of the control is not needed. Similarly to the continuous time case, some geometric pathology occurs when the assumption \( C-1-a \) (Rank condition) is violated so that fold manifolds appear.

Finally, let us note that, for computing the slow manifold, the knowledge of past inputs \( u(k-1), k \geq 1 \) is needed. This is not surprising since it occurs in the linear context and represents the discrete analog of the dependence with respect to \( u(k), k \geq 1 \) of the slow manifold in a continuous time domain.
A complete discussion of these normal forms for nonlinear discrete-time systems and systems under sampling is actually in progress.

References

867


