TM2 - 15:00

AN INTRODUCTION TO MOTION PLANNING UNDER MULTIRATE DIGITAL CONTROL

S. Monaco* and D. Normand-Cyrot**

* Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", via Eudossiana, 18, 00184, Roma, Italy.
** Laboratoire des Systèmes et Systèmes, CNRS-ESE, Plateau de Moulon, 91192 Gif-sur-Yvette Cedex, France.

Abstract

In this paper we propose digital control methods for steering real analytic controllable systems between arbitrary state configurations.

The main idea is to achieve a multirate sampled procedure to perform motions in all the directions of controllability under piecewise constant controls. When it is applied to nonholonomic control systems without drift, the procedure simplifies. In particular, it results in exact steering on chained systems recently introduced in the motion planning literature. A classical example is reported.

1. Introduction

Let us consider a continuous-time nonlinear system:

\[ \dot{x}(t) = f(x) + u g_1(x) + \ldots + u g_m(x) \]  

(1.1)

where the \( f \) and \( g_i \) are real analytic vector fields defined on a connected smooth manifold \( M \) of \( \mathbb{R}^n \) and assumed complete.

Given two state vectors \( x_0 \) and \( x_T \), the problem consists in designing a control procedure steering \( x_0 \) to \( x_T \).

For nonholonomic driftless systems, this problem has widely been investigated over the last few years, especially in literature on motion planning in robotics and several efficient control algorithms have been proposed. Restricting ourselves to the more recent papers which inspire our work, we refer to [6, 8, 9, 10, 11, 18, 19, 21, 23] where a more complete bibliography can be found.

Assuming with \( m < n \) the complete controllability of the system (1.1) to guarantee the existence of a solution (17, 23), the problem is to design a reasonable one (with respect to computing time, complexity, convergence to the solutions, number of paths, control constraints and others).

Among the various proposed approaches, it appears that much can be said about nilpotent or nilpotentizable systems (under coordinate changes and feedback) ([6, 7, 14, 15]). In particular in [8, 18] special classes of systems, which can be steered using sinusoidal inputs, are introduced. These systems, referred to as chained systems, have convenient triangular forms for which such a control reveals many advantages (optimality, small number of paths). Other techniques are based on exponential representations of the trajectories and decomposition in suitable bases of the control Lie algebra associated to (1.1) ([8, 10, 6]). Some attempts to extend these methods to systems with drift can be found in [5, 23].

Let us recall that techniques of linearization cannot be applied to nonholonomic systems which are generally not fully feedback linearizable, and that linear techniques fail since the controllability rank condition of a driftless dynamics cannot be preserved under linearization. With regard to these difficulties, it is not too surprising that digital strategies could lead to efficient solutions.

In fact, multirate digital control is currently investigated in a linear context ([1, 4]) when digital control of continuous systems is investigated. Such methods allow us to achieve, at sampling times, regulation, stabilization, decoupling or tracking objectives set in a continuous-time domain. Moreover, multirate digital control can be applied to controllable continuous-time systems, even to those which are non-minimum phase, to achieve stabilization. Typical discrete-time requirements, such as dead beat or minimum time responses, can also be required.

In a nonlinear context, it has recently been shown that multirate digital control can be proposed as the digital analogue of linearizing feedback laws for systems with a relative degree which is larger than one ([15]). Roughly speaking, if the sampling period is equal to \( \delta \), multirate digital control consists in designing piecewise constant controls over fractions of this sampling period (say \( \delta = \delta / p \) where \( p \) is an integer \( \geq 0 \)). Such a procedure causes an increasing number of freedom degrees on the control, enables to satisfy several objectives. For example, when linearization under sampling is discussed, it has been proved that a multirate procedure of order equal to the relative degree enables to maintain, at sampled instants, the linearizing or decoupling properties of the continuous input/output response. Under controllability assumption of the continuous-time system it can also be shown that full linearization under multirate of order \( n \) can be achieved. Other critical continuous-time situations can also take benefit from such control procedures, such as time delayed or non minimum phase systems ([17]).

Considering the steering problem which essentially requires the complete controllability assumption, it seems interesting trying to apply digital methods.

Even if many questions are still open concerning the existence of solutions and the "right" choice of multirate order, the strict purpose of this paper is to show that digital control can be a possible solution in a number of practical examples. The benefits essentially concern the simplicity of the designed trajectories, the number of paths required for steering (exact steering for chained systems), the computing time.

The paper is organized as follows. The introduction recalls the terminology and the approaches which inspired the present work. Section 2 recalls some basic properties of multirate sampled systems. In fact, exponential representations of the sample dynamics introduced in (16) are here used to compute the steering control. Steering under digital control is outlined in Section 3 for both systems with drift and without drift term. In Section 4 chained systems are considered, in this case an exact solution is explicitly designed. The section ends with a classical example from mobile robotics.

2. Nonlinear Multirate Sampling

Consider a linear analytic continuous-time dynamics

\[ \dot{x}(t) = f(x) + u g_1(x) + \ldots + u g_m(x) \]  

(2.1)
where \( x \in M \) (a smooth manifold of \( \mathbb{R}^n \)), \( u \in \mathbb{R}^m \) and \( g \) are analytic vector fields on \( M \), which are assumed complete. Denote by \( \mathfrak{g} \) the control Lie algebra associated to \((2.1)\) (i.e., the smallest Lie algebra which is invariant under \( f, g \) and containing \( f, g \)) and \( \mathfrak{g} \) the Lie ideal of \( \mathfrak{g} \) generated by the \( g \). The distribution \( \Delta = \text{span} \{ c, e \in \mathfrak{g} \} \) is said to be nonsingular on \( M \) if there exists an integer \( d \) such that:

\[
\text{dim } \Delta(x) = d \quad \forall x \in M
\]

\( \mathfrak{g} \) is said to be nilpotent of order \( p \) (\( \mathfrak{g}^p = 0 \)) if the following decreasing sequence stops:

\[
\mathfrak{g}^1 \subseteq \mathfrak{g}^2 \subseteq \cdots \subseteq \mathfrak{g}^{p-1} \subseteq 0,
\]

In the sequel \( f \) and \( g \) indifferently denote the vector fields and their associated Lie derivatives. At the same time \([f, g]\) denotes the Lie bracket of vector fields

\[
[f, g] = \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}.
\]

Through the paper \( \delta_k \) will denote a sampling period small enough to guarantee the convergence of the series expansions manipulated.

**Usual sampling**

Given a sampling period \( \delta \in [0, \delta_0] \), let us now assume in \((2.1)\), \( m = 1 \) (\( g_1 = g \)) and that the control \( u_1(t) = u(t) \) is constant over time intervals of amplitude \( \delta \):

\[
u(t) = u(t) \quad \text{for} \quad k\delta \leq t < (k+1)\delta \quad k \geq 0 \quad (2.2)
\]

**Definition 2.1** \((\text{[13]}\)\) A nonlinear sampled dynamics of the form

\[
x(k+1) = F^\delta x(k), u(k)) \quad (2.3)
\]

where \( F^\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is analytic for \( \delta \in [0, \delta_0] \), is said the sampled analogue of \((2.1)\) if for the same initialization \( x(0) \), the state evolutions of \((2.1)\) and \((2.3)\) coincide at sampled instants \( t = k\delta \), namely if \( x(t) = x(k) \) for \( k > 0 \).

**Proposition 2.1** For \( \delta \in [0, \delta_0] \), \((2.3)\) admits the exponential representation:

\[
F^\delta (\cdot, u) = e^{\delta [F, u]}(\cdot) \quad (2.4)
\]

**Remark 2.1** For computing purposes, it is interesting to note that, according to \((2.4)\), the dynamics \((2.3)\) can be written as

\[
x(k+1) = x(k) + \sum_{i=1}^n \frac{\delta^i}{i!} x^{(i)}(k) \quad (2.5)
\]

where \( x^{(i)}(k) \) represents the \( i \)-th time derivative of \( x(t) \) computed at time \( t = k\delta \).

Recalling the Baker-Campbell-Hausdorff formula \((\text{[12]}\)\) and related combinatoric expansions it has been shown in \(\text{[16]}\) that the sampled dynamics \((2.4)\) can be regrouped according to the successive powers in \( u \) as follows.

**Proposition 2.2** \((\text{[16]}\)\) For \( \delta \in [0, \delta_0] \), \((2.4)\) satisfies the expansion:

\[
e^{\delta [F, u]} = e^{\delta F} e^{\delta [u]} \sum_{i} \frac{\delta^i}{i!} \mathcal{P}^i \quad (2.6)
\]

where the \( \mathcal{P}^i = \mathcal{P}^i_1, \mathcal{P}^i_2, \ldots, \mathcal{P}^i_n \) are homogeneous Lie polynomial of degree \( i \) in the vector fields \( E_{v} \) where \( E_{v} \in \mathcal{E}^0 \) for \( i \geq 1 \).

Precisely, for the first terms one obtains

\[
\mathcal{P}^i = \mathcal{P}_1^i E_1^i, \mathcal{P}_2^i E_2^i, \mathcal{P}_3^i E_3^i, \ldots \quad (2.7)
\]

where

\[
\mathcal{P}_1^i = E_1^i \quad \mathcal{P}_2^i E_2^i E_2^i = E_2^i \quad (2.8)
\]

with

\[
E_1^i = 1 - \frac{\delta \delta d_f - \delta \delta d_g}{\delta \delta d_f} \quad (2.9)
\]

where \( \mathcal{W} \) denotes the shuffle product \((\text{[12]}\)\).

Composing the successive exponentials one obtains \((\text{[16]}\)\):

**Proposition 2.3** For \( \delta \in [0, \delta_0] \), \( x(0) \in \mathbb{R}^n \) and a sequence of piecewise constant inputs \( u(0), u(1), \ldots, u(p-1) \), the state \( x(p) \), reached at time \( t = p\delta \) under the sampled dynamics action \((2.4)\), is given by:

\[
x(p) = e^{\delta [F, u]} e^{\delta [u]} e^{\delta [u]} \ldots e^{\delta [u]} x(0) \quad (2.10)
\]

where

\[
\mathcal{P}^i = \mathcal{P}_1^i E_1^i, \mathcal{P}_2^i E_2^i, \mathcal{P}_3^i E_3^i, \ldots \quad \text{for} \quad i \geq 1 \quad \text{and} \quad j \geq 2 \quad (\text{2.11})
\]

**Multirate sampling**

Assume now the control \( u(t) \) constant over fraction of the time interval \( \delta \) (say \( \delta = \frac{\delta}{p} \), for \( p \geq 1 \)) and denote \( u(k) \) the constant value of \( u(t) \) over \([k\delta, (k+1)\delta) \) \( k = 0, 1, \ldots, p \).

**Definition 2.2** A nonlinear sampled dynamics with \( p \) inputs of the form

\[
x(k+1) = F^\delta x(k), u_1(k), \ldots, u_p(k)) \quad (2.12)
\]

where \( F^\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is analytic for \( \delta \in [0, \delta_0] \), is said the multirate sampled analogue of \((2.1)\) of order \( p \), if the same initialization, the state evolutions of \((2.1)\) and \((2.11)\) coincide at sampled instants \( t = k\delta \), namely if \( x(t) = x(k) \) for \( k > 1 \).

**Proposition 2.4** For \( \delta \in [0, \delta_0] \) and \( \delta = \frac{\delta}{p} \), the multirate sampled dynamics of order \( p \), associated to \((\text{[1]}\)\), admits the exponential representation:

\[
F^\delta (\cdot, u_1, \ldots, u_p) = e^{\delta F} e^{\delta u_1} e^{\delta u_2} \ldots e^{\delta u_p} \mathcal{P}^i \quad (2.12)
\]

The proof is deduced from \((2.9)\) replacing \( \delta \) by \( \frac{\delta}{p} \) and \( u(i) \) by \( u_{i+1} \) \( i = 0, \ldots, p-1 \).
Remarks (i) Local controllability properties of the sampled dynamics (2.4) can be expressed in terms of the Lie algebra generated by $E_i^0$ for $i,j \geq 1$ ([14, 22]).

(ii) Assuming the nilpotency of $\mathcal{L}_0$ ($\mathcal{L}_0^q = 0$) it results that the series expansions in (2.9) and (2.12) are of finite length. Precisely $E_i^0 = 0$ for $i \geq q$ (no contribution of $u^i$ for $i > q$).

(iii) Assuming the nilpotency of $\mathcal{L}_0$ ($\mathcal{L}_0^q = 0$) then $\mathcal{L}_0^q = 0$. In general, (iii) holds and moreover the expansions in powers of $\delta$ of the $E_i^0$ defined in (2.7) (2.10) are polynomial with respect to $\delta$ of degree at most $q$.

3. Steering Under Digital Control

This section just gives an outline of the general idea for steering, and for the sake of simplicity, we firstly consider a two-input decentralized system configuration under digital control. As we extensively use used in the motion planning literature [4]. Again let us consider system (1.1) and assume that

\[
\begin{align*}
(\mathbf{a}) \quad & \mathcal{L}_0(x) = 0 \quad \text{on } M \quad (n \geq 3) \\
(\mathbf{b}) \quad & G_1(x) = m + 1 \quad \text{on } M \quad \text{with } G_1 = \text{span } \{f, g_1, \ldots, g_n\} \\
(\mathbf{c}) \quad & \mathcal{L}_0 \text{ is nilpotent of order } q \quad (\mathcal{L}_0^q = 0, q \geq 2)
\end{align*}
\]

Denote

\[
\{f, g_1, \ldots, g_m, \ldots, g_n - 1\}
\]

as a basis of the Lie algebra $\mathcal{L}_0$.

A given pair of states $(x_0, x_f)$ will be referred to as a pair of "admissible states" if there exists a sequence of piecewise constant inputs steering $x_0$ to $x_f$ at any $T \geq 0$. In practice, a "steering error" is admitted in the sense that we require the possibility to reach at least $x(T)$ with $\hat{x}(T) - x(T)[out]$. Given a pair of admissible states, the general idea for designing the steering digital control is to equate the state reached at time $T$ under multirate sampled dynamics (its order has to be determined with the final objective $x_f$). Then, the control solution is approximated by means of series inversion ([11]) making use of formal computing languages ([2, 3]).

Another way to proceed is to consider an arbitrary smooth trajectory $\mathcal{L}_0$ joining $x_0$ to $x_f$ (usually referred to in the literature, say $\mathcal{L}_0 = \mathcal{L}_0(x_0 + \delta)\mathcal{L}_0$) and to compute a piecewise constant control producing the "same trajectory". The main idea is that in practice an error margin will be still admitted. For, we begin by writing the sampled trajectory $\mathcal{L}_0$ and the sampled multirate dynamics $x(T)$ respectively at time $T$ in an exponential form with respect to the Control Lie algebra basis $(f, g_1, \ldots, g_n - 1)$ and compute the control solution by equating the exponents.

Thus these two methods differ from the meaning of the approximation performed in practice. The first method would be preferable in a system admitting sampled dynamics of finite degree with respect to the controls and would result in exact steering when exact sampled dynamics can be computed. The second method, which is directly related to the controllability directions, would provide a better approximation in more general cases, in particular when nilpotent systems are considered this procedure simplifies.

The extended system under sampling

Given a pair of admissible states $(x_0, x_f)$, choose a trajectory $\mathcal{L}_0 = \mathcal{L}_0(x_0 + \delta)$ such that $\mathcal{L}_0 = x_0$ and $\mathcal{L}_0 = x_f$ for $\delta \in \mathbf{E}_0, \mathbf{E}_1$. Then compute the "extended equation" defined in [9, 10, 6, 8] as follows:

\[
\mathcal{L}_0 = v_0(\mathcal{L}_0(x_0) + \sum_{i=1}^{n-1} v_i(\mathcal{L}_0(x_0)) (3.2)
\]

The coefficients $v_i(\mathcal{L}_0)$ are easily computed by matrix inversion, namely:

\[
\mathcal{L}_0 = v_0(\mathcal{L}_0(x_0) + \sum_{i=1}^{n-1} v_i(\mathcal{L}_0(x_0)) \mathcal{L}_0^i (3.3)
\]

To express the solution of (3.2) in time $\delta$, one has to apply the sampling procedure proposed in [17] for systems driven by smooth control. For, denoting by $v_i(\mathcal{L}_0)$ the value of the $i$-th time derivative of $v_i$ at time $t = 0$ and defining the successive vector fields

\[
V_1(x_0) = v_0 + \sum_{i=1}^{n-1} v_i(x_0) (3.4)
\]

one obtains

\[
\mathcal{L}_0 = e^{-\sum_{i=1}^{n-1} \mathcal{L}_0^i} V_1(x_0, \ldots, V_1(x_0)) (3.5)
\]

where the $V_i$ are the polynomial (2.7), here defined with respect to the vector fields $V_i$.

Regrouping the exponents with respect to the basis of $\mathcal{L}_0$ one obtains:

\[
\mathcal{L}_0 = e^{-\sum_{i=1}^{n-1} \mathcal{L}_0^i} V_1(x_0, \ldots, V_1(x_0)) (3.6)
\]

where the $\mathcal{L}_0(\delta, \mathcal{L}_0)$ are series on its arguments.

Remarks (i) In the driftless case ($f = 0$), denoting by $(g_1, g_2, \ldots, g_n)$ a basis for the Control Lie algebra $\mathcal{L}_0$, we denote as $\mathcal{L}_0(\delta, \mathcal{L}_0)$ the series satisfying the equality

\[
\mathcal{L}_0(\delta, \mathcal{L}_0) = e^{-\sum_{i=1}^{n-1} \mathcal{L}_0^i} V_1(x_0, \ldots, V_1(x_0)) (3.7)
\]

where the $\mathcal{L}_0(\delta, \mathcal{L}_0)$ are series in their arguments for $\delta \in \mathbf{E}_0, \mathbf{E}_1$. The series $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent, then the series $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpotent. Then $\mathcal{L}_0(\delta, \mathcal{L}_0)$ is nilpoten
The proof follows from (2.11-12) setting \( p = n - 1 \) and the Baker-Campbell-Hausdorff formula.

**Remark** When \( \mathcal{Z} \) is nilpotent, the \( h_i(\delta, \psi) \) are polynomials.

**Proposition 3.1** Given a single input continuous-time dynamics (2.1) satisfying (a), (b), (c), given a pair of "admissible states" \( \{x_0, x_f\} \), there exists a multirate digital control of order \( n - 1 \) or a steering \( x_0 \) to \( x_f \) in one step of amplitude \( \delta \) if the set of \( n \) equations:

\[
(h_i(\delta, \psi(0))) \equiv a_0, \ldots, a_{n-1} \tag{3.8}
\]

admits a solution \( (\delta, u_1, \ldots, u_{n-1}) \).

The proof is achieved by equating the exponents of (3.6) and (3.7).

**Remarks** (i) In practice, the solution, which is an \( n \)-uple \( (\delta, u_1, \ldots, u_{n-1}) \) is approximated by means of formal series inversion (see Gröbner [5] for example) making use of formal computing languages (2, 3).

(ii) In this non zero drift case, the fact that the steering time \( \delta \) is not free but computed as the control from series reversion represents a strong restriction.

**Motion planning for systems without drift term**

From \( G_i \) define the sequence of distributions

\[
G_i = G_{i-1} + 1 + (G_{i-1}) \tag{4.1}
\]

with

\[
[G_i, G_{i-1}] = \text{span} \{[g, h], g \in G_i, h \in G_{i-1}, i \geq 1. \}
\]

System (2.1) is said to be maximally nonholonomic if there exists an integer \( p < n \) such that \( \text{rank} G_i = \text{rank} G_{i-1} \) for all \( i \geq p + 1 \) and \( \text{rank} G_{i-1}(x) = n \) on \( M \). Consider now a maximally nonholonomic driftless dynamics with two inputs, that is, (2.1) with \( m = 2 \) and \( f = 0 \) satisfying conditions (a-b-c). Choose arbitrarily a multirate control of order \( n - 1 \) on the control \( u_2 \) and one on \( u_1 \).

**Lemma 3.2** Given the dynamics (2.1) with \( m = 2 \) and \( f = 0 \), for \( i \in [0, \delta] \), the state reached from \( x_0 \) in one step under the multirate sampled dynamics of order \( (n-1) \) on \( u_2 \) and one on \( u_1 \) can be written as:

\[
x(\delta) = \mathcal{F}^\delta(x_0, u_1, u_2, \ldots, u_{2n-1})
\]

\[
= \mathcal{F}^\delta(\delta, u_1, u_2)g_1 + \cdots + \mathcal{F}^\delta(\delta, u_1, u_2)g_n(x_0) \tag{3.9}
\]

where the \( \mathcal{F}^\delta(\delta, u_1, u_2) \) are series in their arguments.

**Remark** When \( \mathcal{Z} \) is nilpotent, then the \( \mathcal{F}^\delta(\delta, u_1, u_2) \) are polynomial.

Arguing as previously, one obtains:

**Proposition 3.2** Given the dynamics (2.1) with \( m = 2 \) and \( f = 0 \) satisfying (a), (b), (c), given a pair of admissible states \( \{x_0, x_f\} \) if there exists a solution to the set of equations:

\[
(\mathcal{F}^\delta(\delta, u_1, u_2) = \mathcal{F}^\delta(\delta, \psi)) \quad i = 1, \ldots, n \tag{3.10}
\]

then the system can be steered in one step of amplitude \( \delta \) from \( x_0 \) to \( x_f \) using a single rate digital control on \( u_1 \) and a multirate control of order \( n - 1 \) on \( u_2 \).

**Remarks** (i) In this driftless case, a solution is an \( n \)-uple \( (u_1, u_2, \ldots, u_{n-1}) \) and thus the sampling time \( \delta \) is free.

(ii) Depending on \( x_0 \) and \( x_f \), it might be convenient to design a multirate of order \( n - 1 \) on \( u_1 \) rather than on \( u_2 \) even on both controls when the choice \( u_{10} \neq 0 \) or \( u_{20} \neq 0 \) is not possible.

The digital control procedure here proposed has to be compared to dead beat control since the objective is reached in one step. For avoiding high amplitude control but also for solving obstacle avoidance problems for example one can propose to steer a sampled trajectory composed with a sequence of admissible pairs of states. Since the control solution is approximated, it may be convenient for improving the steering to iterate the control procedure over several steps thus designing a steering digital feedback control. Another possibility is also represented by adding extra order on the multirate control \( (n-1) \) for satisfying other constraints such as optimality.

**4. Motion planning of nonholonomic chained systems**

Under the usual assumptions set in the motion planning literature, it has been shown in Section 3 that it is possible to steer under multirate digital control the system in an "admissible" state configuration, at least in a approximated meaning. The remaining difficulties are the choice of the order of the multirate on the various inputs channels depending on the desired objective and the control computations.

An interesting situation occurs when triangular systems are considered, in particular when chained systems are investigated. In this case the previous study is highly simplified. When we look at the literature it clearly appears that a great number of practical examples verify such particular forms. Moreover, if this is not the case, it often occurs that such forms can be obtained under coordinate changes and feedback [9,10,11,18,19] and sufficient conditions to convert systems into chained forms have been given in [18,19]. This considerably enlarges the interest to discuss such forms.

In this section we will consider two inputs single chained systems and show with an example how the control algorithm is simplified in this case and achieves exact steering.

Consider a two input system transformed under coordinates and input changes into the chained form:

\[
x_1 = u_1
\]

\[
x_2 = u_2
\]

\[
x_3 = x_2u_1
\]

\[
x_n = x_{n-1}u_1
\]

and assume that it is controllable on \( M \). In this case it can easily be verified that:

\[
g_1 = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} x_i \frac{\partial}{\partial x_i} \quad g_k = \frac{\partial}{\partial x_k}
\]

\[
\text{adj}_k g_2 = (-1)^k \frac{\partial}{\partial x_{k+2}} \text{ for } k = 0, \ldots, n - 2;
\]

\[
\text{adj}_k g_2 = 0 \text{ for } i \geq n - 1
\]

and

\[
\text{adj}_k \text{adj}_k g_2 = 0 \text{ for } i \geq 0
\]

thus obtaining with \( g := (-1)^2 \text{adj}_k g_2 \), for \( i = 3, \ldots, n \):

\[
\mathcal{Z} := \text{span} \{g_1, g_2, g_3, \ldots, g_n\}
\]
According to the definitions (2.8), the multirate sampled system of order one on \( u_1 \) and \( n-1 \) on \( u_2 \) will be characterized by the vector fields, obtained by substituting \( u_1 \rightarrow f \) and \( g_2 \rightarrow g \).

One finds with \( \delta = \frac{\delta}{n-1} \)

\[
E_1' = 1 - \frac{\delta adj u_1 g_1}{\delta adj u_1 g_1} (\delta g_2) =
\]

\[
\bar{g}_2 + \frac{\delta}{2}(u_1 g_3 + \ldots + \left(\frac{\delta}{n-1}\right)^{n-2} a_n)
\]

(4.2)

\[
E_{i,j}^{(n)} = e^{i \delta adj u_1 g_1} (E_1')^j,
\]

for \( j = 1, \ldots, n-2 \) (4.3)

\[
E_i' = 0 \text{ and } [E_i', E_j'] = 0 \text{ for } i \geq 2, j \geq 1, \text{ } i \neq j
\]

Simple computations, inspired from [16], show that

**Lemma 4.1** If \( u_1 \neq 0 \) then the \( n-1 \) vector fields \( E_1'(x), \ldots, E_{n-1}'(x) \) are linearly independent on \( M \).

**Lemma 4.2** Given any initial state \( x_0 \), the state reached at time \( \delta \) under the action of \( u_1 \) over \([0, \delta] \) and \( u_2 \) over \([i-1, i) \delta \), \( i \in \{1, \ldots, n-1\} \) is given by:

\[
x(\delta) = e^{\delta u_1 g_1} (e^{\delta u_2 g_2} + \ldots + e^{\delta u_{n-1} g_{n-1}}) (x_0)
\]

which can be written because of (4.3) as

\[
x(\delta) = e^{\delta u_1 g_1} (e^{\delta u_2 g_2} + \ldots + e^{\delta u_{n-1} g_{n-1}}) (1)\bigg|_{1 \leq i \leq n-1}(x_0)
\]

(4.6)

(The bar \( \bar{x} \) indicates the evaluation at \( x \) of the function inside the parentheses.)

Because of the triangularity of the chained form, the exponential expansion (4.6) stops and one easily obtains exactly

\[
(x_i(\delta)) = (x_0i_1 + \delta u_1 i_2 + \delta u_2 i_3 + \ldots + \delta u_{n-2} i_{n-1})
\]

(4.7)

Exact steering is thus achieved solving the \( n \) equalities:

\[
\delta u_1 = \delta a_1
\]

\[
\delta = \frac{\delta u_2}{2}(u_2 \geq 2(u_1 + u_2)) = \delta a_2
\]

\[
\frac{\delta^2}{8}(u_2 \geq 2(u_1 + u_2))u_2 = \delta a_3 - \delta x_0 u_4
\]

setting with \( a_1 \neq 0 \) the exact solution:

\[
\begin{cases}
  u_1 = a_1 \\
  u_2 = 2a_2 - u_2
\end{cases}
\]

Figs. 1-2 illustrate the steering action from \( x_0 \) to \( x_f \) in one path of amplitude \( \delta = 1 \). The example tested illustrates the benefits of the digital control proposed with respect to the number of paths required, the simplicity of the control for exact steering, its limited amplitude and the state trajectories (Figs. 1-2). It has to be noted that, after reversing the input changes, the control \( u_1 \) and \( u_2 \) (Fig. 4) become continuous in the sense that they are simulated at the frequency for integrating system (4.9) (Fig. 3).

**REFERENCES**


In order to illustrate the control procedure, let us consider the model of steering a unicycle ([1, 6.10, 19]). The equations are:

\[
\begin{align*}
  \dot{x} &= \cos \theta e_1 \\
  \dot{y} &= \sin \theta e_1 \\
  \dot{\theta} &= e_2
\end{align*}
\]

(4.9)

where \( x \) and \( y \) represent the coordinates of the centre of the unicycle and \( \theta \) represents the rotational angle with respect to \( x \).

Let us transform the system into a chained form, for it is classical ([1, 6.10, 19]) to consider the input changes \( w_1 = \cos \theta e_1 \), and \( w_2 = e_2 \) and the coordinate changes (\( \cos \theta \neq 0 \), \( x_1 = x \), \( x_2 = y \), \( x_3 = \theta \), \( x_4 = y \), one finds:

\[
\begin{cases}
  \dot{x}_1 = u_1 \\
  \dot{x}_2 = u_2 \\
  \dot{x}_3 = x_2 u_1 \\
  \dot{x}_4 = x_3 u_1
\end{cases}
\]

(4.10)

with \( u_1 = w_1 \) and \( u_2 = \cos \theta \). Setting \( \delta = \frac{\delta}{2} \), one obtains the sampled version of system (4.10) under single rate on \( u_1 \) and multirate of order 2 on \( u_2 \), that is:

\[
\begin{align*}
  x_1(\delta) &= x_{01} + \delta u_1 \\
  x_2(\delta) &= x_{02} + \delta \left(\frac{1}{2}(u_1 + u_2)\right) \\
  x_3(\delta) &= x_{03} + \delta x_2 u_1 + \delta \left(\frac{1}{8}(3u_1 + u_2)u_1\right)
\end{align*}
\]

Given a pair of states \( [x_0, x_f] \), setting \( x_f = x_0 + \delta \), one has to solve the three equations:

\[
\begin{align*}
  \delta u_1 &= \delta a_1 \\
  \delta &= \frac{\delta u_2}{2}(u_2 \geq 2(u_1 + u_2)) = \delta a_2 \\
  \frac{\delta^2}{8}(u_2 \geq 2(u_1 + u_2))u_2^2 &= \delta a_3 - \delta x_0 u_4
\end{align*}
\]

Setting for example \( \delta = 1 \), \( x_0 = (0.0,-0.6) \) and \( x_f = (3.3,0) \), Figs. 1-2 illustrate the steering action from \( x_0 \) to \( x_f \) in one path of amplitude \( \delta = 1 \). The example tested illustrates the benefits of the digital control proposed with respect to the number of paths required, the simplicity of the control for exact steering, its limited amplitude and the state trajectories (Figs. 1-2). It has to be noted that, after reversing the input changes, the control \( u_1 \) and \( u_2 \) (Fig. 4) become continuous in the sense that they are simulated at the frequency for integrating system (4.9) (Fig. 3).

Simulated example


