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Some comments about linearization under sampling

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Abstract

Given a partially linearizable continuous-time dynamics, the aim of the present paper is to propose a digital control scheme which preserves this property under sampling. This leads to the introduction of some kind of a "sampled normal form". Starting from such a form and considering the sampled dynamics as a system which is regularly perturbed by the sampling period \( \delta \), a digital control strategy for achieving partial linearization at a fixed order of approximation is proposed.

1 Introduction

Given a nonlinear continuous-time single-input dynamics

\[ \dot{x} = f(x) + g(x)u \quad (1.1) \]

where \( f \) and \( g \) are real analytic vector fields, a well-known control procedure consists in the design of partial linearizing feedback laws. Such control strategies and related ones can be set making reference to some functions considered as dummy outputs associated to (1.1), the corresponding zero dynamics playing a central role in the design algorithm [7]. A basic notion is that of a relative degree, corresponding to the excess of the poles over the zeros of the transfer function in a linear context. In the most favourable case, this degree is equal to the state dimension and thus full linearization can be achieved. In a purely discrete-time nonlinear context, a similar problem has been studied and control solutions have been proposed [12], [6], [8], [9]. An appealing situation, with respect both to theoretical and practical points of view, occurs when control systems under sampling are investigated. An analysis of the structural properties of systems under sampling can be performed making use of quite sophisticated algebraic and combinatoric tools showing interesting connections with the differential geometric context of the existing continuous time results [10], [1]. As to applications, sampled systems are naturally considered when computer control of processes admitting continuous time modelizations is discussed. In such a case, specific digital control schemes have to be designed since the digital implementation of the continuous control schemes by means of zero-holding circuits may significantly degrade the performances. To overcome such difficulties multirate control schemes are proposed in [11].

The aim of the present paper is to propose a digital solution to the partial linearization problem. This is discussed in an approximated sense which means that the series expansions in powers of the sampling period \( \delta \) which are usually manipulated when systems under sampling are considered, are truncated at a certain order.

It is shown that under technical assumptions, it is possible to compute an approximated digital solution. This is achieved in two paths. Let us note that under iterative algorithm an exact solution can be deduced [5].

Assuming the dynamics (1.1) in a normal form [7] with a linearizable part of dimension \( r \), (or relative degree equal to \( r \) with respect to some dummy output function), the first path consists in associating to (1.1) a "sampled normal form". This form enlightens the possibility to preserve under sampling, with respect to a dummy output, the relative degree up to some approximation in \( \delta \). Starting from this sampled normal form, the second path improves the order of approximation, by means of successive coordinate changes and feedback laws. This control procedure, already proposed for regularly perturbed systems [3], [4], [13] can be applied in the present context by considering and manipulating the sampling period as a small parameter.

The paper is organized as follows. Section two introduces the sampled normal form corresponding to a continuous-time system which admits a maximal linearizable part of dimension \( r \) under static state feedback action. Starting from this form, section three discusses the first step of the control algorithm which allows the partial linearization under successive coordinate changes and digital feedback laws. In the last section a simple example illustrates the control procedure.

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2 Normal form under sampling

Let us consider a single input differential state equation \( \Sigma \):

\[
\dot{z} = f(z) + g(z)u
\]

where \( f \) and \( g \) are analytic vector fields defined on \( M \) (a submanifold of \( \mathbb{R}^n \)) and assume that (2.1) is partially feedback linearizable. After an adequate coordinate change, let us rewrite (2.1) as:

\[
\dot{z} = f(z) + g(z)u
\]

and \( f(x) + g(x)u = L_f g u \), where \( f \) and \( g \) are analytic vector fields defined on \( M \).

More precisely,

\[
x_a = (\sigma_1(z), \ldots, \sigma_n(z)) \quad \text{with} \quad \sigma_i = L_f^{(i-1)} \sigma_{i-1} \quad \text{for} \quad i = 1, \ldots, n
\]

and \( L_f \) is a \( r \times r \) matrix such that \( L_f \sigma_{i-1} = 0 \) for \( j = r + 1, \ldots, n \).

Remark: The relative degree associated to the dummy output function \( \sigma_i(z(t)) = \sigma_i(t) \) is equal to \( r \).

Assuming the control \( u(t) \) constant over time intervals of amplitude \( \delta \) (the sampling period), the only times \( t \) for which \( \sigma_i(t) \) is constant are multiples of \( \delta \).

Let us compute the sampled version of (2.2) according to:

\[
z(k+1) = z(k) + \sum_{i=1}^{\delta} z(i)(k)
\]

where \( z(i)(k) \) represents the \( i \)-th time derivative of the state \( z(t) \) computed at time \( t = k\delta \), for \( k \geq R \).

One obtains the sampled analogue \( \Sigma_d \) of (2.2):

\[
x(k+1) = f^d(z(k), u(k)) + \sum_{i=1}^{\delta} \frac{\partial f^d}{\partial u}(z(k), u(k)) \cdot z(i)(k)
\]

where

\[
A_d = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\delta & \delta & \ldots & \delta \\
\vdots & \vdots & \ddots & \vdots \\
(\delta - 1) \delta & (\delta - 2) \delta & \ldots & 0
\end{pmatrix}
\]

and

\[
F_d^d(z, u) = e^{A_d \delta} F_d(z, u)
\]

Given any sampled system of the form

\[
x(k+1) = F_d^d(x(k), u(k))
\]

where \( x \in M \) and \( u \in R \), \( F_d^d : M \times R \rightarrow R \) and \( h^d : M \rightarrow R \) are analytic functions admitting series expansions in \( \delta \), one sets (12):

Definition 2.1: The sampled system (2.6), (2.7) has a relative degree \( r \) on \( M \) if for any state \( x(k) \in M \), \( u(k) \in R \):

\[
(i) \quad \frac{\partial y(k+i)}{\partial u}(z(k), u(k)) = 0 \quad \forall i = 0, \ldots, r - 1
\]

\[
(ii) \quad \frac{\partial y(k+i)}{\partial u}(z(k), u(k)) \neq 0
\]

Moreover referring to the series expansions in \( \delta \) of the composed input-output dynamics, namely:

\[
y(k+i) = h^d \circ F^d(\cdot, u(k+i)) \cdot \ldots \cdot h^d \circ F^d(\cdot, u(k))
\]

if (i) and (ii) hold up to the order \( p \) in \( \delta \), the system will be said to have a relative degree \( r \) at the order \( p \) in \( \delta \).

Lemma 2.2 Given a continuous-time dynamics (2.2), of relative degree \( r \) with respect to \( y = z_1 \), the associated sampled system has a relative degree one, when input-output approximation of order \( r \) at least are considered.

Proof of 2.2 From (2.8) for \( i=1 \), one has:

\[
\frac{\partial y(k+1)}{\partial u}(z(k), u(k)) \neq 0 \quad \forall k \geq 0
\]

which implies \( r = 1 \).
It will be shown hereafter that replacing \( y = x_1 \) by an adequate linear combination of the \( x_i \) (for \( i = 1, \ldots, r \)) it is possible to maintain under sampling the relative degree up to an error of order \( (r + 1) \) in \( \delta \).

**Proposition 2.3:** Given a continuous time dynamics (2.2), in normal form, of relative degree \( r \) with respect to \( y = x_1 \), there exists a modified output function

\[
y' = \sum_{c(x_1, y_1, \ldots, y_r) \in \mathcal{R}} c_6 x_1, \quad (c, x) \in \mathcal{R}
\]

with respect to which the relative degree is equal to \( r \) at the order \( r + 1 \) in \( \delta \).

**Proof of 2.3:** The proof consists in showing that the relative degree associated to the sampled dynamics (2.5) with output \( y' \) is equal to \( r \) when input-output approximation of order \( r \) in \( \delta \) is considered. Since sampled input-output maps are truncated at the order \( r \) in \( \delta \), it is enough to start with the following approximation of the dynamics (2.5):

\[
x_1(k + 1) = A_6 x_1(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)
\]

The order dynamics can be analysed as a linear one and the problem is to find a linear output \( y' = C_6 x_1 \) in such a way that the following conditions hold:

\[
C_6 B_6 = 0, \quad C_6 A_6^{r-1} B_6 = 0
\]

Since, the matrix \( (B_6, A_6 B_6, \ldots, A_6^{r-1} B_6) \) is regular (this is discussed later on) and can exist as:

\[
C_6 = \delta^r (0, \ldots, 0, 1)(B_6, A_6 B_6, \ldots, A_6^{r-1} B_6)^{-1}
\]

**Proposition 2.4:** Given a continuous-time dynamics (2.2), of relative degree \( r \) with respect to \( y = x_1 \), there exists a linear \( \delta \)-dependent coordinate change \( (\xi_6 = \Delta_6 x_1) \) and \( (\zeta_6 = x_1) \) under which the sampled system (2.5) is transformed as:

\[
\begin{align*}
\xi_6(k + 1) &= A_6 \xi_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)) + O(B_6^2) \\
\zeta_6(k + 1) &= F_6^\delta(\xi_6(k), u(k)) + O(\delta) \\
y'(k) &= \zeta_6(k)
\end{align*}
\]

with

\[
A_6 = (\delta I + \delta A) \quad B_6 = \delta B
\]

The equations (2.11) will be referred to as the "sampled normal form" associated to (2.2).

**Proof of 2.4:** Setting

\[
\Delta_6 = \begin{pmatrix}
C_6 \\
C_6 A_6 \\
\vdots \\
C_6 A_6^{r-1}
\end{pmatrix}
\]

with \( \Delta_6 = (\delta I + \Delta A) \) and \( B_6 = \Delta B \), it is easily verified that \( C_6 \), defined in (2.10) also satisfies:

\[
C_6 = (0, \ldots, 0, 1)(\delta_6, \delta_6, \ldots, A_6, A_6^{-1} B_6)^{-1}
\]

whith \((\delta_6, \delta_6, \ldots, A_6, A_6^{-1} B_6)\) regular, since \( A_6 \) is upper triangular with no negative term and \( \delta_6 \) is strictly positive.

Applying the transformation (2.12) to (2.11), one obtains:

\[
\begin{align*}
\xi_6(k + 1) &= \Delta_6 \xi_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)) + O(B_6^2) \\
\zeta_6(k + 1) &= A_6 \zeta_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)) + O(\delta)
\end{align*}
\]

or equivalently:

\[
\zeta_6(k + 1) = A_6 \zeta_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)) + O(\delta)
\]

(2.13)

Moreover, since \( T(\delta) = I + \delta A \) one has:

\[
\begin{align*}
f_6(x) + u_6(x) u(k) &= \Delta_6 f_6(x) + \delta f_6(x) + u_6(x) u(k) + O(\delta) \\
P_6^\delta(x, u(s)) &= P_6^\delta(x, u(s)) + O(\delta)
\end{align*}
\]

and the system (2.5) takes the form (2.11).

**Remarks:**

i. The previous construction is very close to the one used for obtaining a controllable canonical form. In fact, one has:

\[
\Delta_6(k + 1) - \Delta_6(k) = [A_6 \Delta_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k)]
\]

one recovers:

\[
\zeta_6(k) = A_6 \zeta_6(k) + B_6 f_6(x(k)) + u_6(x(k)) u(k))
\]

under \( \delta \) \( \rightarrow \) 0 with \( A \) and \( B \) in a control canonical form.

ii. \( C_6 \) and \( T(\delta) \) only depend on \( \sigma \) but not on the state and the input, so that they can be precomputed.

iii. Holding the continuous feedback:

\[
\begin{align*}
\omega_6(x(k), u(k)) &= \sigma^{-1}(x(k))[\xi_6(k) - f_6(x(k))] \\
y'(z) &= \frac{y(z)}{z^r} + O(z^{r+1})
\end{align*}
\]

(2.14)

(2.15)

3 Approximated partial linearization under digital control

Starting from the sampled normal form (2.11) it is shown hereafter that, under some technical assumptions, it is possible to find a digital solution to the partial linearization problem under coordinates change and feedback. For the sake of simplicity, this is discussed in an approximated sense, the series expansions with respect to the sampling period are truncated at the order \( r + 1 \) in \( \delta \). This represents the first step of
the more general control algorithm which would allow to compute iteratively an exact solution [5]. This control scheme, which has been introduced for regularly perturbed systems ([3], [4] and [13]) is here applied considering the sampling period as a small parameter. For, let us assume that the system is in a sampled normal form and that a control law \( u_0(k) \) is designed for satisfying prefixed objectives on \( x_0 \) (the linearizable part) up to an error \( O(\delta^{r+1}) \). In order to eliminate the \( \delta^{r+1} \) dependent term in \( y^d \) and so to assure, the partial linearization up to \( O(\delta^{r+2}) \), the following technical assumptions are needed:

**A1** : At any sampled time \( k \), the external control \( v \) is known for any sampled time \( k+i \) for \( i = 0, ..., r-1 \).

**A2** : The external control \( v \) does not depend on negative powers in \( \delta \).

**Remarks** :

(i)-A1 is not a constraint for many control problems such as tracking, regulation or stabilization for example.

(ii)-A2 excludes digital control objectives such as dead beat control or minimum time responses.

The procedure we propose is firstly based on the definition a diffeomorphism \( \phi \) in such a way to reject in the last component of the \( x_d \) dynamics the \( \delta \) dependent terms we want to eliminate. One sets:

\[
\chi = (\chi_1, \chi_2, ..., \chi_{r+1})^T = \phi(\xi, u_0) = (I_d + \delta^r \tau)(\xi)
\]

\[(I_d \text{ is the identity function, } \tau \text{ is a smooth function on its arguments}).
\]

Secondly, one sets a digital feedback of the form:

\[
u(\xi, v) = u_0(\xi, v) + u_1(\chi, v)
\]

in such a way to achieve the partial linearization up to order \( r + 1 \) in \( \delta \); \( u_0(\xi, v) \) is computed according to (2.14).

Making reference to the notion of partial feedback linearization as the "maximal part" which can be linearized by using static state feedback [7], one states:

**Theorem 3.1** : Let A1 and A2 be true. Given a partially feedback linearizable continuous-time system \( \Sigma \) of the form (2.2), then there exist a state diffeomorphism \( \phi \) and a digital feedback law \( u \) as in (2.2) which enable to get partial linearization under sampling at the order \( r + 1 \) in \( \delta \).

The proof of the theorem is deduced from the next proposition.

**Proposition 3.2** : Let A1 and A2 be true. Given a partially feedback linearizable continuous-time system \( \Sigma \) of the form (2.2), then there exist a state diffeomorphism \( \phi \) and a digital feedback law \( u \) as in (2.2), such that the closed-loop system under sampling \( \Sigma_s \) (2.5) takes the following form:

\[
\begin{align*}
\chi(k+1) &= A\chi(k) + B\chi(k) + O(\delta^2) \\
\chi(k+1) &= F^*(\chi, u, \delta) + O(\delta)
\end{align*}
\]

\[(the \text{ relative degree associated to } y^d = \chi_1 \text{ is equal to } r \text{ at the order } r + 1 \text{ in } \delta).
\]

**Proof of 3.2** In the previous paragraph, it has been shown that the sampled system \( \Sigma_s \) (2.5) associated to \( \Sigma \) can be transformed under linear \( \delta \) dependent coordinates change and feedback (2.14) into the form (2.11):

\[
\begin{align*}
\xi(k+1) &= A\xi(k) + B\xi(k) + O(\delta^2) \\
\xi(k+1) &= F^*(\xi(k), u(k), \delta)
\end{align*}
\]

Starting from (3.4), one defines the diffeomorphism \( \chi = \phi(\xi, u) \) in order to reject the nonlinear term

\[
T_a + \delta^2 B^x \text{ in the last linearized component } \chi_s.
\]

Precisely, the \( \xi_s \) component remains unchanged \( (\chi_s = \xi_s) \) and the \( \xi_e \) component is modified as follows:

\[
\begin{pmatrix}
x_1(k) \\
x_2(k) \\
x_3(k) \\
\vdots \\
x_{r+1}(k)
\end{pmatrix}
= \begin{pmatrix}
\xi_1(k) + \delta^r \tau_1(k) + p_1^1(k) \\
\xi_2(k) + \delta^r \tau_2(k) + p_1^2(k) \\
\vdots \\
\xi_{r+1}(k) + \delta^r \tau_{r+1}(k) + p_1^{r+1}(k)
\end{pmatrix}
\]

where

\[
p_j^1(k) = \left(T_a + \delta^2 B^x \right)_j L_2 u_0 \text{ for } j = 1, ..., r.
\]

Note that \( \xi(k+i) \) and \( u_0(k+i) \), which appear in the transformation \( \phi \) can be computed according to the approximated dynamics

\[
\xi(k+1) = A\xi(k) + B\xi(k) + u_0(k+i)
\]

After tedious but easy manipulations one obtains:

\[
\begin{pmatrix}
x_1(k+1) = A_1x_1(k) + B_1x_1(k) + O(\delta^2) \\
\vdots \\
0 = O(\delta^2)
\end{pmatrix}
\]

2395
\[ x_d(k+1) = F^*(x(k), u(k)) + O(\delta^2) \]  

(3.6)

In order to achieve the partial linearization up to \( O(\delta^{r+2}) \) the corrective feedback \( u_1 \) appearing in (3.6) must be chosen as:

\[ u_1(k) = -2x_2(x(k))^{-1}[y_1^{\delta}(\delta) + 3x(k) + 1 - 3c(x(k+1)) \]  

thus obtaining (3.3).

### Remarks:

(i) \( -\delta \in [0, \delta_0] \) also ensures that \( \phi \) is a diffeomorphism.

(ii) the zeros dynamics is unchanged under sampling only up to an error of order one in \( \delta \).

### 4 An illustrative example

Let us consider the following simple dynamics:

\[
\begin{align*}
x_1 &= x_2 \\
x_2 &= x_1^2 + x_2^3 + u
\end{align*}
\]

which is fully input-output linearizable with respect to \( y = x_1 \), under the feedback

\[ u = -x_1 - x_2^3 + v \]

The approximated sampled dynamics is:

\[
x(k+1) = A_d x(k) + B_d y(k) + u(k) + O(\delta^3)
\]

(4.2)

\[
\begin{align*}
A_d &= \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, & B_d^1 &= \begin{pmatrix} \frac{\delta^2}{2} \\ \frac{\delta^3}{2} \end{pmatrix}, & B_d^2 &= \begin{pmatrix} \frac{\delta^2}{2} \\ \frac{\delta^3}{2} \end{pmatrix}
\end{align*}
\]

It is important to note, that when the continuous time relative degree is equal to two, the matrix \( A_d \) is equal to the matrix \( A_1 \).

The relative degree associated to (4.2), with the output \( y = x_1 \), is equal to 1 at order 2 in \( \delta \). In fact, one computes:

\[
y(k+1) = x_1(k+1) = x_1(k) + 4x_2(k) + \frac{\delta^2}{2} [x_1^2(k) + x_2(k) + u(k)) + O(\delta^3)
\]

Dummy output function: Referring hereafter, to section 2, it is shown that the full linearization can be maintained at the order 2 in \( \delta \) with respect to a dummy output function which is a linear combination of \( y \) and \( y_1 \), namely (2.10):

\[
y^d(k) = C_{d1} y + \delta^2 C_{d2}
\]

which satisfies conditions:

\[
C_{d1} B_d^1 = C_{d1} A_d B_d^1 = \delta^2 \neq 0
\]

which correspond to a relative degree equal to two with respect to \( y^d \) at order 2 in \( \delta \). Thus the full linearization is maintained at order 2 in \( \delta \).

More precisely, the input-output response map associated to \( y^d \) is equal to:

\[
y^d(k+2) = 2y^d(k+1) + y^d(k) = \delta^2 [x_1^2(k) + x_2^2(k) + u(k)) + O(\delta^3)
\]

### Sampled normal form: Referring again to section 2, the system (4.2) can be transformed into the sampled normal form:

\[
\xi(k+1) = A_3 \xi(k) + B_3 \eta(k) + u(k) + O(\delta^5)
\]

(4.3)

\[
\begin{align*}
A_3 &= A_d, & B_3 &= (0, \delta)^T
\end{align*}
\]

under the coordinates change (2.12):

\[
T_3(\delta) = \begin{pmatrix} 1 & 0 \\ \frac{-\delta}{\delta_0} & 1 \end{pmatrix}
\]

where \( \xi = T_3(\delta)x \).

**Feedback stabilization:** Fixe, for example at sampling instant, the behaviour of \( y^d(x) \) equal to

\[
x^2 - 2x + 1 - \delta^3 \lambda x - \delta^2 \lambda^2
\]

in order to stabilize the system and also verify assumption A.2 and we obtain:

\[
u_0(k) = -[x_1^2(k) + x_2^2(k) + \lambda_1 y^d(k+1) + \lambda_2 y^d(k)]
\]

which obtaining:

\[
y^d(k+2) - 2y^d(k+1) + y^d(k) = \lambda_1 y^d(k+1) + \lambda_2 y^d(k) + O(\delta^3)
\]

with

\[
y^d(0) = \xi_1(k), \text{ and } y^d(k+1) = \xi_1(k) + \delta \xi_2(k)
\]

or again in the state space

\[
\xi(k+1) = A_3 \xi(k) + B_3 \lambda y^d(k+1) + \lambda_2 y^d(k) + u_1(k)) + \lambda_3 y^d(k)
\]

(4.3)

(4.3)

**Full linearization at order 3 in \( \delta \):** Referring hereafter, to section 3, it is possible to fully linearize the system (4.3) at order 3 in \( \delta \). For, we must eliminate the nonlinear term

\[
2T_3(\delta) B_3 [x_1(k) x_2(k) + x_2(k) x_1^2(k) + x_2^3(k) + u(k)]
\]

with

\[
T_3(\delta) B_3 = \begin{pmatrix} \frac{-\delta^3}{2} \\ \frac{\delta^3}{2} \end{pmatrix}
\]

in two steps:

**First step:** Define the diffeomorphism \( \phi \) as follows:

\[
\phi(x(k), u(k)) = \xi_1(k)
\]

\[
\xi_2(k) - \frac{\delta^3}{2} [2x_1(k)x_2(k) + 2x_2(k)x_1^2(k) + x_2^3(k) + u(k)]
\]

2396
thus obtaining:
\[
\chi(k+1) = A_x\chi(k) + B_x(\lambda_1 y(k+1) + \lambda_2 y(k) + \delta u(k))
\]
\[
+ \begin{pmatrix} 0 \\ \delta^2 \delta^2 \end{pmatrix} + O(B_x^2) \tag{4.4}
\]
with
\[
\begin{aligned}
\delta^2 &= x_1(k)x_2(k) + x_2(k)x_2(k) + x_2^2(k) + u_0(k) - \frac{1}{6} \\
&= \frac{1}{6} - \delta^2 x_2(k) + u_0(k)
\end{aligned}
\]
\[
\begin{aligned}
&\text{Second step: Computing the control corrective term } u_1, \text{ for obtaining the following behaviour} \\
&\chi(k+1) = A_x\chi(k) + B_x(\lambda_1 y(k+1) + \lambda_2 y(k)) + O(B_x^2)
\end{aligned}
\]
\[
\begin{aligned}
&\text{one obtain from (4.4)} \\
&u_1(k) = -\delta^2 \delta^2
\end{aligned}
\]
In order to compute \(\delta^2 \delta^2\) at the sampling instant \(k\), it is necessary to know \(u_0(k+1)\) and \(x(k+1)\) at this instant. For, it is sufficient to expand \(\xi(k+1)\) with an error of order \(1\) in \(\delta\), that is:
\[
\begin{aligned}
&\xi_1(k+1) = \xi_1(k) + \delta \xi_2(k) = \xi_1(k) + O(\delta) \\
&\xi_2(k+1) = \xi_2(k) + (\lambda_1 \xi_1(k) + \lambda_2 \xi_2(k)) = \xi_2(k) + O(\delta)
\end{aligned}
\]
thus obtaining \(x(k+1)\) as:
\[
\begin{aligned}
&x_1(k+1) = \xi_1(k) + \frac{\delta}{2} \xi_2(k) + \xi_1(k) + O(\delta) \\
&x_2(k+1) = \xi_2(k) + O(\delta)
\end{aligned}
\]
Moreover as
\[
\begin{aligned}
y'(k+1) &= x_1(k) + O(\delta) \\
u_0(k+1) &= -(x_2^2(k) + x_2^2(k) + \lambda_1 \xi_1(k) + \lambda_2 \xi_2(k)) + O(\delta)
\end{aligned}
\]
Thus we get:
\[
\begin{aligned}
&u_1(k) = -\frac{\delta}{6} [x_1(k)x_2(k) + x_2(k)x_2(k) + \lambda_1 \xi_1(k) + \lambda_2 \xi_2(k)]
\end{aligned}
\]
In conclusion, the full linearization is preserved at order \(3\) in \(\delta\) with respect to \(y\), as it is shown by the following relations:
\[
\begin{aligned}
y'(k+1) &= \chi_1(k) + \delta \chi_2(k) + O(\delta^3) \\
y'(k+2) &= \lambda_2 y(k+1) + \delta \delta^2 (k+1) + \delta \delta^2 (k+1) + O(\delta^3)
\end{aligned}
\]
\[
\text{References}
\]