Quadratic forms and approximate feedback linearization in discrete time

J.-P. Barbot, S. Monaco & D. Normand-Cyrot

Published online: 08 Nov 2010.

To cite this article: J.-P. Barbot, S. Monaco & D. Normand-Cyrot (1997) Quadratic forms and approximate feedback linearization in discrete time, International Journal of Control, 67:4, 567-586, DOI: 10.1080/002071797224081

To link to this article: http://dx.doi.org/10.1080/002071797224081

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.
Quadratic forms and approximate feedback linearization in discrete time

J.-P. BARBOT, S. MONACO and D. NORMAND-CYROT

Quadratic feedback linearization is discussed for discrete-time systems, i.e. approximate feedback linearization up to the second order with respect to input and state variables. After introducing in discrete time two equivalence classes, invariant under quadratic static state feedback and diffeomorphism, two canonical quadratic representations are proposed. Then it is shown that any nonlinear discrete-time dynamics, which are controllable in their first approximations, can be linearized up to the second order under dynamic feedback. The results stated here can be considered as the general discrete-time analogues of the continuous-time ones (Kang and Krener 1992).

1. Introduction

On the basis of nonlinear single-input discrete-time dynamics of the form

\[ x(k + 1) = F(x(k), u(k)) \]  

where \( M \) is an \( n \)-dimensional real differential manifold, \( F \) (defined from \( M \times \mathbb{R} \) to \( M \)) is an analytic map with controllable linear approximation around the equilibrium \((0, 0)\), we study quadratic feedback linearization. This is a problem set and solved by Kang and Krener (1992) in a continuous-time context.

Linearization under feedback in discrete time has been studied by Monaco and Normand-Cyrot (1983), Grizzle (1986a, b), Lee et al. (1987), Armanda et al. (1994), Monaco and Normand-Cyrot (1995). Based on the results of Monaco and Normand-Cyrot (1995), where necessary and sufficient conditions for linearization are given, involving a family of vector fields, we study the approximate feedback linearization up to the second order with respect to both the state and control variables. Approximate feedback linearization has been extensively studied (Krene 1984), Krener et al. 1987, Hauser 1991, Hermes 1991, Kang and Krener 1992, Guzzella and Isidori 1993, Godbole and Sastry 1993, Kang 1994a, b). It appears particularly appropriate in a discrete-time context where only approximated solutions can be computed to solve several control problems.

Starting from (1), we assume that a preliminary linear feedback and diffeomorphism have been applied to obtain

\[ \bar{x}(k + 1) = A\bar{x}(k) + Bv(k) + f(\bar{x}(k), u(k)) \]

with

Received 10 August 1995. Revised 2 October 1996

† Laboratoire des Signaux et Systèmes, CNRS-ESF, Plateau de Moulon, 91190 Gif-sur-Yvette, France. e-mail: barbot@iss.supelec.fr and cyrot@iss.supelec.fr.
‡ Dipartimento di Informatica e Sistemistica, Università di Roma 'La Sapienza', Via Eudossiana 18, 00184 Rome, Italy. e-mail: monaco@riscedis.ing.uniroma1.it.
Such a structure, with an invertible drift term, will be assumed throughout the paper. We note that the controllability of the linear approximation is a usual assumption in studying linear equivalence under feedback, and the preliminary application of a feedback and coordinates change does not imply any loss of generality.

It has been shown by Monaco and Normand-Cyrot (1989) that nonlinear difference dynamics admits an exponential representation when the drift-term $F(x, 0)$ (or $F(x, u_0)$ for $u_0 \in \mathbb{R}$) is invertible. Such an exponential form is described in terms of an infinite family of vector fields, defined on $M$, on the basis of which we can set analysis and synthesis problems in a differential geometric framework. This is a mathematical context commonly used in continuous time and more recently introduced in discrete time (Monaco and Normand-Cyrot 1984, Mokadem 1989, Jakubzyk and Sontag 1990, Albertini and Sontag 1993, Barbot 1990, Barbot et al. 1995).

Approximate feedback linearization in discrete time was first studied by Lee and Marcus (1986), who gave solutions in terms of control-dependent rank conditions. The problem is studied here in terms of the already-mentioned canonical vector fields. The study follows the lines Kang and Krener (1992), thus underlining the existence, as proposed by Monaco and Normand-Cyrot (1995), of a unified approach to study continuous- and discrete-time problems. Quadratically feedback equivalent discrete-time dynamics turn out to be equivalence classes associated with two series of characteristic numbers. The nullity of these numbers provides an algebraic test to check linear feedback equivalence at the second order. As in Kang and Krener (1992), two canonical forms are introduced to represent quadratically equivalent dynamics under static state feedback. On these bases it is shown that they are quadratically linear equivalent under dynamic state feedback. A constructive design procedure is described for the controller.

The results proposed can be generalized to the MIMO system owing to simple, though tedious, extended notations.

The paper is organized as follows. In §2 preliminaries concerning linear and linear feedback equivalences in discrete time are recalled in exact and approximate meaning. In §3 quadratic equivalences under coordinate change and feedback are studied for nonlinear discrete-time dynamics with a controllable linear part. Characteristic numbers that identify each class of quadratically feedback equivalent dynamics are introduced in §4. Two canonical forms associated with each class are proposed in §5. Quadratic linearization under dynamic feedback is proved in §6. Throughout the paper, simple examples illustrate the main concepts.
2. Preliminaries on feedback linearization

The purpose of this section is to recall some results concerning exact and approximate linear feedback equivalences.

2.1. Exponential representation

Consider the discrete-time dynamics (1) and assume that the drift term $F_0(.) \equiv F(.,0)$ is invertible, so that for $u$ in a neighbourhood of 0, $F(x,u)$ is invertible. Let $F_0(.)$ be the $i$th composition of $F_0$.

As did Monaco and Normand-Cyrot (1989, 1995), let us define the following vector field parametrized by $u$

$$G_i^0(.,u) = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} F(.,u+\epsilon) \cdot F^{-1}(.,u)$$

and denote by $G_i^0$ its transport along the free evolution $F_0^i$

$$G_i^0(.,u) = \left\{I,F_0^i(.,u)\right\}_{|_{G_i^0(.,u)}}$$

Consider, moreover, their expansions with respect to $u$

$$G_i^0(.,u) = G_i^0(.) + \sum_{j} \frac{u^j}{j!} G^j_{i+1}(.) \quad (4)$$

The $G_i^0(.) : M \to T_x M$ are smooth vector fields referred to as the canonical vector fields associated with the dynamics (1) (Monaco and Normand-Cyrot 1989).

It has been shown (Monaco and Normand-Cyrot 1995) that the discrete-time drift invertible dynamics (1) admit the exponential representation

$$x(k+1) = \exp \left[ \left( k \right) \left( x(k) \right) \right]$$

$$x(k) = \exp \left[ \left( 0 \right) G^0(.,u(0)) \right] \cdots \exp \left[ \left( k \cdot 1 \right) G^0(.,u(k - 1)) \right]$$

where $G_i^0(x,u)$ are smooth vector fields parametrized by $u$ and deduced from the $G_i^0(.)$ as

$$uG_i^0(.,u) = \sum_{j} \frac{u^j}{j!} B_i(G_1^0,\ldots,G_j^0) (I_j) \quad (7)$$

where $B_i(G_1^0,\ldots,G_j^0)$ is a homogeneous Lie polynomial of degree $i$ for $i \geq 1$ and $G_i^0$ stands for the Lie derivative $L_{G_i^0}$.

For the first terms, one has

$$B_i(G_j^0) = G_j^0$$

$$B_2(G_1^0,G_2^0) = G_2^0$$

$$B_3(G_1^0,G_2^0,G_3^0) = G_3^0 + \frac{1}{4}[G_1^0,G_2^0]$$

where $[.]$ denotes the usual Lie bracket of vector fields.

2.2. Exact and approximate linear feedback equivalence

Let us denote by $T := (\phi(x), \gamma(x,v))$ a smooth transformation composed of a smooth coordinates change from $M$ to $\mathbb{R}^n$. 

and a smooth static-state feedback from \( M \times \mathbb{R} \) to \( \mathbb{R} \)
\[
u = \gamma(x, v)
\]

**Definition 1**: Nonlinear discrete-time dynamics are (locally) linear feedback equivalent if there exists a smooth transformation (around \((0, 0)\)), which changes them into linear controllable ones.

According to this definition, the following theorem is recalled.

**Theorem 1** (Monaco and Normand-Cyrot 1995): The discrete-time drift invertible dynamics (1) are locally linear feedback equivalent around \((0, 0)\) if and only if

(a) \( \text{span} \{G_x^\phi, G_y^\phi, \ldots\} \subset \text{span} \{G_1^\phi\} \)

(b) \( \text{span} \{G_1^\phi, \ldots, G_{n-1}^\phi\} \) is involutive around 0

(c) \( \text{rank} (G_1(0), \ldots, G_{n-1}(0)) = n \)

The conditions of Theorem 1 can be weakened when dealing with approximate feedback linearization. For this, one considers the quadratic approximation around an equilibrium pair \((0, 0)\) of the dynamics (1) both with respect to \(x\) and \(u\). By denoting by \((\cdot)^{[\phi]}\) any homogeneous function of degree \(j\) in \(x\) and \(u\), one has

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) + f^{[\phi]}(x(k), u(k)) + O(x, u)^3 \\
uG_i^\phi(x) &= uG_1^\phi + uG_i^\phi(x) + O(x, u)^3 \\
u^2G_i^\phi(x) &= u^2G_2^\phi + O(x, u)^3
\end{align*}
\]

with

\[
f^{[\phi]}(x, u) = ug^{[\phi]}(x) + f^{[\phi]}(x) + u^2h^{[\phi]}\]

where

\[
g^{[\phi]}(x) = \left( \frac{\partial^2 F(x, u)}{\partial u \partial x} \right)_{x=0} f^{[\phi]}(x) = \frac{1}{2} \left( \frac{\partial^2 F(x, 0)}{\partial x^2} \right)_{x=0} x^{\phi 2}
\]

\[
h^{[\phi]} = \frac{1}{2} \frac{\partial^2 F(0, u)}{\partial u} = G_1^{\phi} \]

\[
A = \frac{\partial F(x, 0)}{\partial x} = G_1^{\phi} \quad B = \frac{\partial F(0, u)}{\partial u} = G_1^{\phi} \\
G_i^{[\phi]}(Ax) = g^{[\phi]}(x) + AX, \quad G_2^{[\phi]} = G_1^{[\phi] + \frac{\partial G_1^{[\phi]}(x)}{\partial x} \cdot G_i^{[\phi]} = h^{[\phi]} \}
\]

Barbot et al. (1995) studied approximate feedback linearization at any order of approximation around an equilibrium manifold (see Xu and Hauser (1992) in the continuous-time case). With reference to quadratic approximation around an equilibrium point one can state the following definition.
**Definition 2:** Nonlinear discrete-time dynamics are (locally) quadratically linear feedback equivalent if there exist a feedback and a smooth transformation (around \((0, 0)\)), which change them into linear controllable ones up to an approximated error of order three with respect to \(x\) and \(u\).

Denoting by \(G^l_i\) the approximation up to the order \(l\) of \(G_i\), one has the following proposition.

**Proposition 1:** The discrete-time drift invertible dynamics (1) are (locally) quadratically linear feedback equivalent (around \((0,0)\)), if and only if

(a) \(\text{span} \{G^0_i \} \subset \text{span} \{G^l_i \} \)

(b) \(\text{span} \{G^0_i, \ldots, G^{l-1}_i \} \) is involutive at the order 0, around 0

(c) \(\text{rank} (G^0_i, \ldots, G^{l-1}_i) = n \)

Note that in (b) involutivity at order 0 must be checked with reference to the constant approximations of the vector fields and their successive brackets.

As pointed out by Monaco and Normand-Cyrot (1995), the same formulation as that in Theorem 1 may be used for linear feedback equivalence to continuous-time dynamics of the form

\[ \dot{x} = f(x, u) = f_0(x) + \sum_{i=1}^{l} \frac{d}{dt} g_i(x) \]  

(11)

once the \(G_i^k\) have been replaced by the \(\text{ad}_{g_i} \).

The same argument works for approximated feedback linearization so that the results of Barbot et al. (1995) and Proposition 1 above represent the generalization to dynamics (11) of the results of Kang and Krener (1992).

### 3. Quadratic feedback equivalence in discrete time

The purpose of this section is to study quadratic feedback equivalence.

**Definition 3:** Two nonlinear discrete-time dynamics of the form (1) are quadratically feedback equivalent if there exist a change of coordinates and a static feedback under which their approximations coincide up to the quadratic terms.

We note that, as pointed out in the introduction, under the controllability assumption, the canonical form (2) may be assumed. In our study we will therefore refer to the following structure:

\[ \tilde{z}(k + 1) = A \tilde{z}(k) + B x(k) + f_1(\tilde{z}(k)) + g_1(\tilde{z}(k)) v(k) + h_1(\tilde{z}(k))^3 + O(\tilde{z}^4) \]  

(13)

so that, starting from the same linear part, only the terms which can modify the quadratic part are taken into account. With this in mind and without any loss of generality, feedback and the coordinates transformation will be assumed of the form

\[ v = w(\tilde{z}, u) = u + \alpha_1(\tilde{z}) + \beta_1(\tilde{z}) u + \gamma_1(\tilde{z}) u^2 \]  

(13)

\[ x = \phi(\tilde{z}) = \tilde{z} + \phi_1(\tilde{z}) \]  

(14)

It can easily be verified that the class of transformations given by (13) and (14) has a group structure for the usual composition, provided that truncations at order two in both the state and control variables are performed. Setting
\[ T_1(\xi, u) = \begin{cases} 
  x = \xi + \phi \mathcal{E}(\xi) \\
  v = u + \alpha \mathcal{E}(\xi) + \beta \mathcal{E}(\xi)u + \gamma \mathcal{E}u^2 
\end{cases} \] (15)

\[ T_2(x, \mu) = \begin{cases} 
  z = x + \phi \mathcal{E}(x) \\
  u = \mu + \alpha \mathcal{E}(x) + \beta \mathcal{E}(x)\mu + \gamma \mathcal{E}u^2 
\end{cases} \]

one easily computes the composition \( T_2 \cdot T_1 \) as

\[ T_2 \cdot T_1(\xi, u) = \begin{cases} 
  z = \xi + \phi \mathcal{E}(\xi) + \phi \mathcal{E}(\xi) + O(\xi^2) \\
  v = \mu + \alpha \mathcal{E}(\xi) + \alpha \mathcal{E}(\xi) + (\beta \mathcal{E} + \beta \mathcal{E}u) \mu + \gamma \mathcal{E}u^2 + O(\xi, \mu)^3 
\end{cases} \]

and the inverse transformation as

\[ T_1^{-1}(x, v) = \begin{cases} 
  \xi = x - \phi \mathcal{E}(x) + O(x^3) \\
  u = v - \alpha \mathcal{E}(x) - \beta \mathcal{E}(x)v - \gamma \mathcal{E}v^2 + O(x, v)^3 
\end{cases} \] (16)

With this in mind, it is a matter of computation to verify that under (13) and (14) the dynamics (12) be expressed as

\[ x(k+1) = Ax(k) + Bu(k) + f_\mathcal{E}(A, x)(k) + g_\mathcal{E}(A, x)u(k) + h_\mathcal{E}u(k)^2 + O(x, u)^3 \] (13)

with

\[ f_\mathcal{E}(x) = f_\mathcal{E}(x) - A\phi \mathcal{E}(x) + \phi \mathcal{E}(A, x) + \alpha \mathcal{E}(x)B \] (18)

\[ g_\mathcal{E}(x) = g_\mathcal{E}(x) + \beta \mathcal{E}(x)B - \beta \mathcal{E}(x)B, B \] (19)

\[ h_\mathcal{E} = h_f + B \eta_f + \frac{1}{2} \frac{\partial \phi}{\partial \mathcal{E}} B \] (20)

These arguments constitute the proof of the following proposition.

**Proposition 2:** The two dynamics (12) and (17) are quadratically equivalent under (13) and (14), if and only if \((\phi \mathcal{E}, \alpha \mathcal{E}, \beta \mathcal{E}, \gamma \mathcal{E})\) satisfy

1. \( f_\mathcal{E}(x) - f_\mathcal{E}(x) = - A\phi \mathcal{E}(x) + \phi \mathcal{E}(A, x) + \alpha \mathcal{E}(x)B \) (21)
2. \( g_\mathcal{E}(x) = g_\mathcal{E}(x) + \beta \mathcal{E}(x)B - \beta \mathcal{E}(x)B, B \) (22)
3. \( h_\mathcal{E} = h_f + B \eta_f + \frac{1}{2} \frac{\partial \phi}{\partial \mathcal{E}} B \) (23)

**Remark 1:** In terms of the canonical vector fields \( G_i \), an equivalent formulation of (b) and (c) in Proposition 2 is

\[ (b) \quad G_1[x] - G_1[x] = \beta \mathcal{E}(x)G_1[x]B \]

\[ (c) \quad G_2 + G_1[x] = G_2[x] - G_1[x]G_1[x] = G_1[x] + \frac{1}{2} \frac{\partial \phi}{\partial \mathcal{E}} (G_1[x])^2 \]

By making use of the concept of Ritt bracket (Ritt 1950) associated with the composition of functions, Proposition 2 may be reformulated as in Corollary 1 below. For this, let us consider the space of analytic functions from \( \mathbb{R}^n \) to \( \mathbb{R}^d \) of the form \( L_d + f(\cdot) \), and define the operation of substitution of one function into another.
Denoting the Ritt bracket by $[f;g]$, given by $\left[ f; g \right] = f \ast g - g \ast f$ 

the following reformulation of Proposition 2 can be given.

**Corollary 1:** *The two dynamics (12) and (17) are quadratically feedback equivalent under (13) and (14) if and only if $\left( \phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr] \bigr) \right.$ and $\left. \bigl[ \beta \bigl[ \gamma \bigr] \bigr) \right.$ satisfy*

(a) $f^T(\alpha) - f^T(\alpha) = \alpha^T(\alpha)B + \left[ A_0, \phi^T(\alpha) \right]$

(b) $g^T(\alpha) - g^T(\alpha) = \beta^T(\alpha)B + \frac{\partial}{\partial u} \left[ \phi^T(\alpha), A + Bu \right]$

(c) $h^T(\alpha) - h^T(\alpha) = B_0^T(\alpha) + \frac{1}{2} \frac{\partial}{\partial u} \left[ \phi^T(\alpha), A + Bu \right]$

with $A = A - I_d$.

We conclude this section by noting that as in Kang and Krener (1992), the result stated here may be formulated in terms of an equivalence relation. For this, let $W$ and $V$ be the finite dimensional linear spaces of transformations of the form $\left( \phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr] \bigr)$ and quadratic elements of the form $\left( \phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr) \right.$, respectively.

Let $U$ be the map defined from $W$ to $V$ as

$U(\phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr) = \left( - A_0 \phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr) + \phi \bigl[ \alpha \bigl[ \beta \bigl[ \gamma \bigr] \bigr) + \alpha^T(\alpha)B, \beta^T(\alpha)B - \left[ A_0, \phi^T(\alpha), B \right] \right)$

It is easy to verify that the map $U$ is linear and allows us to define an equivalence relation. Denoting by $V_0 = U(W)$ the image of $W$ under $U$, one has the following proposition.

**Proposition 3:** *The discrete-time dynamics (12) and (17), represented by the triplets $\left( f^T, \gamma^T, h^T \right)$ and $\left( f^T, \gamma^T, h^T \right)$, respectively, are quadratically feedback equivalent if and only if*

\[ (f^T, \gamma^T, h^T) \in (f^T, \gamma^T, h^T) + V_0 \]

4. **Characteristic numbers**

We now introduce a family of numbers which are invariant under a transformation of the form (13) and (14) and characterize the equivalence classes associated with the linear space $V_0$. It must be noted that such numbers give an intrinsic characterization of quadratic equivalence because they do not depend on the preliminary feedback used to transform the system into the form (2).

As, roughly speaking, these numbers characterize the $u$-dependence of a specific input–output map, it is rather intuitive to think that two kinds of characteristic numbers are needed in a discrete-time context, e.g. a first one related to the action of $u$ and a second one related to $u'$.

Let $C = (1, 0, \ldots, 0)$ be the matrix which projects any element of $\mathbb{R}^n$ into its first component.
**Definition 4:** Given the discrete-time dynamics (12), we define the characteristic numbers

\[ a_1^{(r)} = C[\bar{G}_1, G_1^{[r]}] \quad \text{with} \quad 1 \leq t < n - 1, 0 \leq r \leq t - 1 \]  
\[ a_2^{(r)} = C_G^{[r]} \quad \text{with} \quad 0 \leq l < n - 1 \]  

Denoting by \( Z \) the linear finite-dimensional space composed of elements of the form \( (f^{[1]}, \bar{G}_1^{[1]}, G_2^{[1]}) \), the following lemma holds.

**Lemma 1:** The characteristic numbers \( a_1^{(r)} \) and \( a_2^{(r)} \) are linear maps from \( Z \) to \( \mathbb{R} \) (or equivalently from \( V \) to \( \mathbb{R} \)).

**Proof of Lemma 1:** To prove the linearity of \( a_1^{(r)} \) and \( a_2^{(r)} \) from \( Z \) to \( \mathbb{R} \), one has to rewrite the approximations of the \( G_1 \) and \( G_2 \) in terms of \( (f^{[1]}, \bar{G}_1^{[1]}, G_2^{[1]}) \). In fact, because of the expressions of \( G_1 \) in terms of the \( G_2 \) recalled in § 2 and taking into account (12), one has

\[ G_1 = G_1^{[1]} + G_1^{[1]}(x) + O(x^2) \]
\[ G_2 = G_2^{[1]} + O(x) \]

with

\[ G_2^{[1]} = h_3^{[1]} - DA^{-1}B \]
\[ G_1^{[1]} = A' G_1^{[1]} = A'B \]

\[ G_1^{[1]}(x) = A' G_1^{[1]}(A^{-1}x) + \sum_{i=0}^{n-1} A' \left[ \frac{\partial (f^{[1]})}{\partial x} \right]_{A^{1-i}x} A^{1-i} B \]  

Consequently, from Definition 4 one has

\[ a_1^{(r)} = C[G_1, G_1^{[1]}] = C[G_1^{[1]}, G_1^{[1]}] + C[G_1^{[1]}, G_1^{[1]}] \]
\[ a_2^{(r)} = C_G^{[r]} \]

which proves the linearity of \( a_1^{(r)} \) and \( a_2^{(r)} \) from \( Z \) to \( \mathbb{R} \).

As

\[ G_1^{[1]} = g_1^{[1]}(A^{-1}x) \quad \text{and} \quad G_2^{[1]} = \frac{\partial g_1^{[1]}(x)}{\partial x} A^{-1} B + h^{[1]} \]

the linearity of \( a_1^{(r)} \) and \( a_2^{(r)} \) from \( V \) to \( \mathbb{R} \) follows. \( \square \)

Hereafter, necessary and sufficient conditions to achieve linear feedback equivalence up to quadratic approximations are given in terms of the characteristic numbers.

**Theorem 2:** Discrete-time dynamics (12) are quadratically feedback equivalent to a linear dynamics

\[ x(k + 1) = Ax(k) + Bu(k) \]
with $A$ and $B$ given by (3) if and only if the two series of characteristic numbers are equal to zero.

**Proof of Theorem 2:**

**Sufficiency.** First, from the choice of $A$ and $B (= G_1^{a_{[1]}})$, condition (c) of Proposition 1 is always verified.

Assuming

$$d'_2 = 0, \quad \text{for } 0 \leq i < n - 1$$

that is

$$CG_2^{a_{[i]}} = CA'G_2^{a_{[i]}} = 0$$

the parallelism between $G_1^{a_{[i]}}$ and $G_2^{a_{[i]}}$ (condition (a) of Proposition 1) follows from the conditions

$$CG_1^{a_{[i]}} = CA'B = 0$$
$$CG^{-1}_1^{a_{[i]}} = CA^{-1}B \neq 0$$

which are characteristic for the canonical linear part.

The condition

$$\left[G_1^{a_{[i]}}, G_1^{a_{[i]}}\right] = 0, \quad \text{for } i \in \{1, \ldots, n - 2\} \quad \text{and} \quad r \in \{0, \ldots, r - 1\}$$

implies, because of (29) and (30), the property

$$\left[G_1^{a_{[i]}}, G_1^{a_{[i]}}\right] = \sum_{i=0}^{n-2} \alpha G_1^{a_{[i]}} + O(x)$$

and thus the involutivity (condition (b) of Proposition 1) is verified.

**Necessity.** Assuming condition (a) of Proposition 1 and because of (29) and (30), one has

$$d'_2 = CA'\left(-\frac{\partial G_1^{a_{[i]}}}{\partial x} AB + h^{a_{[i]}}\right) = CA'G_1^{a_{[i]}} = 0$$

for $i \in \{0, \ldots, n - 2\}$. Moreover, by assuming condition (b) of Proposition 1 for $0 \leq i < n - 1$ and $0 \leq r < t$, one has

$$\left[G_1^{a_{[i]}}, G_1^{a_{[i]}}\right] = \sum_{i=0}^{n-2} \alpha G_1^{a_{[i]}}$$

and consequently, because of (29), one has

$$\left[G_1^{a_{[i]}}, G_1^{a_{[i]}}\right] = \sum_{i=0}^{n-2} \alpha G_1^{a_{[i]}}, G_1^{a_{[i]}} = 0$$

**Remark 2:** From Theorem 2 one deduces that a non-degenerate nonlinear one-dimensional system is always quadratically feedback equivalent to a linear one (in this case $n - 1 = 0$ and consequently (25) and (26) are always verified). For a nonlinear two-dimensional non-degenerate system, the conditions of quadratic feedback equivalence to a linear one are reduced to ($d'_2 = 0$).
Example 1—Quadratically linearizable dynamics: Let us consider the discrete-time dynamics

\[
\begin{align*}
    x_1(k+1) &= x_1(k) + x_2(k) - x_1(k)^2 \\
    x_2(k+1) &= x_2(k) + x_3(k) + 2x_1(k)x_2(k) + x_2(k)^2 \\
    x_3(k+1) &= x_3(k) + u(k) + u(k)^2 + x_3(k)^2
\end{align*}
\]

From (27) (see the Proof of Lemma 1), the \( G_i^f \) with \( 1 \geq i \geq 0, 2 \geq j \geq 1 \) and \( 1 \geq l \geq 0 \) are equal to

\[
\begin{align*}
    G_1^0 &= (0,0,1)^T, \quad G_1^1 = (0,0,0)^T \\
    G_2^0 &= (0,0,1)^T, \quad G_2^1 = (0,1,1)^T \\
    G_3^0 &= (0,1,1)^T, \quad G_1^1 = (0,0,1)^T \\
    G_2^1 &= (0,0,2)^T
\end{align*}
\]

Now, by computing the characteristic numbers, one finds

\[
\begin{align*}
    a_1^1 &= C[G_1^0,G_1^0] = (1,0,0)(0,0,2)^T = 0 \\
    a_2^1 &= CG_2^1 = (1,0,0)(0,0,1)^T = 0 \\
    a_3^1 &= CG_3^1 = (1,0,0)(0,1,1)^T = 0
\end{align*}
\]

and thus all the characteristic numbers are equal to zero. Consequently, one concludes from Theorem 2 that the system (31) is quadratically feedback equivalent to the canonical linear dynamics (28). In fact, applying the quadratic coordinates change

\[
\begin{align*}
    \xi_1(k) &= x_1(k) \\
    \xi_2(k) &= x_2(k) - x_1(k)^2 \\
    \xi_3(k) &= x_3(k)
\end{align*}
\]

and the quadratic feedback

\[
v(k) = u(k) + u(k)^2 + x_3(k)^2\]

one obtains

\[
\begin{align*}
    \xi_2(k+1) &= \xi_2(k) + \xi_1(k) \\
    \xi_3(k+1) &= \xi_3(k) + \xi_2(k) \\
    \xi_3(k+1) &= \xi_3(k) + v(k)
\end{align*}
\]

Hereafter it is shown that quadratically feedback equivalent dynamics are the equivalence classes associated with the characteristic numbers.

**Theorem 3:** Two discrete-time dynamics (12) and (17) are quadratically feedback equivalent if and only if their characteristic numbers are equal.

**Proof of Theorem 3:** Let \((a_1^r, a_2^r)\) and \((a_1^l, a_2^l)\) be the characteristic numbers associated with (12) and (17), respectively.
Necessity. Suppose that (12) and (17) are quadratically state feedback equivalent. From Proposition 3 it follows that
\[
(f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1) \in (f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1) + V_0
\]
so that there exists \((f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1) \in V_0\) satisfying
\[
(f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1) = (f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1) + (f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1)
\]
and because of the linearity
\[
(a', d') = (a', d') + (a', d')
\]
where \((a', d') = (0, 0)\) from Theorem 2. Consequently, the dynamics (12) are quadratically state feedback equivalent to the dynamics (17) if \((a', d')\) coincide with \((a', d')\).

Sufficiency. Assume that \((a', d') = (a', d')\). Because of the linearity, one obtains
\[
(f_{\Delta}^1 - f_{\Delta}^1 g_{\Delta}^1 h_{\Delta}^1 - g_{\Delta}^1 h_{\Delta}^1) \in V_0
\]
so that the equality of the two series \((a', d')\) and \((a', d')\) implies that the dynamics (12) are quadratically feedback equivalent to (17).

5. Quadratic canonical forms

It is shown in this section that a suitable structure of the homogeneous quadratic terms in (12) may be assumed. Such a structure remains unchanged under feedback and quadratic transformation of the form (13) and (14).

Theorem 4: Under feedback and coordinates change of the form (13) and (14), any discrete-time dynamics can be uniquely transformed (2) into one of the following equivalent forms:

\[
x(k + 1) = Ax(k) + Bv(k) + f_{\Delta}^1 h_{\Delta}^1 (k) + O(x, v)^3
\]

with

\[
h_{\Delta}^1 = (h_{\Delta}^1_{+1}, \ldots, h_{\Delta}^1_{+n-1}, 0)^T
\]

\[
f_{\Delta}^1 = (f_{\Delta}^1(x), \ldots, f_{\Delta}^1(x))^T
\]

\[
g_{\Delta}^1 = \sum_{j=2}^{n} d_{j, x_j}, \quad \text{for } i \in \{2, \ldots, n-1\}
\]

the \(d_{j, x_j}\) and \(h_{\Delta}^1\) associated one by one with the two series; or

\[
x(k + 1) = Ax(k) + Bv(k) + f_{\Delta}^1 h_{\Delta}^1 (k) + O(x, v)^3
\]

with

\[
h_{\Delta}^1 = (h_{\Delta}^1_{+1}, \ldots, h_{\Delta}^1_{+n-1}, 0)^T
\]

\[
f_{\Delta}^1 = (f_{\Delta}^1(x), \ldots, f_{\Delta}^1(x))^T
\]

\[
f_{\Delta}^1 = \sum_{j=2}^{n} c_{j, x_j}, \quad \text{for } i \in \{1, \ldots, n-1\}
\]
the $c_{i,j}$ and $h_{k,l}$ associated one by one with the two series.

**Proof:** For the proof, see the Appendix.

For computational reasons (i.e. linearity in the state), one generally uses the form (32) instead of (36); see, for example, Barbot et al. (1994).

**Example 2—Dynamics that are not quadratically linearizable under static state feedback:** Let us consider the discrete-time dynamics

$$
\begin{align*}
    x_1(k+1) &= x_1(k) + x_2(k) - x_1(k)^2 \\
    x_2(k+1) &= x_3(k) + 2x_1(k)x_2(k) + x_2(k)^2 + u(k)^2 \\
    x_3(k+1) &= x_3(k) + u(k) + u(k)^2 + x_3(k)^2
\end{align*}
$$

(40)

From (27) the $G_{i,j}^{[l]}$ with $1 \geq i \geq 0$, $2 \geq j \geq 1$ and $1 \geq l \geq 0$ are equal to

$$
\begin{align*}
    G_{1}^{[0]} &= (0,0,1)^T, & G_{1}^{[1]} &= (0,0,0)^T \\
    G_{2}^{[0]} &= (0,1,1)^T, & G_{2}^{[1]} &= (1,2,1)^T \\
    G_{3}^{[0]} &= (0,1,1)^T, & G_{3}^{[1]} &= (0,0,2x_3)^T
\end{align*}
$$

and the characteristic numbers are equal to

$$
\begin{align*}
    a_1^0 &= CG_{1}^{[1]}G_{1}^{[0]} = (1,0,0)(0,0,2)^T = 0 \\
    a_2^0 &= CG_{2}^{[0]} = (1,0,0)(0,1,1)^T = 0 \\
    a_3^0 &= CG_{3}^{[0]} = (1,0,0)(1,2,1)^T = 1
\end{align*}
$$

Thus, because $a_3^0$ is not equal to zero, one concludes from Theorem 2 that the dynamics (40) are not quadratically feedback equivalent to the canonical linear dynamics (28). As a matter of fact, by applying the quadratic diffeomorphism

$$
\begin{align*}
    \xi_1(k) &= x_1(k) \\
    \xi_2(k) &= x_2(k) - x_1(k)^2 \\
    \xi_3(k) &= x_3(k)
\end{align*}
$$

and the quadratic feedback

$$
\nu(k) = u(k) + u(k)^2 + x_3(k)^2
$$

one obtains

$$
\begin{align*}
    \xi_1(k+1) &= \xi_1(k) + \xi_2(k) \\
    \xi_2(k+1) &= \xi_2(k) + \xi_3(k) + \nu(k)^2 \\
    \xi_3(k+1) &= \xi_3(k) + \nu(k)
\end{align*}
$$

(41)

which is the quadratic canonical form proposed in Theorem 4. Thus it is proved that the given system belongs to the equivalence class characterized by (41) which is not feedback linearizable.
6. Dynamic state feedback linearization

The purpose of this section is to show that the conditions for achieving quadratically state feedback linearization can be relaxed if dynamic state feedback is used.

A discrete-time quadratic dynamic feedback is defined by the set of equations
\[
\begin{align*}
    w(k+1) &= \alpha[x(k),w(k),\mu(k)] + \beta[x(k),w(k),\mu(k)] \\
    u(k) &= \alpha[x(k),w(k),\mu(k)] + \beta[x(k),w(k),\mu(k)]
\end{align*}
\]
where \( w \in \mathbb{R}^q, u, \mu \in \mathbb{R} \) and \( \alpha, \beta \) are functions of appropriate dimensions and \( \alpha, \beta \) are scalar functions.

**Definition 5:** The discrete-time system (1) is said to be **quadratically feedback linearizable under dynamic state feedback** if there exists a feedback of the form (42) such that the extended system
\[
\begin{align*}
    x(k+1) &= f(x(k),[\alpha[x(k),w(k),\mu(k)] + \beta[x(k),w(k),\mu(k)]]) \\
    w(k+1) &= \alpha[x(k),w(k),\mu(k)] + \beta[x(k),w(k),\mu(k)]
\end{align*}
\]
is quadratically linear equivalent.

The following theorem shows that quadratic feedback linearization can always be achieved under dynamic feedback.

**Theorem 5:** Any discrete-time dynamics of the form (1) with a controllable linear part are quadratically feedback linearizable around \((0,0)\) under dynamic state feedback.

As the proof is constructive and it provides the computation of the controller, it will be proposed hereafter.

**Proof:** Because of the controllability assumption and Theorem 4, the proof may be achieved starting from the canonical form (32). Given a dynamic quadratically state feedback of the form (42)
\[
\begin{align*}
    w_1(k+1) &= w_1(k) + w_2(k) \\
    & \vdots \\
    w_m(k+1) &= w_m(k) + \mu(k)
\end{align*}
\]
and
\[
u(k) = \alpha[x(k),w(k)] + \beta[x(k),w(k)]\mu(k) + \gamma[k^2(k) + w_1(k)]
\]

The state dynamic extension (44) is a particular case of (42) where \( \alpha, \beta \) is equal to zero. Under feedback the dynamics (32) are transformed into
\[
\begin{bmatrix}
    x(k+1) \\
    w(k+1)
\end{bmatrix} = A(k) \begin{bmatrix}
    x(k) \\
    w(k)
\end{bmatrix} + B(k)\mu(k)
\]
\[
+ \begin{bmatrix}
    g_1[k^2(k) + w_1(k)] + h[k^2(k) + B(u(k) - w_1(k))] \\
    0
\end{bmatrix} + O(x, w, \mu)^3
\]
where $A_1$ and $B_1$ are still matrices of the form (3). The third term in the right-hand side has entries which must be restricted to their quadratic part with respect to the variables $x$, $w$ and $\mu$, i.e.

$$\left[ c_0^k y(x(k))_{w_1}(k) + h_{x_2}^k w_2^z(k) + h_{x_3}^k w_3^z(k) \right]$$

Now, let the diffeomorphism $z = \Phi(x) = x + w(x)$, defined on $\mathbb{R}^{2n-1}$, be

$$z_1(k) = x_1(k)$$
$$z_2(k) = x_2(k) + h_{x_2}^k w_2^z(k)$$
$$z_3(k) = x_3(k) + \gamma_{x_2}^k \Phi(x(k))_{w_1}(k) + h_{x_3}^k w_3^z(k) + \left[ z_1(k) - x_2(k) \right]$$
$$\vdots$$
$$z_n(k) = x_n(k) + \gamma_{x_2}^k \Phi(x(k))_{w_1}(k) + h_{x_n}^k w_n^z(k) + \left[ z_1(k) - x_2(k) \right]$$
$$z_{n+1}(k) = w_1(k)$$
$$\vdots$$
$$z_{2n-1}(k) = w_{n-1}(k)$$

Any term of the form $z_i(k+1) - x_i(k+1)$ can be expressed as a term at time $k$ of an order greater than or equal to two. For example, $z_2(k+1) - x_2(k+1)$ is equal to $h_{x_3}^k (w_1(k) + w_3^z(k))^2$.

After some easy though tedious manipulations, one obtains

$$z_1(k+1) = x_1(k) + x_2(k) + h_{x_2}^k w_2^z(k)$$
$$= z_1(k) + z_2(k) + O(z)^3$$
$$z_2(k+1) = x_2(k) + x_3(k) + \gamma_{x_2}^k \Phi(x(k))_{w_1}(k) + h_{x_3}^k w_3^z(k) + h_{x_2}^k w_2^z(k) + O(z)^3$$
$$= z_2(k) + \left[ z_1(k) - x_2(k) \right] + x_3(k) + \gamma_{x_2}^k \Phi(x(k))_{w_1}(k) + h_{x_3}^k w_3^z(k) + O(z)^3$$
$$\vdots$$

$$z_n(k+1) = x_n(k) + w_1(k) - \left[ z_n(k) - x_n(k) \right] + h_{x_n}^k w_n^z(k) + \left[ z_1(k) - x_2(k) \right] + O(z)^3$$

$$\vdots$$

$$z_{2n-1}(k) = w_{n-1}(k) + O(z)^3$$

(46)
Setting now
\[\alpha f(x(k)) + \beta f(\dot{x}(k))\mu(k) + \gamma f^2(k) = [dx(k) - x(k)] - h_f\dot{w}^2(k) - [e_x(k) - x(k + 1)] \tag{47}\]
where, as before, \(z_n(k + 1) - x_n(k + 1)\) can be expressed as a term, at time \(k\), of order greater than or equal to two in \(w, x\) and \(\mu\), one obtains
\[z_n(k + 1) = z_n(k) + z_{n+1}(k) + O(z, \mu)^3\]
For the remaining components one has
\[
\begin{align*}
z_{n+1}(k + 1) &= z_{n+1}(k) + z_{n+2}(k) \\
\vdots &= \vdots \\
z_{2n-1}(k + 1) &= z_{2n-1}(k) + \mu(k)
\end{align*}
\tag{48}\]
Equations (460 and (48) show that, under the feedback defined by (45) and (47), the system takes the form
\[z(k + 1) = A_1z(k) + B_1\mu(k) + O(z, \mu)^3\]
and the proof is complete.

**Example 2**—**Dynamical feedback linearization**: Considering again Example 2 (dynamics (40)) now transformed under feedback and coordinate changes into (41), and applying the dynamic quadratically state feedback of the form (42), one finds
\[w_1(k + 1) = w_1(k) + \mu(k)\]
\[u(k) = w_1(k) - w_1(k)^2 - 2w_1(k)\mu(k) - \mu(k)^2\]
Here one has reduced the dynamic feedback extension to its minimum order which is related to the first characteristic number. Now, using the extended coordinates change
\[
\begin{align*}
\chi_1(k) &= \xi_1(k) \\
\chi_2(k) &= \xi_2(k) \\
\chi_3(k) &= \xi_3(k) + w_1^2(k) \\
\chi_4(k) &= w_1(k)
\end{align*}
\]
one obtains
\[
\begin{align*}
\chi_1(k + 1) &= \chi_1(k) + \chi_2(k) \\
\chi_2(k + 1) &= \chi_2(k) + \chi_3(k) + O(\chi, \mu)^3 \\
\chi_3(k + 1) &= \chi_3(k) + \chi_4(k) + O(\chi, \mu)^3 \\
\chi_4(k + 1) &= \chi_4(k) + \mu(k)
\end{align*}
\]
As shown in the proof of Theorem 5, the dynamic feedback and the coordinates change are simultaneously computed. This is due to the fact that the input \( u(k) \) is computed to cancel quadratic terms after the coordinates change.

**Appendix**

**Proof of Theorem 4:** As far as (32) is concerned, let us consider two discrete-time dynamics of the form (12); that is, according to the definition of \( V \), two triplets \((0, g^1_i, h^1_i)\) and \((0, g^2_i, h^2_i)\). They are quadratically equivalent if and only if

\[
(0, g^1_i, h^1_i) \sim g^2_i, h^2_i (0, g^2_i, h^2_i) \in V_0
\]  

(A1)

The first part of the proof shows that two systems of the form (32) are not quadratically feedback equivalent to each other. We will show that if they are, then \( g^1_i \) and \( h^1_i \) are necessarily equal to zero.

Let us assume that there exists \( i \in \{1, \ldots, n-1\} \) such that \( h^1_i \neq 0 \); then the characteristic number \( a^1_i \) is not equal to zero and because of Theorem 2 the dynamics are not quadratically feedback linearizable.

On the other hand, assuming the existence of \( i \in \{2, \ldots, n-1\} \) such that \( g^1_i \neq 0 \), then all the characteristic numbers \( a^1_i \) are equal to zero and this leads to an absurdity, as shown hereafter by induction.

**Step 1.** According to Definition 4, one sets

\[
a^1_i = C[g^1_i, G^0_i] = C(ADAD^{-1}B - DB)
\]

with \( A \) and \( B \) given in (3), \( C = (1, 0, \ldots, 0) \) and

\[
D = \frac{\partial h^1_i}{\partial x} \bigg|_{x=0}
\]

With easy computations, one obtains

\[
A_j = (I_d + \tilde{A})^j = \sum_{i=0}^{j} C_i \tilde{A}^i, \quad \text{with} \quad C_j = \frac{j}{(j-i)! i!}
\]

From the structure of the matrices \( C \) and \( D \) one deduces that \( a^1_i = 0 \) implies \( d_{2,n} = 0 \).

**Step 2.** By performing further computations one finds

\[
a^2_i = C[g^2_i, G^0_i] = C(\tilde{A}^2 \tilde{D} \tilde{A}^{-1}B - DAB) = CA^2 \tilde{D} \tilde{A}^{-1}B = 0
\]

because of the structure of \( C \) and \( D \). Moreover, because \( d_{2,n} \) is equal to zero

\[
a^2_i = C[g^2_i, G^0_i] = C(\tilde{A}^2 \tilde{D} \tilde{A}^{-1}B - ADB) = CA^2 \tilde{D} \tilde{A}^{-1}B = 0
\]

Thus, one obtains

\[
(0, 0) = \Omega(d_{3,n-1}, d_{3,n}) \quad \text{where the matrix}
\]

\[
\Omega = \Omega_{ij} = (A^{i+j}B)_{n+2-i}
\]
is non-singular by construction. It results that \( a_i^{2,1} = a_i^{2,0} = 0 \) implies that \( d_{i,j-1} = d_{i,j} = 0 \).

**Step j.** Assume that the rows of the \( D \) matrix are equal to zero up to the \( j \)th one. By assuming also that \( a_i^{2,k} = 0 \) for \( 0 \leq k \leq j - 1 \), then we will show that the \( (j+1) \)th row of \( D \) is equal to zero, that is \( d_{j+1,j} = 0 \) for \( l \in \{ n-j, \ldots , n \} \).

Arguing as above, one computes
\[
\begin{align*}
\alpha_0^j &= CA'DA^{-j-1}B \\
\vdots \\
\alpha_{j-1}^j &= CA'DA^{-2}B
\end{align*}
\]

By rewriting these equalities in a matrix form as
\[
0 = \Omega(d_{j+1,n-j}, \ldots , d_{j+1,n})^T
\]

with \( \Omega \), the \( j \times j \) matrix defined here by
\[
\Omega = \left( \Omega_{ij} \right) = (A^{-j-1}B)_{n-j+1}
\]

the same arguments as before (i.e. \((A, B)\) linear controllable dynamics) are used to show that \( \Omega \) is invertible. Finally, one concludes that \( a_i^{2,k} = 0 \) for \( 0 \leq k \leq j - 1 \) implies \( d_{j+1,n-k} = 0 \) for \( 0 \leq k < j \).

The second part of the proof shows that any system is quadratically feedback equivalent to a dynamics of the form (32).

From Lemma 1, the two series of characteristic numbers \( (a_i^{1,1}, a_i^{1,2}) \) define a linear map from \( V \) to \((n-1)n/2\). Moreover, to any discrete-time canonical form (32) correspond two series, and to two series corresponds one canonical form (32). Thus, because the matrix \( D \) and the vector \( \vec{h}_0^{(k)} \) are realized with the same number of elements \( \mathbb{R}^{(n-1)n/2} \) as the two series \( (a_i^{1,1}, a_i^{1,2}) \), there is a linear bijection between the two series and the canonical form (32). Consequently, from Theorem 3, the first part of the proof is complete.

So far as (36) is concerned, let us consider two discrete-time dynamics of the form (36) described by the triplets \((\vec{f}_{[#]}^{(k)}0, \vec{h}_{[#]}^{(k)})\) and \((\vec{f}_{[#]}^{(l)}0, \vec{h}_{[#]}^{(l)})\), respectively.

From Proposition 3, one has \((\vec{f}_{[#]}^{(k)}0, \vec{h}_{[#]}^{(k)})\) quadratically state feedback equivalent, if and only if
\[
(\vec{f}_{[#]}^{(l)}0, \vec{h}_{[#]}^{(l)}) = (\vec{f}_{[#]}^{(l)}0, \vec{h}_{[#]}^{(l)}) \in V_0 \quad (A.2)
\]

The first part of the proof shows that any two systems given by (36) are not feedback quadratically equivalent to each other. To prove this, we will show that if they had been, then necessarily \( \vec{f}_{[#]}^{(k)} \) and \( \vec{h}_{[#]}^{(k)} \) should have been equal to zero.

Let us assume that there exists \( i \in \{1, \ldots , n - 1 \} \) such that \( \vec{h}_{[#]}^{(k)} \neq 0 \) then the characteristic number \( a_i^{2,1} \neq 0 \), and because of Theorem 2 the dynamics are not quadratically feedback linearizable.

On the other hand, assuming that there exists \( i \in \{1, \ldots , n - 2 \} \) such that \( \vec{f}_{[#]}^{(k)} \neq 0 \) and all the characteristic numbers \( a_i^{2,1} \) are equal to zero, which leads to an absurdity, as shown hereafter by induction.
Step 1. According to Definition 4 and (27), we set
\[ a_{1,0}^1 = C^\top \begin{bmatrix} G_{1,1}^1 \end{bmatrix} \begin{bmatrix} E \end{bmatrix} = CE(AB \otimes B) = 0 \]
with \( E = \mathcal{O}_{02} \mathcal{F} \mathcal{X}_{02}^\top \) and from (39) one has \( E_{i,j} = 0 \) for \( k \neq j \) or \( j \leq i + 1 \) and \( E_{i,j} = c_{i,j} \) otherwise. Thus, one obtains
\[ a_{1,0}^1 = c_{1,0} = 0 \quad (A3) \]

Step 2. Secondly, one computes \( a_{1}^2 \) as
\[ a_{1,0}^2 = C(E(A^{-1}B \otimes AB) + AE(A^{-2}B \otimes B)) \]
and from the particular form of \( f[B] \) and because of (A3) one obtains
\[ a_{1,0}^2 = c_{1,0} \quad (A4) \]

Similarly, for \( a_{0}^2 \) one has
\[ a_{0,0}^2 = C \begin{bmatrix} E(B \otimes AB) + AE(A^{-1}B \otimes B) - E(AB \otimes B) \end{bmatrix} = 0 \quad (A5) \]
and because of the particular structure of \( E \), this gives
\[ a_{0,0}^2 = c_{2,0} = 0 \quad (A6) \]

From (A3) and (A6), one deduces
\[ c_{1,0} = c_{2,0} = 0 \]

Step \( j \). Assume that all the \( a_{ij}^k = 0 \) for \( 0 < i \leq j \) and \( 0 \leq k < j \) implies \( C_{m+1,j-1} = 0 \) for \( l, n \in \mathbb{N} \) and \( l + m < j \). One has
\[ a_{1,0}^{j+1} = C \begin{bmatrix} E(A^{-1}B \otimes A^{-j}B) + AE(A^{-2}B \otimes A^{-j}B) \end{bmatrix} = 0 \]
\[ a_{1,1}^{j+1} = C \begin{bmatrix} E(B \otimes A^{-j}B) + AE(A^{-1}B \otimes A^{-j}B) \end{bmatrix} = 0 \]
\[ a_{1,2}^{j+1} = C \begin{bmatrix} E(A^{-2}B \otimes A^{-j}B) + AE(A^{-1}B \otimes A^{-j}B) \end{bmatrix} = 0 \]
\[ \vdots \]
\[ a_{1,j-1}^{j+1} = C \begin{bmatrix} E(A^{-j}B \otimes A^{-j}B) + AE(A^{-1}B \otimes A^{-j}B) \end{bmatrix} = 0 \]

By taking into account the structures of \( E \), one has
\[ a_{1,0}^{j} = C \begin{bmatrix} E(A^{-1}B \otimes A^{-j}B) \end{bmatrix} = 0 \]
\[ a_{1,1}^{j} = C \begin{bmatrix} AE(A^{-1}B \otimes A^{-j+1}B) \end{bmatrix} = 0 \]
\[ \vdots \]
\[ a_{1,j-1}^{j} = C \begin{bmatrix} A^{-j+1}E(A^{-1}B \otimes B) \end{bmatrix} = 0 \]
Quadratic forms and approximate feedback linearization

\[ a_i^{jk} = c_{i+j} \left( A^{-1} B \right)_{i+j} + c_{i+j+1} \left( A^{-2} B \right)_{i+j+1} + \ldots + c_{i+j+n} \left( A^{-n} B \right)_n = 0 \]
\[ a_i^{jk} = c_{i+j+1} \left( A^{-1} B \right)_{i+j+1} + \ldots + c_{i+j+n} \left( A^{-n} B \right)_n = 0 \]
\[ a_i^{jk} = c_{i+j+n} \left( A^{-n} B \right)_n = 0 \]

Finally, one concludes that all the \( a_i^{jk} = 0 \) for \( 0 < i \leq j \) and \( 0 \leq k < j \) implies \( c_{i+j+n-1} = 0 \) for \( i, n \in \mathbb{N} \) and \( i + m < j \). This proves the independence of each form (36).

The second part of the proof which shows that any system is quadratically feedback equivalent to the dynamics (36) is omitted, because it can be worked out as in the previous case for (32).

\[ \text{References} \]


