IDENTIFIABILITY ANALYSIS OF AN EPIDEMIOLOGICAL MODEL IN A STRUCTURED POPULATION

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Abstract. We investigate the parameter identifiability problem for a system of nonlinear transport PDE, representing the spread of a disease with a long infectious but undetectable period in an animal population. After obtaining the expression of the model input-output relationships, we formulate the identifiability problem. We then give sufficient conditions on the initial and boundary conditions of the system that guarantee the parameter identifiability on a finite time horizon. We finally illutrate our findings with numerical simulations.

Key words. identifiability, PDE, transport equations, epidemiological model

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1. Introduction.

Epidemiological models are useful tools to describe the spread of a disease in a population, to predict its evolution and control its outbreak. They usually derive from the classical SIR model, a compartmental model in which the population is structured in susceptible, infected and recovered individuals. Depending on the interactions between host and pathogen, as well as their time and space scales, several models have been built [9], dating back to the Kermack-McKendrick model in which a constant population size is assumed [7].

The model we investigate in this paper is a simplified version of a model developed to study the spread of scrapie in a sheep flock [13]. Scrapie is a fatal disease with a long incubation period compared to the animal lifespan, during which animals are infectious but cannot be detected. Our model represents such a disease with a long infectious but undetectable period. At the end of this period, infected animals are removed from the flock. We will assume in this paper that they are culled, but they could also become immune, results would still hold. We consider here a managed flock, that is a well-mixed population, not a wild animal population, so the space dimension can be omitted. The population is however described by densities structured in age and infection load, which leads to a non-linear integro-partial differential dynamical system of transport type.

We address the identifiability of the model parameters. The inputs being given, the model is identifiable if a unique set of parameters corresponds to the observed outputs. It is an inverse problem that consists in establishing that the map from parameters to outputs is into, thus checking that the model is suitably parametrised. This property is an important prerequisite to the model identification, in which parameters are estimated from observed data.

There is a well-established theory for the identifiability of controlled and uncontrolled dynamical systems described by ordinary differential equations [4]. Three main approaches have been used: (i) the state isomorphism method in a linear [15] and nonlinear [14, 5] context; (ii) the Taylor series expansion method [12]; (iii) the algebro-differential elimination method [11], aiming at obtaining the expression of the

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input-output map of the system. There is no well-established theory and few studies on the parameter identifiability problem for infinite dimension dynamical systems. Some results on convolutive systems, which include the delay-differential equations, were shown [3]. Results on partial differential systems are scarce and deal with particular models, for instance the Schrödinger equation [2].

Our aim is to check the identifiability of our epidemiological model, by an approach inspired by the elimination method. The paper is organised as follows: Section 2 presents the model, recalls well-posedness results and establishes an input-output relation; Section 3 is devoted to the identifiability results, the detailed proofs being postponed in the Appendices. Appendix A deals with technical lemmas that are useful to prove Proposition 3.8. The following appendices demonstrate the paper main theorems. The results are illustrated by simulations in Section 4; finally, conclusions are drawn in Section 5.

2. The epidemiological model.

2.1. Model description. The mathematical model is formulated in terms of population densities structured according to disease status (susceptible and infected), to age and, for infected animals, to infection load, a variable related to incubation. Age a belongs to a finite interval $[0, A]$, animals reaching age A being systematically culled. The infection load variable θ lies in the interval $[0, 1]$. During the incubation period, θ grows exponentially with time according to

$$\frac{d\theta}{dt} = c\,\theta, \tag{2.1}$$

where $c > 0$ denotes the infection load growth rate. Newly infected animals get first infection loads $\theta_0 \in]0, 1[$, which are distributed according to a probability density function (p.d.f.) Θ. It satisfies

$$\Theta \in \mathcal{A}_0(0, 1), \tag{2.2}$$

where $\mathcal{A}_0(0, 1)$ is the set of real-analytic functions on $]0, 1[$ continuous on $[0, 1]$ with zero values at 0 and 1. The cumulative distribution function F_Θ is given by

$$F_\Theta(\theta) = \int_0^\theta \Theta(u)\ du.$$

The onset of clinical signs corresponds to an infection load of 1 and is immediately followed by culling. The p.d.f. Θ is used to allow variable incubation periods among the infected population. Indeed, the incubation period τ is related to the first infection load θ_0 by $\tau = -\frac{1}{c}\ln\theta_0$. So it is distributed according to a p.d.f., denoted X, which is a real-analytic function on $\mathbb{R}^{+*}$, continuous on $\mathbb{R}^+$ and related to Θ by

$$X(\tau) = c\,e^{-c\tau}\Theta(e^{-c\tau}). \tag{2.3}$$

An alternative option would have been to structure the infected population according to the age of infection as in [1, 6] instead of the infection load. Whatever the parametrisation, it yields a distributed delay structure.

For $(t, a, \theta) \in \mathbb{R}^+ \times [0, A] \times [0, 1]$, we use $S(t, a)$, $I(t, a, \theta)$ and $K(I)(t)$ to denote respectively the density of susceptible animals, the density of infected animals and the total number of infected animals at time t. Recall that

$$\forall t \in \mathbb{R}^+, \quad K(I)(t) = \int_0^A \int_0^1 I(t, a, \theta)\ d\theta da.$$

Densities S and I satisfy the following transport equations

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial a} = -\mu S - \beta S K(I), \tag{DS}$$

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a} + \frac{\partial (c\,\theta\, I)}{\partial \theta} = -\mu I + \beta \Theta S K(I). \tag{DI}$$

This is a modified version of the classical Kermack-McKendrick SI-epidemiological PDE model [7]. The rate of infection is given by $\beta\, K(I)(t)$, where the positive parameter β is the constant horizontal transmission rate. The positive parameter μ in the model is the basic mortality rate. We assume in this model that there is no vertical transmission, i.e. that infected animals give birth to susceptible individuals. Moreover, according to (2.1,2.2), infected animals have a positive infection load. Consequently, the following boundary conditions are associated to (DS, DI):

$$\begin{cases} S(t,0) = n(t), \\ I(t,0,\theta) = 0, \quad I(t,a,0) = 0, \end{cases} \quad \forall\,(t,a,\theta) \in [0,+\infty[\times[0,A]\times[0,1], \tag{BC}$$

where n is the birth function. The initial conditions are given by

$$\begin{cases} S(0,a) = S_0(a), \\ I(0,a,\theta) = I_0(a,\theta), \end{cases} \quad \forall\,(a,\theta) \in [0,A]\times[0,1]. \tag{IC}$$

We denote by $(\mathcal{P})$ the problem (DS, DI, BC, IC).

The initial condition functions S_0 and I_0, and the birth function n are the inputs of the system. The system outputs correspond to the observations, collected on a given finite time horizon $T > 0$. They consist of the total population density on $[0,T]\times[0,A]$ given by

$$\mathfrak{p}(t,a) = S(t,a) + \int_0^1 I(t,a,\theta)\ d\theta, \tag{2.4}$$

the incidence, i.e. the disease-induced mortality outflow, defined on $[0,T]\times[0,A]$ as

$$\mathfrak{i}(t,a) = c\,I(t,a,1), \tag{2.5}$$

and the basic mortality outflow, defined on $[0,T]\times[0,A]$ as $\mathfrak{m}(t,a) = \mu\,\mathfrak{p}(t,a)$. Indeed, infected animals cannot be distinguished from susceptible animals during the incubation period and are culled when showing clinical signs.

Measuring the basic mortality outflow allows the immediate estimation of parameter μ. If μ is age dependent, the identification of the basic mortality function $\mu(a)$ in a similar model is treated in [8]. In the sequel the vector of outputs of the system is reduced to

$$\mathfrak{s}(t,p) = \begin{pmatrix} \mathfrak{p}(t,p) \\ \mathfrak{i}(t,p) \end{pmatrix}.$$

2.2. Well Posedness. Let $H = L^2([0,A],\mathbb{R}) \times L^2\big([0,A]\times[0,1],\mathbb{R}\big)$ and $H^+ = L^2\left([0,A],\mathbb{R}^+\right) \times L^2\big([0,A]\times[0,1],\mathbb{R}^+\big)$. We use $Pc(I,J)$ to denote the set of piecewise continuous functions defined on a subset I of $\mathbb{R}^+$ or $(\mathbb{R}^+)^2$, taking values in an interval $J \subset \mathbb{R}$. The following theorem holds [10].

THEOREM 2.1. *Let* $(S_0, I_0) \in H^+$, $T > 0$ *and* $n \in Pc([0,T], \mathbb{R}^+)$. *Then Problem* $(\mathcal{P})$ *has a unique mild solution* $(S(t), I(t)) \in C([0,T], H^+)$ *such that for* $t \in [0,T]$

$$S(t,a) = \begin{cases} S_0(a-t)e^{-(\mu t+\beta\int_0^t K(I)(s)\ ds)} & \text{for } a \geqslant t, \\ n(t-a)e^{-(\mu a+\beta\int_{t-a}^t K(I)(s)\ ds)} & \text{for } a \leqslant t, \end{cases} \tag{2.6}$$

in $L^2([0,A],\mathbb{R})$ *and*

$$I(t,a,\theta) = \begin{cases} S_0(a-t)e^{-\mu t}\int_0^t e^{c(s-t)}\Theta\left(\theta e^{c(s-t)}\right)\beta K(I)(s)e^{-\beta\int_0^s K(I)(u)du}ds \\ \quad + I_0(a-t, \theta e^{-ct})e^{-(\mu+c)t} \qquad \text{for } a \geqslant t, \\ n(t-a)e^{-\mu a}\int_{t-a}^t e^{c(s-t)}\Theta\left(\theta e^{c(s-t)}\right)\beta K(I)(s)e^{-\beta\int_{t-a}^s K(I)(u)du}ds \\ \qquad \text{for } a \leqslant t, \end{cases} \tag{2.7}$$

in $L^2\big([0,A]\times[0,1],\mathbb{R}\big)$.

For convenience, more concise formulations of equations (2.6) and (2.7) are introduced. Let

$$M(t,a) = \begin{cases} S_0(a-t)\,e^{-\mu t} & \text{if } a \geqslant t, \\ n(t-a)\,e^{-\mu a} & \text{if } t \geqslant a, \end{cases} \quad \text{and } r(s) = \max(s,0).$$

Then

$$S(t,a) = M(t,a)e^{-\beta\int_{r(t-a)}^t K(I)(s)\ ds}, \tag{2.8}$$

and

$$\begin{aligned} I(t,a,\theta) &= I_0(r(a-t), \theta e^{-ct})e^{-(\mu+c)t} \\ &+ M(t,a)\int_{r(t-a)}^t e^{c(s-t)}\,\Theta\left(\theta e^{c(s-t)}\right)\beta\, K(I)(s)\ e^{-\beta\int_{r(t-a)}^s K(I)(u)du}\ ds. \end{aligned} \tag{2.9}$$

Equation (2.7) and the positivity of the solution semigroup have the subsequent immediate and useful consequences.

COROLLARY 2.2. *If* $K(I)(t) = 0$ *for some* t *in* $[0,T]$, *then the function* $K(I)$ *is zero on* $[0,T]$. The next corollary results from Theorem 2.1.

COROLLARY 2.3. *Let* $(S_0, I_0) \in H^+$ *and* $n \in Pc([0,T], \mathbb{R}^+)$. *Consider the operator* L *defined on* $C([0,T])$ *with values in* $C([0,T])$ *by*

$$\begin{aligned} L(f)(t) =& e^{-\mu t}\int_0^{A-\min(t,A)}\int_0^{e^{-ct}} I_0(u,v)\ dv\ du + \\ & \int_0^A M(t,a)\left(\int_{r(t-a)}^t F_\Theta(e^{c(s-t)})\,\beta f(s)\, e^{-\int_{r(t-a)}^s \beta f(u)du}\ ds\right)\ da. \end{aligned} \tag{2.10}$$

Then L *has a unique fixed point* f *in* $C([0,T])$ *given by* $f = K(I)$, *where* (S,I) *is the mild solution of* $(\mathcal{P})$.

These well-posedness results allow us to derive useful information on the model output presented in next section.

2.3. Input-Output relationships. An alternative expression of the incidence (2.5) can be deduced from (2.6,2.7) as follows. For $(t,a) \in [0,T] \times [0,A]$ and $t \leqslant a$, one has

$$\mathfrak{i}(t,a) = S_0(a-t)e^{-\mu t} \int_0^t c e^{c(s-t)} \Theta\left(e^{c(s-t)}\right) \beta K(I)(s) e^{-\beta \int_0^s K(I)(u)du} ds \\ + cI_0(a-t, e^{-ct})e^{-(\mu+c)t}, \quad (2.11)$$

and, for $(t,a) \in [0,T] \times [0,A]$ and $t \geqslant a$,

$$\mathfrak{i}(t,a) = n(t-a)e^{-\mu a} \int_{t-a}^t c e^{c(s-t)} \Theta\left(e^{c(s-t)}\right) \beta K(I)(s) e^{-\beta \int_{t-a}^s K(I)(u)du} ds. \quad (2.12)$$

A standard strategy to investigate identifiability problems is to seek differential relationships between the input and the output of the model [11, 14, 15]. Unfortunately, in equation (2.12), the lack of regularity of n does not allow us to differentiate function $\mathfrak{i}$. For this reason, we introduce the function y defined on

$$\mathcal{D} = \{(t,a) \in [0,T] \times [0,A],\ a \leqslant t\},$$

by

$$y(t,a) = c \int_{t-a}^t e^{c(s-t)} \Theta\left(e^{c(s-t)}\right) \beta K(I)(s) e^{-\beta \int_{t-a}^s K(I)(u)du}\ ds. \quad (2.13)$$

This function plays an important role since it enjoys good regularity properties and, as soon as $n(t-a) \neq 0$ for $(t,a) \in \mathcal{D}$, one has

$$y(t,a) = \frac{\mathfrak{i}(t,a)}{n(t-a)e^{-\mu a}},$$

and therefore $y(t,a)$ is measured. In the sequel we denote

$$\mathcal{N} = \{t \in [0,T],\ n(t) \neq 0\}, \quad \underline{n} = \inf \mathcal{N}, \\ \mathcal{D}_{\mathcal{N}} = \{(t,a) \in \mathcal{D},\ t-a \in \mathcal{N}\},\ D = \partial_a + \partial_t\ .$$

The following result on y will be used in the sequel.

Proposition 2.4. *On $\mathcal{D}$, y and Dy are C^1, $\partial_a y$ is differentiable and they satisfy*

$$D\partial_a y(X(a) - y) = \partial_a y(X'(a) - Dy), \quad (2.14)$$

where X is defined in (2.3).

The proof of this proposition is postponed in section A.1 in appendix.

3. Identifiability. A model is said to be identifiable if, for given and known inputs, the observed outputs correspond to a unique set of parameters. Identifiability hence ensures a good parametrisation of the model. We consider that the inputs are known and that the outputs are observed. The parameter identifiability problem is thoroughly exposed in Section 3.1. Section 3.2 is devoted to the formulation of the main results of the paper, which are proved in Section 3.3.

3.1. The parameter identifiability problem. Parameters appearing in Problem $(\mathcal{P})$ are μ, c, β, Θ, n, I_0 and S_0. The last three correspond to the known inputs. Moreover, as mentioned in 2.1, the basic mortality rate μ is also known. Therefore, the identifiability problem only deals with the remaining parameters, defining the parameter vector

$$p = \begin{pmatrix} c \\ \beta \\ \Theta \end{pmatrix} \in \mathbf{P},$$

where the set of the unknown parameters $\mathbf{P}$ is defined as $\mathbf{P} = (\mathbb{R}^{+*})^2 \times \mathcal{A}_0(0,1)$. The system $(\mathcal{P})$ with output vector $\mathfrak{s}$, input functions $u = (S_0, I_0, n)$ and parameter vector p will be denoted as S_p^u.

The regularity results in the next proposition follow easily from Theorem 2.1 and Eq. (2.11,2.12).

Proposition 3.1. *Assume that*

$$u \in Pc([0,A],\mathbb{R}^+) \times Pc([0,A] \times [0,1],\mathbb{R}^+) \times Pc([0,T],\mathbb{R}^+).$$

Then, for $p \in \mathbf{P}$, the system S_p^u has a unique solution $(S,I) \in C([0,T],H^+)$ such that

$$\begin{gathered} \mathfrak{s} \in C\big([0,T],L^2([0,A],\mathbb{R}^+)\big), \quad \forall\, t \in [0,T],\ \mathfrak{s}(t) \in Pc([0,A],\mathbb{R}^+)^2, \\ \forall\, t \in [0,T], \quad S(t) \in Pc([0,A],\mathbb{R}^+) \text{ and } I(t) \in Pc([0,A] \times [0,1],\mathbb{R}^+). \end{gathered}$$

Definition 3.2 (Structural model). *The structural model $\mathcal{M}^u$ associated to $u \in Pc([0,A],\mathbb{R}^+) \times Pc([0,A] \times [0,1],\mathbb{R}^+) \times Pc([0,T],\mathbb{R}^+)$ is the mapping defined on $\mathbf{P}$ given by $\mathcal{M}^u : p \mapsto S_p^u$.*

The parameter identifiability of the model is defined as follows.

Definition 3.3 (Identifiable model). *For all $T' \leqslant T$, the model $\mathcal{M}^u$ is said to be identifiable on $[0,T']$ if, for all $(p,\bar{p}) \in \mathbf{P}^2$,*

$$\{\mathfrak{s}(t,p) = \mathfrak{s}(t,\bar{p}) \text{ on } [0,T']\} \Rightarrow p = \bar{p},$$

where $\mathfrak{s}(.,p)$ (respectively $\mathfrak{s}(.,\bar{p})$) is the output of $\mathcal{M}^u(p)$ (respectively $\mathcal{M}^u(\bar{p})$).
In other words, the parameter-to-output mapping is injective on $\mathbf{P}$.

3.2. Main results. Let the constants $\theta^\star$ and $c^\star$ be defined as

$$\begin{aligned} \theta^\star &= \sup\{\theta \in]0,1[,\ \exists a^\star \in]0,A[,\ I_0(a^\star,\theta) > 0\}, \\ c^\star &= -\frac{1}{\underline{m}} \ln \theta^\star, \quad \text{where } \underline{m} = \min(A,T). \end{aligned} \tag{3.1}$$

Remark 1. *It follows from the definition of $\theta^\star$ that $\theta^\star > 0$, since there are some infected animals in the initial population. Hence $0 \leqslant c^\star < +\infty$ and $I_0(a,\theta) = 0$ for $(a,\theta) \in]0,A[\times]e^{-c^\star \underline{m}},1[$.*

Consider the three following conditions on (BC,IC),

(H_1) $\exists\,\tilde\theta\in]0,\theta^\star[,\ \exists\,\tilde T\in[0,\underline{m}]\cap[0,A[$ such that $\tilde E=\{a\in[0,A-\tilde T],\ S_0(a)\neq 0\}\neq\emptyset$ and $a\mapsto \dfrac{I_0(a,\tilde\theta)}{S_0(a)}$ is not constant on $\tilde E$.

(H_2) $\exists\, t'\in[0,\underline{m}]$ such that $]\underline{n},t']\subset\mathcal{N}$ and $t\mapsto S_0(A-t)$ has two discontinuity points $t_1<t_2\in\mathcal{N}\cap[0,t'[$.

(H_3) The birth function n is such that $\mathcal{N}$ is a *finite* union of disjointed intervals that are not reduced to singleton sets since n is piecewise continuous.

We are now in a position to state the main results of the paper.

THEOREM 3.4. *Assume that condition (H_1) is satisfied and let $\tilde c=-\frac{1}{\tilde T}\ln\tilde\theta$. Then*

$$\forall (p,\bar p)\in(\tilde Q)^2,\quad (\forall t\in[0,T],\ \mathfrak{s}(t,p)=\mathfrak{s}(t,\bar p))\Rightarrow(c=\bar c),\tag{3.2}$$

where $\tilde{\mathbf{Q}}=]\tilde c,+\infty[\times\mathbb{R}^{+*}\times\mathcal{A}_0(0,1)\subset\mathbf{P}$.

THEOREM 3.5. *Assume that*

$$\textit{either } \{\underline{n}=0 \textit{ and } (H_3)\}$$
$$\textit{or } \{\underline{n}>0 \textit{ and } (H_2) \textit{ and } (H_3)\}.$$

Then model $\mathcal{M}^u$ is identifiable on $\mathbf{Q}^\star\subset\mathbf{P}$, where $\mathbf{Q}^\star=]c^\star,+\infty[\times\mathbb{R}^{+*}\times\mathcal{A}_0(0,1)$.

THEOREM 3.6. *The model $\mathcal{M}^u$ is not identifiable on the set $\mathbf{R}^\star\subset\mathbf{P}$, where $\mathbf{R}^\star=]0,c^\star[\times\mathbb{R}^{+*}\times\mathcal{A}_0(0,1)$.*

THEOREM 3.7. *Assume that (H_3) holds and let $\mathcal{G}\subset\mathcal{A}_0(0,1)$ be such that for all $(\Theta,\bar\Theta)\in\mathcal{G}^2$*

$$\left(\exists(\alpha,\bar\alpha)\in(\mathbb{R}^{+*})^2,\ \frac{1}{\alpha}\bar X'-\frac{1}{\bar\alpha}X'=X-\bar X\right)\Rightarrow\left(\alpha=\bar\alpha \textit{ and } X=\bar X.\right)\tag{3.3}$$

Then $\mathcal{M}^u$ is identifiable on $\mathbf{Q}^\star_{\mathcal{G}}\subset\mathbf{P}$, where $\mathbf{Q}^\star_{\mathcal{G}}=]c^\star,+\infty[\times\mathbb{R}^{+*}\times\mathcal{G}$.

REMARK 2. *Theorem 3.7 has a very strong practical interest, because when dealing with parameter identification on experimental data, Θ is indeed restricted to a parametric family of p.d.f., such as for instance the two-parameter family of Beta p.d.f. with support in $[0,1]$. For this family, condition 3.3 holds.*

3.3. Outline of proofs.

3.3.1. Consequence of equal outputs for two parameter vectors p and $\bar p$.

Let (S_0,I_0) and n be given as in 3.1 and consider $(p,\bar p)\in\mathbf{P}2$ such that, for all $t\in[0,T]$,

$$\mathfrak{s}(t,p)=\mathfrak{s}(t,\bar p).\tag{3.4}$$

In the sequel, the population densities, the p.d.f. of first infection load and incubation period, the output vector of $\mathcal{M}^u(\bar p)$ shall be denoted as $\bar S$, $\bar I$, $\bar\Theta$, $\bar X$ and $\bar{\mathfrak{s}}$; more generally, all the quantities wearing a bar will be related to $\mathcal{M}^u(\bar p)$. The same quantities without bar will be related to $\mathcal{M}^u(p)$. Moreover, we define

$$\Delta=X-\bar X \text{ on } \mathbb{R}^+.$$

Note that (3.4) implies $\bar{\mathfrak{s}} = \mathfrak{s}$ and, by (2.13), $\bar{y} = y$ on $\mathcal{D}_{\mathcal{N}}$.

Let us first consider the equality between incidences for animals born after the beginning of the observation period ($t \geqslant a$). The regularity results in Section 2.3 and the input-output relationship obtained in Proposition 2.4 allow us to eliminate some variables and to obtain relationships between the two sets of parameters p and $\bar{p}$. This is summarised in Proposition 3.8 below.

PROPOSITION 3.8. *If* (3.4) *holds, then*

$$\text{either} \quad X = \bar{X} \text{ on } \mathbb{R}^+,$$

$$\text{or} \quad \exists\,(\alpha,\bar{\alpha}) \in (\mathbb{R}^{+*})2 \;/ \qquad \alpha \neq \bar{\alpha} \text{ and } \frac{1}{\alpha}\bar{X}' - \frac{1}{\bar{\alpha}}X' = X - \bar{X} \text{ on } \mathbb{R}^{+*}.$$

In this last case, $t \mapsto \beta K(I)(t)$ and $t \mapsto \bar{\beta}\, K(\bar{I})(t)$ are non zero constant functions on $\mathcal{N}$, whose values are α and $\bar{\alpha}$ respectively. The proof of this proposition is given in Section A.2 in appendix.

Then the equality between incidences for animal born before the beginning of the observation period ($a > t$) can be exploited to obtain the following proposition, whose proof can be found in A.3.

PROPOSITION 3.9. *Assume that* (3.4) *holds and $c > \bar{c}$. Then, for all $\theta \in [e^{-c\underline{m}}, 1]$, the function g_θ, given by*

$$g_\theta : a \mapsto \frac{\int_\theta^{\theta^{\bar{c}/c}} I_0(a,u)\; du}{S_0(a)},$$

is constant on $\{a \in [0, A + \frac{1}{c}\ln\theta]\,,\, S_0(a) \neq 0\}$.

3.3.2. Proof of Theorem 3.4. Theorem (3.2) is proved by contradiction. We show that if the outputs are equal on $[0,T]$, assuming that $c > \bar{c}$ leads to a contradiction with condition (H_1), and consequently $c = \bar{c}$. The details of the proof are given in Appendix B.

3.3.3. Proof of Theorem 3.5. The proof is split into two steps. The first one consists in establishing the identifiability of X by contradiction, as shown in the following two propositions.

PROPOSITION 3.10. *If* (3.4) *is satisfied,* (H_3) *holds and $\underline{n} = 0$, then $X = \bar{X}$.*

PROPOSITION 3.11. *If* (3.4) *is satisfied,* (H_2) *and* (H_3) *hold, and $\underline{n} > 0$, then $X = \bar{X}$.*

The proofs are postponed in appendices C.1 and C.2, and use Proposition 3.8 as a key ingredient.

The second step is the proof of the identifiability of β, as stated in Proposition 3.12.

PROPOSITION 3.12. *Assume that* (3.4) *is satisfied and $X = \bar{X}$. Then $\beta = \bar{\beta}$.*

This proposition is proved in C.3. As a consequence, we obtain the following corollary, whose proof is in C.4.

COROLLARY 3.13. *Assume that $X = \bar{X}$ and $\beta = \bar{\beta}$. Then $K(I)(t) = K(\bar{I})(t)$ for all $t \in [0,T]$.*

Then we use the propositions that were proved in the first two steps of the demonstration to show that the model $\mathcal{M}^u$ is identifiable on $\mathbf{Q}^\star$. The detailed proof is given in C.5 in Appendix. We first prove that $c = \bar{c}$. It easily follows that $\Theta = \bar{\Theta}$, which completes the proof of the identifiability of the model $\mathcal{M}^u$ on $\mathbf{Q}^\star$.

3.3.4. Proof of Theorem 3.6. To prove that the model $\mathcal{M}^u$ is not identifiable on $\mathbf{R}^\star$, we build a counter example, that is two vectors of parameters $p \neq \bar{p} \in \mathbf{R}^\star$ such that $\mathfrak{s} = \bar{\mathfrak{s}}$ on $[0,T]$. These vectors are such that $\beta = \bar{\beta}$, $0 < \bar{c} < c < c^\star$ and Θ and $\bar{\Theta}$ are p.d.f. in $\mathcal{A}_0(0,1)$ related by

$$\bar{\Theta}(\theta) = \frac{c}{\bar{c}}\,\theta^{\frac{c-\bar{c}}{\bar{c}}}\,\Theta\left(\theta^{\frac{c}{\bar{c}}}\right). \tag{3.5}$$

This relationship ensures that the two incubation time p.d.f. X and $\bar{X}$ are identical. The details of the proof can be found in Appendix D.

3.3.5. Proof of Theorem 3.7. The assumption on $\mathcal{G}$ and 3.8 immediately yield $X = \bar{X}$. Then the proof of the theorem follows the same steps as the proof of Theorem 3.5.

4. Numerical simulations. In this section, we illustrate our identifiability results through two simulation scenarios. Scenario 1 corresponds to the non identifiability case under the assumptions of Theorem 3.6. Scenario 2 represents Theorem 3.7 for the Beta distribution family.

For both scenarios, system (DS, DI, BC, IC) is integrated with parameter values given in Table 4.1. The birth function n is constant. The initial susceptible population density follows an exponential distribution $S_0(a) \propto e^{-\mu a}$. The initial infected population density $I_0(a,\theta)$ is uniformly distributed over $[a^{\min}, a^{\max}] \times [\theta^{\min}, \theta^{\max}]$. Scaling coefficients are adjusted to obtain the initial population sizes given in Table 4.1. Parameter values are chosen to mimic realistic epidemiological situations.

Table 4.1
Parameter values used for the simulations.

Parameter definition	symbol	value
initial population size	–	600 indiv.
initial infected population size	–	30 indiv.
— age range	$[a^{\min}, a^{\max}]$	$[0.625, 1, 04]$ years
basic mortality rate	μ	0.15 year^{-1}
horizontal transmission rate	β	$3\,10^{-3}$ (indiv. year)$^{-1}$
birth rate	n	70 indiv/year
maximum lifespan	A	13 years
observation period	T	4 years
Scenario 1 specific parameters		
initial infection load range	$[\theta_1^{\min}, \theta_1^{\max}]$	$[0.125, 0.18]$
infection load growth rates	$(c_1, \bar{c}_1)$	$(0.35, 0.28)$ year^{-1}
first infection load distribution Θ_1: mean	m_{Θ_1}	0.35
— standard deviation	σ_{Θ_1}	0.05
Scenario 2 specific parameters		
initial infection load range	$[\theta_2^{\min}, \theta_2^{\max}]$	$[0.68, 0.73]$
infection load growth rates	$(c_2, \bar{c}_2)$	$(0.35, 0.12)$ year^{-1}
first infection load distribution Θ_2: mean	m_{Θ_2}	0.35
— standard deviation	σ_{Θ_2}	0.05
first infection load distribution $\bar{\Theta}_2$: mean	$m_{\bar{\Theta}_2}$	0.7
— standard deviation	$\sigma_{\bar{\Theta}_2}$	0.05

4.1. Scenario 1. We build two parameter vectors $p_1 \neq \bar{p}_1$ of $\mathbf{R}^\star$ for which the observed incidences $\mathfrak{i}(t,a)$ are the same on the observation time interval $[0,T]$. The only differences between the two parameter vectors p_1 and $\bar{p}_1$ are the infection load growth rates c_1 and $\bar{c}_1$, and the first infection load distributions Θ_1 and $\bar{\Theta}_1$. Θ_1 is a Beta distribution with mean m_{Θ_1} and standard deviation σ_{Θ_1}. The first infection load distribution $\bar{\Theta}_1$ is related to Θ_1 by (3.5). Parameter values ensure that c_1 and $\bar{c}_1$ are in $]0, c_1^\star[$, $c_1^\star = 0.42$ being defined in (3.1).

As a consequence of Theorem 3.6 the model is not identifiable on $[0,T]$. This is illustrated in Fig. 4.1, that represents the total incidence $\int_0^A \mathfrak{i}(t,a)da$ over time for both parameter vectors p_1 and $\bar{p}_1$. The two incidence curves coincide up to time T, but become different on a longer time horizon.

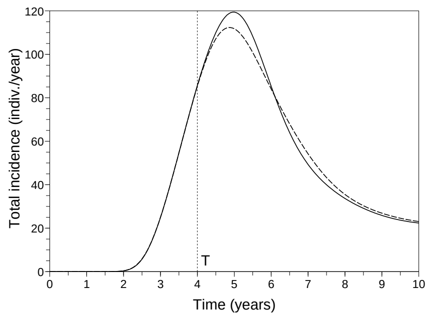


FIG. 4.1. *Scenario 1 – Total incidence* $\int_0^A \mathfrak{i}(t,a)da$ *over time* t *for the two parameter sets given in Table 4.1:* (c_1,Θ_1) *plain line &* $(\bar{c}_1,\bar{\Theta}_1)$ *dashed line. Up to time* $T = 4$*, the model is not identifiable and the incidence outputs coincide.*

Moreover, the proof of Theorem 3.6 states that $K(I) = K(\bar{I})$ on $[0,T]$. However, the infected densities are different, as shown in Fig. 4.2.

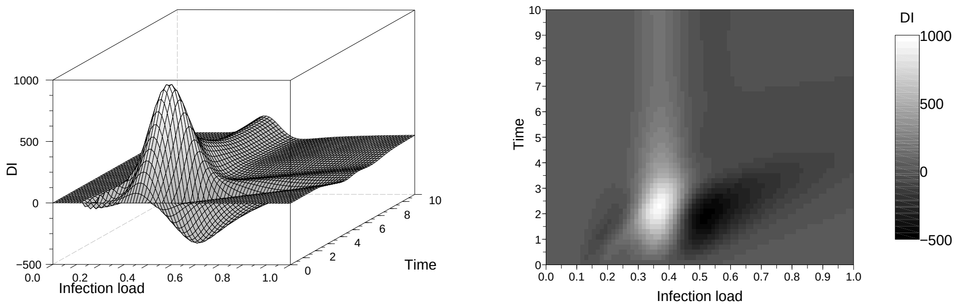


FIG. 4.2. *Scenario 1 – Difference* $DI(t,\theta) = \int_0^A (I-\bar{I})(t,a,\theta)da$ *between the two infected densities obtained with the two parameter sets given in Table 4.1. Up to time* $T = 4$*, the model is not identifiable, but the infected densities differ.*

4.2. Scenario 2. The differences between the parameter vectors p_2 and $\bar{p}_2$ are again the infection load growth rates c_2 and $\bar{c}_2$, and the first infection load distributions Θ_2 and $\bar{\Theta}_2$. They are both Beta distributions with the same standard deviations $\sigma_{\Theta_2} = \sigma_{\bar{\Theta}_2}$, but different means $m_{\Theta_2} \neq m_{\bar{\Theta}_2}$. Parameters c_2 and $\bar{c}_2$ are adjusted to obtain the same mean incubation period of 3 years for the distribution given in (2.3). First infection load and incubation period distributions are represented in Fig. 4.3.

With such similar incubation period distributions, one could fear the model not to be identifiable. However, parameter values ensure that c_2 and $\bar{c}_2$ are in $]c_2^\star, +\infty[$, with $c_2^\star = 0.08$. Theorem 3.7 then guarantees that the model is identifiable. This is

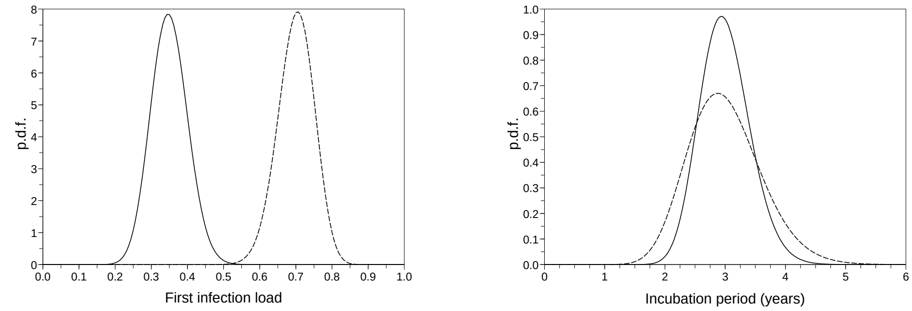
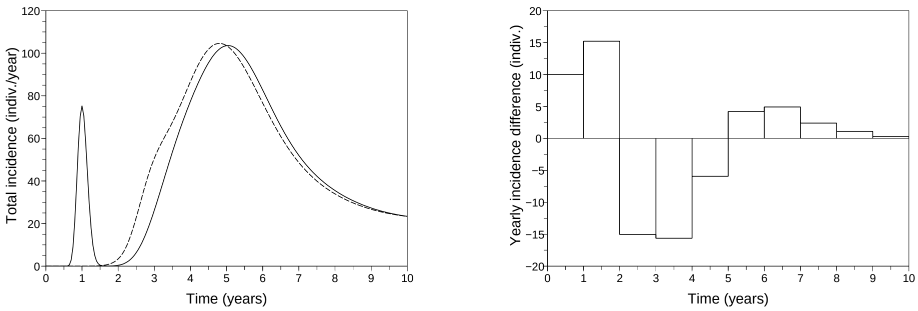


FIG. 4.3. *Scenario 2 – Distributions represented for the two parameter sets given in Table 4.1:* (c_2, Θ_2) *plain line &* $(\bar{c}_2, \bar{\Theta}_2)$ *dashed line.*

illustrated in Fig. 4.4 that represents the total incidence for both parameter sets. Total incidences, which are instantaneous flow measurements, exhibit notable differences. It is even more obvious on the yearly cumulated incidences, which are closer to the data collected in realistic situations.



(a) Total incidence $\int_0^A \mathrm{i}(t,a)da$.

(b) Difference between the yearly cumulated incidences.

FIG. 4.4. *Scenario 2 – Incidence outputs correspond to the two parameter sets given in Table 4.1:* (c_2, Θ_2) *plain line &* $(\bar{c}_2, \bar{\Theta}_2)$ *dashed line. The model is identifiable.*

5. Conclusions. We showed in Theorems 3.4 and 3.5 that model $\mathcal{M}^u$ is identifiable under realistic hypotheses on the inputs, i.e. the birth function and the initial conditions. Hypothesis (H_1) holds in many practical situations, such as a patchy structure for $I(.,\theta)$. (H_2) is a technical assumption that is not too restrictive on the initial conditions. Finally, hypothesis (H_3) includes seasonal birth functions, that correspond to real situations in many animal populations.

Moreover, Theorem 3.7 ensures that, given a suitable parametric family for the first infection load distribution and assuming that the infection load growth rate verifies a common sense condition, model $\mathcal{M}^u$ is identifiable under the realistic hypothesis (H_3). The condition on the infection load growth rate states that it should be bigger a threshold value $c^\star$ that depends on the initial condition I_0. The biological interpretation of this condition is clear: for such growth rates, some initially infected animals necessarily die of the disease (i.e. their load reaches value 1) during the observation period.

Finally, we are convinced that hypothesis (H_2) could be made more realistic, or even unnecessary and plan to follow this direction in future work.

Appendix A. Technical lemmas.

A.1. Proof of Proposition 2.4. The proof of Proposition 2.4 is a direct consequence of the following lemma.

LEMMA A.1. *Consider the function $\tilde{y}$ defined on $\mathcal{D}$ by*

$$\tilde{y}(t,a) = c\int_{t-a}^{t} e^{2c(s-t)}\,\Theta'(e^{c(s-t)})\,\beta\,K(I)(s)\,e^{-\beta\int_{t-a}^{s}K(I)(u)du}\,ds.$$

Then y, Dy, $\tilde{y}$ are C^1 on $\mathcal{D}$ and $\partial_a y$ is differentiable on $\mathcal{D}$. So

$$\partial_a y(t,a) = \beta\,K(I)(t-a)\,(X(a) - y(t,a)), \tag{A.1}$$

$$\partial_t y(t,a) = -\beta\,X(a)\,K(I)(t-a) - c\,y(t,a) + \beta\,K(I)(t-a)\,y(t,a) - c\,\tilde{y}(t,a), \tag{A.2}$$

$$D\partial_a y = \beta\,K(I)(t-a)\,(X'(a) - Dy). \tag{A.3}$$

Proof. From equation (2.7) of Theorem 2.1 the function $t \mapsto K(I)(t)$ is differentiable on $[0,T]$ and has a piecewise continuous derivative. Consequently,

$$t \mapsto e^{-\beta\int_0^t K(I)(u)du} \in C^1([0,T]),$$

and $y(t,a)$ and $\tilde{y}(t,a)$ have partial derivatives in a and t on $\mathcal{D}$, given by

$$\begin{aligned}
\partial_a y &= c\,e^{-ca}\Theta(e^{-ca})\,\beta K(I)(t-a)\\
&\quad - \beta\,K(I)(t-a)\,c\int_{t-a}^{t} e^{c(s-t)}\,\Theta(e^{c(s-t)})\,\beta\,K(I)(s)\,e^{-\beta\int_{t-a}^{s}K(I)(u)du}\,ds\\
&= c\,e^{-ca}\Theta(e^{-ca})\,\beta K(I)(t-a) - \beta K(I)(t-a)\,y(t,a)\\
&= \beta K(I)(t-a)\left(c\,e^{-ca}\Theta(e^{-ca}) - y\right),
\end{aligned}$$

which proves (A.1) and

$$\begin{aligned}
\partial_t y &= -ce^{-ca}\Theta(e^{-ca})\beta K(I)(t-a) - c^2\int_{t-a}^{t} e^{c(s-t)}\Theta(e^{c(s-t)})\beta K(I)(s)e^{-\beta\int_{t-a}^{s}K(I)(u)du}ds\\
&\quad - c^2\int_{t-a}^{t} e^{2c(s-t)}\,\Theta'(e^{c(s-t)})\,\beta\,K(I)(s)\,e^{-\beta\int_{t-a}^{s}K(I)(u)du}\,ds\\
&\quad + (\beta\,K(I)(t-a))\,c\int_{t-a}^{t} e^{c(s-t)}\,\Theta(e^{c(s-t)})\,\beta\,K(I)(s)\,e^{-\beta\int_{t-a}^{s}K(I)(u)du}\,ds\\
&= -c\,e^{-ca}\Theta(e^{-ca})\,\beta K(I)(t-a) - cy + \beta K(I)(t-a)y - c\tilde{y},
\end{aligned}$$

which proves (A.2).

Moreover, standard results on integrals depending on parameters imply that the functions y and $\tilde{y}$ are continuous on $\mathcal{D}$. From equations (A.1,A.2) we deduce that $\partial_a y$ and $\partial_t y$ are continuous functions on $\mathcal{D}$ and consequently y is C^1 on this set. Similar arguments prove that $\tilde{y}$ is also C^1. Summing (A.1) and (A.2) leads to $Dy = -c\,y - c\,\tilde{y}$, which proves that Dy is C^1. Since y is C^1 and $t \mapsto K(I)(t)$ is differentiable, equation (A.1) implies that $\partial_a y$ is differentiable. Applying the operator D to (A.1), since $D\big(K(I)(t-a)\big) = 0$, leads to (A.3). □

Combining (A.1) and (A.3) on $\mathcal{D}$ to eliminate $\big(K(I)(t-a)\big)$ leads to (2.14).

A.2. Proof of Proposition 3.8. It starts with the proof of technical lemmas A.2, A.3, A.4 and A.5, which are valid if (3.4) holds.

Let us define the following quantities, using the notations given in Section 3.3.1:

$$M_y(t,a) = \begin{pmatrix} Dy \\ y \end{pmatrix}, \text{ and also } M_{\partial_a y},$$
$$Y(a) = \begin{pmatrix} X'(a) \\ X(a) \end{pmatrix} \text{ and } \bar{Y}(a) = \begin{pmatrix} \bar{X}'(a) \\ \bar{X}(a) \end{pmatrix},$$

and finally for $x > 0$,

$$R(x) = \begin{vmatrix} X'(x) & \bar{X}'(x) & \Delta(x) \\ X^{(2)}(x) & \bar{X}^{(2)}(x) & \Delta'(x) \\ X^{(3)}(x) & \bar{X}^{(3)}(x) & \Delta^{(2)}(x) \end{vmatrix}. \tag{A.4}$$

Note that from (3.4) we have $M_y = M_{\bar{y}}$ and $M_{\partial_a y} = M_{\partial_a \bar{y}}$.

REMARK 3. *Hypothesis* (2.2) *on* Θ *and* $\bar{\Theta}$ *implies that* X, $\bar{X}$, Δ *and all their derivatives are real-analytic functions on* $\mathbb{R}^{+*}$. *Consequently, either they have isolated zeros in* $\mathbb{R}^{+*}$ *or they are identically equal to zero.*

LEMMA A.2. *The equality* (3.4) *implies that*

$$\partial_a Dy(t,a)\,(X(a) - \bar{X}(a)) - \partial_a y(t,a)\,(X'(a) - \bar{X}'(a)) = 0 \quad \forall (t,a) \in \mathcal{D}_{\mathcal{N}}. \tag{A.5}$$

Proof. From (3.4) we have $y = \bar{y}$, so from Proposition 2.4 the two following equalities hold for all $(t,a) \in \mathcal{D}_{\mathcal{N}}$.

$$D\partial_a y(X(a) - y) = \partial_a y(X'(a) - Dy),$$
$$D\partial_a y(\bar{X}(a) - y) = \partial_a y(\bar{X}'(a) - Dy).$$

Subtracting these two equations we obtain (A.5). □

LEMMA A.3. *If* (3.4) *holds one gets for all* $(t,a) \in \mathcal{D}_{\mathcal{N}}$

$$[X'\bar{X} - X\bar{X}'] - y\,[X' - \bar{X}'] + Dy\,[X - \bar{X}] = 0. \tag{A.6}$$

Proof. Let $(t,a) \in \mathcal{D}_{\mathcal{N}}$. Then either $M_{\partial_a y}(t,a) \neq 0$ or $M_{\partial_a y}(t,a) = 0$.

In the first case, as Eq. (A.1,A.3) yield $\det(M_{\partial_a y}(t,a), Y(a) - M_y(t,a)) = 0$, and similarly for $\bar{Y}$, it follows that $\det\big(\bar{Y}(a) - M_y(t,a), Y(a) - M_y(t,a)\big) = 0$ and consequently (A.6) holds.

In the second case, Eq. (A.1) yields

$$\beta\, K(I)(t-a)(X(a) - y(t,a)) = 0.$$

From Corollary 2.2 we know that $\beta\, K(I)(t-a) \neq 0$. So $X(a) = y(t,a) = \bar{X}(a)$. Using (A.3) we similarly obtain $X'(a) = \bar{X}'(a)$ and (A.6) also holds. □

LEMMA A.4. *If* (3.4) *holds one gets for all* $(t,a) \in \mathcal{D}_{\mathcal{N}}$

$$\begin{aligned} &[X'\bar{X} - X\bar{X}']\,[X' - \bar{X}'] - [X^{(2)}\bar{X} - X\bar{X}^{(2)}]\,[X - \bar{X}] \\ &\quad -y\,\Big([X' - \bar{X}']^2 - [X^{(2)} - \bar{X}^{(2)}]\,[X - \bar{X}]\Big) = 0. \end{aligned} \tag{A.7}$$

Proof. Consider $(t,a) \in \mathcal{D}_{\mathcal{N}}$. Since n is piecewise continuous, there exists an interval $\mathcal{V}(a)$ such that $\{a\} \subsetneq \mathcal{V}(a) \subset [0,A]$, and $\{t\} \times \mathcal{V}(a) \subset \mathcal{D}_{\mathcal{N}}$. Therefore we differentiate Eq. (A.6) w.r.t. a, which yields, for all $(t,a) \in \mathcal{D}_{\mathcal{N}}$,

$$[X^{(2)}\bar{X} - X\bar{X}^{(2)}] - \partial_a y[X' - \bar{X}'] - y[X^{(2)} - \bar{X}^{(2)}] + Dy\,[X' - \bar{X}'] + \partial_a Dy\,[X - \bar{X}] = 0.$$

Using Eq. (A.5), which is valid on $\mathcal{D} \supset \mathcal{D}_{\mathcal{N}}$, we have

$$[X'^{(2)}\bar{X} - X\bar{X}^{(2)}] - y[X^{(2)} - \bar{X}^{(2)}] + Dy\,[X' - \bar{X}'] = 0 \quad \forall (t,a) \in \mathcal{D}_{\mathcal{N}}. \tag{A.8}$$

Combining (A.6) and (A.8) one gets (A.7) on $\mathcal{D}_{\mathcal{N}}$. □,

LEMMA A.5. *If* (3.4) *holds then* $R(x) = 0$ *for all* $x \in \mathbb{R}^{+*}$.

Proof. We perform algebro-differential elimination of y in (A.6) and (A.7) using operator D to obtain the following equality

$$\begin{aligned}(X-\bar{X})^3 \Big(&-\bar{X}\bar{X}^{(2)}X^{(3)} + \bar{X}X^{(2)}\bar{X}^{(3)} - X^{(2)}\bar{X}^{(2)}X' + (\bar{X}')^2X^{(3)} \\ &+ X\bar{X}^{(2)}X^{(3)} - X'\bar{X}'X^{(3)} + \bar{X}^{(3)}(X')^2 + X'(\bar{X}^{(2)})^2 \\ &- X^{(2)}X\bar{X}^{(3)} - \bar{X}^{(2)}X^{(2)}\bar{X}' - \bar{X}^{(3)}X'\bar{X}' + (X^{(2)})^2\bar{X}'\Big) = 0,\end{aligned}$$

which rewrites after some calculation

$$\Delta(x)^3\, R(x) = 0. \tag{A.9}$$

Using similar arguments as in the proof of Lemma A.7, Eq. (A.9) is valid on an open interval of $[0,A]$ and can be extended to $\mathbb{R}^{+*}$ consequently to Remark 3. The proof is ended by contradiction: assume there exists $x_0 > 0$ such that $R(x_0) \neq 0$. By continuity, this is still valid on a neighbourhood $\mathcal{V}(x_0) \subset \mathbb{R}^{+*}$ and equality (A.9) implies that $\Delta(x) = 0$ for all $x \in \mathcal{V}(x_0)$ and finally, since the third column of the determinant is null, $R(x) = 0$ on $\mathcal{V}(x_0)$ which is impossible. □

We now proceed with the proof of Proposition 3.8. Lemma A.5 and (3.4) imply that, for all $x > 0$, there exists $\lambda(x), \mu(x), \nu(x) \in \mathbb{R}$ such that

$$\left\{\begin{array}{l} \lambda X' + \mu\bar{X}' + \nu\Delta = 0, \\ \lambda X^{(2)} + \mu\bar{X}^{(2)} + \nu\Delta' = 0, \\ \lambda X^{(3)} + \mu\bar{X}^{(3)} + \nu\Delta^{(2)} = 0, \end{array}\right. \tag{A.10}$$

where λ, μ, ν are minors of determinant (A.4). We can choose ν associated to $\Delta^{(2)}$, given by $\nu = X'\bar{X}^{(2)} - \bar{X}'X^{(2)}$. Then two cases may arise.

Case 1. Assume that $\nu(x) = 0$ *for all* $x > 0$. The function $\bar{X}'$ is a non zero function on $\mathbb{R}^{+*}$, otherwise, by continuity, $\bar{X}$ would be constant and equal to zero on $\mathbb{R}^{+}$. Therefore, we can find $x_1 > 0$ such that $\bar{X}'(x_1) \neq 0$. By continuity, this is still true in a neighbourhood $\mathcal{V}(x_1)$ of x_1. Then, for all $x \in \mathcal{V}(x_1)$,

$$\left(\bar{X}'(x)\right)^2 \times \frac{d}{dx}\left(\frac{X'}{\bar{X}'}\right) = 0,$$

which implies that there exists a constant c_0 such that $X' = c_0\,\bar{X}'$ on $\mathcal{V}(x_1)$. From Remark 3, we get $X' = c_0\,\bar{X}'$ on $\mathbb{R}^{+*}$ and $X = c_0\,\bar{X}$ on $\mathbb{R}^{+*}$ since $X(0) = \bar{X}(0) = 0$. Taking into account that $\int_0^{+\infty} X(x)dx = \int_0^{+\infty} \bar{X}(x)dx = 1$, we have $c_0 = 1$ and finally $X = \bar{X}$ on $\mathbb{R}^{+*}$.

Case 2. Assume that there exists $x_2 > 0$ and a neighbourhood $\mathcal{V}(x_2) \subset \mathbb{R}^{+*}$ such that $\nu(x) \neq 0$ for all $x \in \mathcal{V}(x_2)$. Then, from system (A.10), we deduce that the following equations are satisfied on $\mathcal{V}(x_2)$,

$$\tilde{\lambda} X' + \tilde{\mu}\bar{X}' = \Delta, \tag{A.11}$$

$$\tilde{\lambda} X^{(2)} + \tilde{\mu}\bar{X}^{(2)} = \Delta', \tag{A.12}$$

$$\tilde{\lambda} X^{(3)} + \tilde{\mu}\bar{X}^{(3)} = \Delta^{(2)}, \tag{A.13}$$

where $\tilde{\lambda} = -\frac{\lambda}{\nu}$, $\tilde{\mu} = -\frac{\mu}{\nu}$. Differentiating (A.11) and subtracting (A.12) yields, for $x \in \mathcal{V}(x_2)$,

$$\tilde{\lambda}' X' + \tilde{\mu}'\bar{X}' = 0. \tag{A.14}$$

In the same way, differentiating (A.11) twice and subtracting (A.13) yields

$$\tilde{\lambda}^{(2)} X' + \tilde{\mu}^{(2)}\bar{X}' + 2(\tilde{\lambda}' X^{(2)} + \tilde{\mu}'\bar{X}^{(2)}) = 0. \tag{A.15}$$

Finally, differentiating (A.14) and combining it (A.15), we get

$$\tilde{\lambda}^{(2)} X' + \tilde{\mu}^{(2)}\bar{X}' = 0 \text{ on } \mathcal{V}(x_2). \tag{A.16}$$

From (A.14) and (A.16), we have $W = 0$ on $\mathcal{V}(x_2)$ where

$$W = \begin{vmatrix} \tilde{\lambda}' & \tilde{\mu}' \\ \tilde{\lambda}^{(2)} & \tilde{\mu}^{(2)} \end{vmatrix}.$$

Otherwise, there would exist an open subset $\mathcal{V} \subset \mathcal{V}(x_2)$ such that $W(x) \neq 0$ for $x \in \mathcal{V}$. The unique solution of system (A.14,A.16) would be $\left(X', \bar{X}'\right) = (0, 0)$ on $\mathcal{V}$. This would imply $\nu(x) = 0$ on $\mathcal{V}$, which is impossible. We now distinguish the two following cases.

Case 2.1. If there exists an open subset $\mathcal{V} \subset \mathcal{V}(x_2)$ on which $\tilde{\lambda}'(x) \neq 0$, then $W = 0$ on $\mathcal{V}(x_2)$ implies that $\frac{d}{dx}(\tilde{\mu}'/\tilde{\lambda}') = 0$ in $\mathcal{V}$. Consequently, there exists a constant c_0 such that $X' = c_0\,\bar{X}'$ on $\mathcal{V}$ and we can conclude as in *Case 1* that $X = \bar{X}$ on $\mathbb{R}^+$.

Case 2.2. If $\tilde{\lambda}' = 0$ on $\mathcal{V}(x_2)$, then $\tilde{\lambda}$ is a constant function on $\mathcal{V}(x_2)$ whose value is denoted $\tilde{\lambda}_0$. Since $\bar{X}'$ has isolated zeros, Remark 3 and (A.14) imply that $\tilde{\mu}$ is also a constant function on $\mathcal{V}(x_2)$ whose value is denoted $\tilde{\mu}_0$. Consequently, on $\mathcal{V}(x_2)$, equalities (A.11) and (A.12) become respectively

$$\tilde{\lambda}_0 X' + \tilde{\mu}_0\bar{X}' = \Delta, \tag{A.17}$$

$$\tilde{\lambda}_0 X^{(2)} + \tilde{\mu}_0\bar{X}^{(2)} = \Delta'.$$

By Remark 3, these equalities can be extended to $\mathbb{R}^{+*}$ and can be used to simplify (A.7). On $\mathcal{D}_{\mathcal{N}}$ one therefore has

$$[X'\bar{X} - X\bar{X}'][X' - \bar{X}'] - [X^{(2)}\bar{X} - X\bar{X}^{(2)}][X - \bar{X}] = (\tilde{\lambda}_0 X + \tilde{\mu}_0\bar{X})(\bar{X}^{(2)}X' - \bar{X}'X^{(2)}),$$

$$[X' - \bar{X}']^2 - [X^{(2)} - \bar{X}^{(2)}]\,[X - \bar{X}] = (\tilde{\lambda}_0 + \tilde{\mu}_0)\,(\bar{X}^{(2)}X' - \bar{X}'X^{(2)}),$$

and

$$\left(-y(\tilde{\lambda}_0 + \tilde{\mu}_0) + \tilde{\lambda}_0 X + \tilde{\mu}_0\bar{X}\right)\left(\bar{X}^{(2)}X' - \bar{X}'X^{(2)}\right) = 0.$$

By Remark 3, since $\nu \neq 0$, we conclude that

$$-y(\tilde{\lambda}_0 + \tilde{\mu}_0) + \tilde{\lambda}_0 X + \tilde{\mu}_0 \bar{X} = 0 \quad \text{on } \mathcal{D}_{\mathcal{N}}. \tag{A.18}$$

Then, either $\tilde{\lambda}_0+\tilde{\mu}_0 = 0$, and integrating (A.17) yields $\Delta = X-\bar{X} = 0$. Or $\tilde{\lambda}_0+\tilde{\mu}_0 \neq 0$ and consequently for all $(t, a) \in \mathcal{D}_{\mathcal{N}}$

$$y(t,a) = \frac{\tilde{\lambda}_0 X(a) + \tilde{\mu}_0 \bar{X}(a)}{\tilde{\lambda}_0 + \tilde{\mu}_0}.$$

This expression used in (A.1) yields, for all $(t, a) \in \mathcal{D}_{\mathcal{N}}$,

$$\tilde{\lambda}_0 X'(a) + \tilde{\mu}_0 \bar{X}'(a) = \tilde{\mu}_0\, \beta\, K(I)(t-a)\, (X(a) - \bar{X}(a)). \tag{A.19}$$

Denoting $\mathcal{T} = \{a \in [0, A],\ \Delta(a) \neq 0\}$, we easily check that 0 is in the closure of $\mathcal{T}$. Moreover, equation (A.19) implies that $(t, a) \mapsto \beta\, K(I)(t-a)$ is a constant on $\{(t, a) \in \mathcal{D}_{\mathcal{N}},\ a \in \mathcal{T}\}$ and consequently, for all $a \in \mathcal{T} \cap [0, T]$, $t \mapsto \beta\, K(I)(t)$ is constant on $\mathcal{N} \cap [0, T-a]$. Since 0 is in the closure of $\mathcal{T}$, we conclude that $t \mapsto \beta\, K(I)(t)$ is constant on $\mathcal{N}$. We denote α this constant, which is different from zero according to Corollary 2.2. By the same arguments we also prove that $t \mapsto \bar{\beta}\, K(\bar{I})(t)$ is a positive constant on $\mathcal{N}$ that we denote $\bar{\alpha}$. Then (A.11) and (A.19) yield $\alpha = \frac{1}{\tilde{\mu}_0}$. Similarly, $\bar{\alpha}$ is positive and such that $\bar{\alpha} = -\frac{1}{\tilde{\lambda}_0}$. Substituting these values in (A.17) yields the desired result.

A.3. Proof of Proposition 3.9. Let $t \leqslant \underline{m}$. From the equality of the incidences $\bar{\mathfrak{i}} = \mathfrak{i}$ and from (2.11) we obtain that, for all $(t, a') \in [0, \underline{m}] \times [0, A]$ such that $S_0(a') \neq 0$ and $0 \leqslant a' \leqslant A - t$,

$$\begin{aligned}\frac{cI_0(a', e^{-ct})e^{-ct} - \bar{c}I_0(a', e^{-\bar{c}t})e^{-\bar{c}t}}{S_0(a')} &= \int_0^t X(s-t)\, \beta K(I)(s)\, e^{-\beta \int_0^s K(I)(u)du}\, ds \\ &\quad - \int_0^t \bar{X}(s-t)\, \bar{\beta} K(\bar{I})(s)\, e^{-\bar{\beta} \int_0^s K(\bar{I})(u)du}\, ds.\end{aligned} \tag{A.20}$$

Let $t \mapsto h(t)$ be the function in the right member of (A.20). Integrating this equality with respect to t and performing a change of variable, one gets, for all $(t, a') \in [0, \underline{m}] \times [0, A]$ such that $S_0(a') \neq 0$ and $0 \leqslant a' \leqslant A - t$,

$$\frac{\int_{e^{-ct}}^{e^{-\bar{c}t}} I_0(a', s)\ ds}{S_0(a')} = \int_0^t h(s)\ ds.$$

Finally, the change of variable $\theta = e^{-ct}$ implies that, for all $\theta \in [e^{-c\underline{m}}, 1]$ and all $a' \in [0, A + \frac{1}{c}\ln\theta]$ such that $S_0(a') \neq 0$,

$$g_\theta(a') = \int_0^{-\frac{1}{c}\ln\theta} h\,(s)\ \ ds,$$

which proves the desired result.

Appendix B. Proof of Theorem 3.4.

Assume that I_0 satisfies condition (H_1) and let $(p, \bar{p}) \in \tilde{Q}^2$, then Theorem 3.4 is proved by contradiction. Assuming that $c > \bar{c}$ and the outputs are equal on $[0, T]$, let

$\theta \in [e^{-c\underline{m}}, 1[$ and let the sequence $\{\theta_n\}_{n\in\mathbb{N}}$ be defined by $\theta_0 = \theta$ and $\theta_{n+1} = (\theta_n)^{\frac{\bar{c}}{c}}$. As $c > \bar{c}$, $\theta_n \in [\theta, 1] \subset [e^{-c\underline{m}}, 1[$ for all $n \in \mathbb{N}$ and $[\theta, 1[= \cup_{n\in\mathbb{N}}[\theta_n, \theta_{n+1}[$. Moreover, for all $n \in \mathbb{N}$, $[0, A + \frac{1}{c}\ln\theta_n] \subset [0, A + \frac{1}{c}\ln\theta_{n+1}]$. Hence from Proposition 3.9, it follows that

$$a \mapsto \sum_{n\in\mathbb{N}} g_{\theta_n}(a) = \frac{\int_\theta^1 \; I_0(a,u)\; du}{S_0(a)}$$

is constant on $\{a \in [0, A+\frac{1}{c}\ln\theta],\, S_0(a) \neq 0\} = \bigcap_{n\in\mathbb{N}}\{a \in [0, A+\frac{1}{c}\ln\theta_n],\, S_0(a) \neq 0\}$. Therefore, for all $\theta \in [e^{-c\underline{m}}, 1[$ and all $a, a' \in [0, A + \frac{1}{c}\ln\theta]$ such that $S_0(a) \neq 0$,

$$\frac{\int_\theta^1 \; I_0(a,u)\; du}{S_0(a)} = \frac{\int_\theta^1 \; I_0(a',u)\; du}{S_0(a')}. \tag{B.1}$$

Differentiating (B.1) w.r.t. θ it follows that for all $\theta \in [e^{-c\underline{m}}, 1[$ and for all $a, a' \in [0, A + \frac{1}{c}\ln\theta[$ such that $S_0(a) \neq 0$,

$$\frac{I_0(a,\theta)}{S_0(a)} = \frac{I_0(a',\theta)}{S_0(a')}.$$

But since $c > \tilde{c}$ and $\tilde{T} \leqslant \underline{m}$ we have $\tilde{\theta} \in [e^{-c\underline{m}}, 1[$ and $\tilde{T} = -1/\tilde{c}\ln(\tilde{\theta}) > -1/c\ln(\tilde{\theta})$, so that $[0, A - \tilde{T}] \subset [0, A + \frac{1}{c}\ln(\tilde{\theta})[$ and therefore (H_1) is contradicted. Consequently, $c = \bar{c}$.

Appendix C. Proof of Theorem 3.5.

C.1. Proof of Proposition 3.10. By contradiction, assume that there exists $x_0 > 0$ such that $X(x_0) - \bar{X}(x_0) \neq 0$. Then, from Proposition 3.8, $t \mapsto \beta\, K(I)(t)$ and $t \mapsto \bar{\beta}\, K(\bar{I})(t)$ are constant positive functions on $\mathcal{N}$ with values

$$\alpha \neq \bar{\alpha}. \tag{C.1}$$

Therefore, Eq. (2.4) can be rewritten as

$$\int_0^A S(t,a)\; da + \frac{\alpha}{\beta} = \int_0^A \bar{S}(t,a)\; da + \frac{\bar{\alpha}}{\bar{\beta}}, \quad \forall t \in \mathcal{N}. \tag{C.2}$$

Since $\underline{n} = 0$ and $S_0 = \bar{S}_0$, letting t tend to 0 in (C.2) yields $\frac{\alpha}{\beta} = \frac{\bar{\alpha}}{\bar{\beta}}$ and

$$\int_0^A S(t,a)\; da = \int_0^A \bar{S}(t,a)\; da \quad \forall\, t \in \mathcal{N}. \tag{C.3}$$

From hypothesis (H_3), let $t' > 0$ be such that $]0, t'[\subset \mathcal{N}$. Then, on $]0, t'[\times[0, A]$, S satisfies the following PDE

$$\partial_t S + \partial_a S = -\mu\, S - \alpha\, S.$$

Integrating w.r.t. a on $[0, A]$ leads to

$$\frac{\partial}{\partial t}\int_0^A S(t,a)\; da + S(t,A) - n(t) = -(\mu+\alpha)\int_0^A S(t,a)\; da, \quad \forall t \in]0, t'[.$$

The same holds for $\bar{S}$. Using (C.3) and its derivative on $]0, t'[$ one gets

$$S(t,A) - \bar{S}(t,A) = (\bar{\alpha} - \alpha)\int_0^A S(t,a)\; da, \quad \forall t \in \mathcal{N}.$$

Letting t tend to 0, one has $\alpha = \bar{\alpha}$, which contradicts (C.1).

C.2. Proof of Proposition 3.11. We follow the same steps as in the proof of Proposition (3.10) until Eq. (C.2).

Eq. (DI) can be integrated w.r.t. a and θ so as to obtain the following integro-differential equation for $K(I)$ on $[0,T]$

$$\frac{d}{dt}K(I)(t)+\int_0^1 I(t,A,\theta)\,d\theta+\int_0^A \mathfrak{i}(t,a)\,da=-\mu\,K(I)(t)+\beta\,K(I)(t)\int_0^A S(t,a)\,da.$$

Substituting the constant value $\beta K(I)=\alpha$, one gets

$$\int_0^1 I(t,A,\theta)\;d\theta+\int_0^A \mathfrak{i}(t,a)\;da=-\frac{\mu\,\alpha}{\beta}+\alpha\int_0^A S(t,a)\;da,\quad \forall t\in\mathcal{N}.$$

The same holds for $K(\bar{I})$. Subtracting these two equations and using (2.4, 3.4) yields

$$(\bar{S}-S)(t,A)=\alpha\left(\int_0^A S(t,a)\;da-\frac{\mu}{\beta}\right)-\bar{\alpha}\left(\int_0^A \bar{S}(t,a)\;da-\frac{\mu}{\bar{\beta}}\right),\tag{C.4}$$

for $t\in\mathcal{N}$. From (2.8), for all $t\in[0,T]$,

$$\int_0^A S(t,a)da=\int_0^{\min(t,A)} n(t-a)e^{-\mu a-\beta\int_{t-a}^t K(I)(u)du}da+\int_{\min(t,A)}^A M(t,a)e^{-\mu t-\beta\int_0^t K(I)(u)du}\;da,$$

and, therefore,

$$\begin{aligned}\int_0^A S(t,a)da=&\int_0^{\min(t,A)} n(t-a)e^{-\mu a-\beta\int_{t-a}^t K(I)(u)du}da\\&+\int_{\min(t,A)}^A S_0(a-\min(t,A))e^{-\mu t-\beta\int_0^t K(I)(u)du}da.\end{aligned}$$

From hypothesis (H_3), let $t<A$ be such that $]\underline{n},t]\subset\mathcal{N}$. Then

$$\begin{aligned}\int_0^t n(t-a)e^{-\mu a-\beta\int_{t-a}^t K(I)(u)du}da&=\int_0^{t-\underline{n}} n(t-a)\,e^{-(\mu+\alpha)a}\;da,\\&=\int_{\underline{n}}^t n(u)e^{-(\mu+\alpha)(t-u)}du,\\ S(t,A)&=S_0(A-t)G(t),\\ \int_t^A S_0(a-t)\,e^{-\mu t+\beta\int_0^t K(I)(u)du}\;da&=\left(\int_0^{A-t} S_0(u)\;du\right)G(t),\end{aligned}$$

where $G(t)=e^{-\left(\mu t+\beta\int_0^t K(I)(u)du\right)}$. $\bar{G}(t)$ is similarly defined for $\bar{p}$. Eq. (C.4) rewrites, for all t such that $]\underline{n},t]\subset\mathcal{N}$,

$$\begin{aligned}S_0(A-t)(G-\bar{G})(t)=&\,\alpha\int_{\underline{n}}^t n(u)e^{-(\mu+\alpha)(t-u)}du-\bar{\alpha}\int_{\underline{n}}^t n(u)e^{-(\mu+\bar{\alpha})(t-u)}du\\&+(\alpha G(t)-\bar{\alpha}\bar{G}(t))\int_0^{A-\min(t,A)} S_0(u)du+\mu\left(\frac{\alpha}{\beta}-\frac{\bar{\alpha}}{\bar{\beta}}\right).\end{aligned}\tag{C.5}$$

Thanks to hypothesis (H_2), (C.5) is valid on a neighbourhood of $[t_1, t_2]$. Moreover, the right member of (C.5) is a continuous function of t and so is $t \mapsto (G - \bar{G})(t)$. Hence the discontinuity of $t \mapsto S_0(A - t)$ at t_1 and t_2 implies that

$$G(t_1) = \bar{G}(t_1), \quad G(t_2) = \bar{G}(t_2).$$

Since $[t_1, t_2] \subset \mathcal{N}$, $G(t) = G(t_1)e^{-(\mu+\alpha)(t-t_1)}$ and $\bar{G}(t) = \bar{G}(t_1)e^{-(\mu+\alpha)(t-t_1)}$ for all $t \in [t_1, t_2]$, so $e^{-(\mu+\alpha)(t_2-t_1)} = e^{-(\mu+\bar{\alpha})(t_2-t_1)}$ and consequently $\alpha = \bar{\alpha}$. This contradicts Eq. (C.1).

C.3. Proof of Proposition 3.12. We first prove the following lemma.

LEMMA C.1. *Assume that* (3.4) *is satisfied and* $X = \bar{X}$. *Then*

$$\beta\, K(I)(t) = \bar{\beta}\, K(\bar{I})(t), \quad \forall t \in \mathcal{N}.$$

Proof. Substituting $X = \bar{X}$ in Eq. (A.1), one has for all $(\xi, a) \in \mathcal{N} \times [0, A]$

$$\partial_a y(\xi + a, a) = \beta K(I)(\xi) \big(X(a) - y(\xi + a, a)\big),$$
$$\partial_a y(\xi + a, a) = \bar{\beta} K(\bar{I})(\xi) \big(X(a) - y(\xi + a, a)\big).$$

Term to term subtraction yields

$$\big(\beta K(I)(\xi) - \bar{\beta} K(\bar{I})(\xi)\big) \big(X(a) - y(\xi + a, a)\big) = 0. \tag{C.6}$$

By contradiction, assume that there exists $\xi_0 \in \mathcal{N}$ such that $\beta K(I)(\xi_0) \neq \bar{\beta} K(\bar{I})(\xi_0)$. Since n is piecewise continuous and $\xi \mapsto (\beta K(I) - \bar{\beta} K(\bar{I}))(\xi)$ is continuous, there exists an interval $\mathcal{V}(\xi_0)$ included in $\mathcal{N}$, containing ξ_0, not reduced to a singleton set, such that $(\beta K(I) - \bar{\beta} K(\bar{I}))(\xi) \neq 0$ for all $\xi \in \mathcal{V}(\xi_0)$. Therefore, (C.6) reduces to

$$X(a) = y(\xi + a, a), \quad \forall (\xi, a) \in \mathcal{V}(\xi_0) \times [0, A]. \tag{C.7}$$

This implies that $\partial_t y(\xi + a, a) = 0$ for $(\xi, a) \in \mathcal{V}(\xi_0) \times [0, A]$. Consequently, equation (A.1) becomes $\partial_a y(\xi + a, a) = 0$ on $\mathcal{V}(\xi_0)$ and differentiating (C.7) w.r.t a yields

$$X'(a) = \partial_t y(\xi + a, a) + \partial_a y(\xi + a, a) = 0, \quad \forall\, a \in [0, A].$$

It follows that $X \equiv 0$ on $[0, A]$. Then Remark 3 implies that X is null on $\mathbb{R}^+$, which contradicts its definition as a p.d.f. ◻

We now proceed with the proof of Proposition 3.12. If $\underline{n} = 0$, then 0 is in the closure of $\mathcal{N}$ and $\beta = \bar{\beta}$ from Lemma C.1.

Assume now that $\underline{n} > 0$ and let $t \in [0, \underline{n}]$. Then, from Eq. (2.4)

$$K(I)(t) - K(\bar{I})(t) = \int_0^A \bar{S}(t, a)\, da - \int_0^A S(t, a)\, da, \; \forall t \in [0, \underline{n}]. \tag{C.8}$$

Since for $a \in [0, t]$, $n(t - a) = 0$, multiplying (C.8) by β and using (2.6), one gets

$$\begin{aligned} \beta(K(I) - K(\bar{I}))(t) = \beta e^{-\mu t} &\left(\int_0^{A - \min(t, A)} S_0(a) da \right) \\ &\times \left(e^{-\bar{\beta} \int_0^t K(\bar{I})(\xi) d\xi} - e^{-\beta \int_0^t K(I)(\xi) d\xi} \right). \end{aligned} \tag{C.9}$$

Consider the continuous functions $f:\mathbb{R}^2 \to]0,1]$ and $g:[0,T]\to]0,1]$ defined by

$$f:(x,y)\mapsto \begin{cases} -\frac{e^{-x}-e^{-y}}{x-y} \text{ if } x\neq y, \\ e^{-x} \text{ if } x=y, \end{cases} \tag{C.10}$$

$$g:t\mapsto \exp\left(-\int_0^t \beta e^{-\mu s}\left(\int_0^{A-\min(s,A)} S_0(a)da\right) f\left(\int_0^s \bar{\beta}K(\bar{I})(\xi)d\xi, \int_0^s \beta K(I)(\xi)d\xi\right) ds\right).$$

Eq. (C.9) can be rewritten as

$$\beta\big(K(I)(t)-K(\bar{I})(t)\big) = \frac{g'(t)}{g(t)}\left(\int_0^t \bar{\beta}K(\bar{I})(\xi)d\xi - \int_0^t \beta K(I)(\xi)d\xi\right). \tag{C.11}$$

We end the proof by contradiction. Let us assume that $\beta > \bar{\beta}$. Then we have

$$\beta\big(K(I)-K(\bar{I}))(t)\big) \leqslant \beta(K(I)-\bar{\beta}K(\bar{I}))(t),$$

and, consequently to (C.11), we get

$$-\frac{g'(t)}{g(t)}\left(\int_0^t \beta K(I)(\xi)d\xi - \int_0^t \bar{\beta}K(\bar{I})(\xi)d\xi\right) \leqslant \beta K(I)(t) - \bar{\beta}K(\bar{I})(t), \tag{C.12}$$

which implies that $t\mapsto g(t)\int_0^t \big(\beta K(I)-\bar{\beta}K(\bar{I})\big)(\xi)d\xi$ is increasing on $[0,\underline{n}]$.

At $t=0$, one has $(\beta K(I)-\bar{\beta}K(\bar{I}))(0) = (\beta-\bar{\beta})K(I_0)>0$ and, by a continuity argument, there exists $0<\varepsilon_0<\underline{n}$ such that $\beta K(I)-\bar{\beta}K(\bar{I})$ is positive on $[0,\varepsilon_0]$. Since $0<g<1$, for all $t\in[\varepsilon_0,\underline{n}]$

$$\int_0^t \big(\beta K(I)-\bar{\beta}K(\bar{I})\big)(\xi)\ d\xi \geqslant g(t)\int_0^t \big(\beta K(I)-\bar{\beta}K(\bar{I})\big)(\xi)\ d\xi \geqslant B_0,$$

where $B_0 = g(\varepsilon_0)\int_0^{\varepsilon_0}\big(\beta K(I)(\xi)-\bar{\beta}K(\bar{I})(\xi)\big)d\xi > 0$. Using this inequality and the expression of $-\frac{g'(t)}{g(t)}$ in (C.12), we deduce that

$$\beta e^{-\mu t}\left(\int_0^{A-\min(t,A)} S_0(a)da\right) f\left(\bar{\beta}\int_0^t K(\bar{I})(\xi)d\xi, \beta\int_0^t K(I)(\xi)d\xi\right) B_0 \\ \leqslant \beta K(I)(t)-\bar{\beta}K(\bar{I})(t),\ \forall t\in[\varepsilon_0,\underline{n}].$$

Evaluating the above expression at $t=\underline{n}$ yields a contradiction with Lemma C.1.

C.4. Proof of Corollary 3.13. From Lemma C.1 and Proposition 3.12 we deduce that $K(I)=K(\bar{I})$ on $\mathcal{N}$. We first show that $K(I)(t)=K(\bar{I})(t)\quad \forall t\in[0,\underline{n}]$. If $\underline{n}=0$, this is obviously true. If $\underline{n}>0$, Eq. (C.11) rewrites

$$K(I)(t)-K(\bar{I})(t) = \frac{g'(t)}{g(t)}\left(\int_0^t K(I)(\xi)d\xi - \int_0^t \bar{K}(\bar{I})(\xi)d\xi\right),\quad \forall t\in[0,\underline{n}],$$

and, therefore,

$$\int_0^t K(I)(\xi)d\xi - \int_0^t \bar{K}(\bar{I})(\xi)d\xi = \big(K(I)(0)-K(\bar{I})(0)\big)\ e^{\int_0^t \frac{g'}{g}(s)ds} = 0,\ \forall t\in[0,\underline{n}].$$

After differentiating the above equation w.r.t. t, one gets $K(I) = K(\bar{I})$ on $[0, \underline{n}]$.

Consider $E = \{t \in [0,T] / \forall s \in [0,t],\ K(I)(s) = K(\bar{I})(s)\}$. We now prove that $E = [0,T]$. Since $[0, \underline{n}] \subset E$ and $K(I)$ and $K(\bar{I})$ are continuous on $[0,T]$, E is a nonempty closed subset of $[0,T]$. Let $s \in E$. Using hypothesis (H_3), we can choose $\varepsilon > 0$ small enough so that either $n > 0$ on $]s, s+\varepsilon[\cap[0,T]$ or n is identically equal to 0 on $]s, s+\varepsilon[\cap[0,T]$. We show that $K(I) = K(\bar{I})$ on $[s, s+\varepsilon[\cap[0,T]$. In the first case, since $K(I) = K(\bar{I})$ on $\mathcal{N}$, the desired result is obviously true. In the second case, after performing the change of variables $b = t - a$, we have for $t \in]s, s+\varepsilon[\cap[0,T]$

$$(K(I) - K(\bar{I}))(t) = \int_0^t \left(n(b)\, e^{-\mu(t-b)} f\Big(\int_b^t \beta K(\bar{I})(\xi)d\xi, \int_b^t \beta K(I)(\xi)d\xi\Big) \times \int_b^t \beta\,(K(I) - K(\bar{I}))(\xi)d\xi\right)db$$
$$+ \left(\int_0^{A-\min(t,A)} S_0(a)da\right) f\left(\int_0^t \beta K(\bar{I})(\xi)d\xi, \int_0^t \beta K(I)(\xi)d\xi\right) \int_0^t \beta\,(K(I) - K(\bar{I}))(\xi)d\xi,$$

where f is defined in (C.10). Since $n(b) = 0$ for $b < \underline{n}$ and $K(I) = K(\bar{I})$ on $[0,s]$, we get for $t \in]s, s+\varepsilon[\cap[0,T]$,

$$(K(I) - K(\bar{I}))(t) = \int_{\underline{n}}^s \left(n(b)\, e^{-\mu(t-b)} f\Big(\int_b^t \beta K(\bar{I})(\xi)d\xi, \int_b^t \beta K(I)(\xi)d\xi\Big) \times \int_s^t \beta\,(K(I) - K(\bar{I}))(\xi)d\xi\right)db$$
$$+ \left(\int_0^{A-\min(t,A)} S_0(a)da\right) f\left(\int_0^t \beta K(\bar{I})(\xi)d\xi, \int_0^t \beta K(I)(\xi)d\xi\right) \int_s^t \beta\,(K(I) - K(\bar{I}))(\xi)d\xi,$$

and finally for $t \in]s, s+\varepsilon[\cap[0,T]$,

$$K(I)(t) - K(\bar{I})(t) = H(t) \int_s^t \big(K(I)(\xi) - K(\bar{I})(\xi)\big)\ d\xi, \tag{C.13}$$

where

$$H : t \mapsto \beta \left(\int_{\underline{n}}^s n(b)\, e^{-\mu(t-b)} f\left(\int_b^t \beta K(\bar{I})(\xi)d\xi, \int_b^t \beta K(I)(\xi)d\xi\right) db\right.$$
$$\left. + \left(\int_0^{A-\min(t,A)} S_0(a)da\right) f\left(\int_0^t \beta K(\bar{I})(\xi)d\xi, \int_0^t \beta K(I)(\xi)d\xi\right)\right).$$

Since $(K(I) - K(\bar{I}))(s) = 0$, by a standard Gronwall argument, $K(I) = K(\bar{I})$ on $[s, s+\varepsilon[\cap[0,T]$. Therefore E is also an open subset of $[0,T]$ and $E = [0,T]$.

C.5. Proof of the identifiability of $\mathcal{M}^u$ on $\mathbf{Q}^\star$. We first prove that $c = \bar{c}$. From Proposition 3.11 and Corollary 3.13 we have $X = \bar{X}$, $\beta = \bar{\beta}$ and $K(I) = K(\bar{I})$. By using (2.11) and (3.4) implies that for $(t,a) \in [0,T] \times [0,A]$, $a \geqslant t$

$$cI_0(a-t, e^{-ct})\, e^{-(\mu+c)t} = \bar{c}I_0(a-t, e^{-\bar{c}t})\, e^{-(\mu+\bar{c})t}.$$

Performing the coordinate change $(t,a) \to (t, u = a - t)$ and dividing each member by $e^{-\mu t}$, this equality rewrites

$$cI_0(u, e^{-ct})\, e^{-ct} = \bar{c}I_0(u, e^{-\bar{c}t})\, e^{-\bar{c}t},\ \text{for } (t,u) \in [0,T] \times [0,A].$$

Note that for $u > A - t$, both members are zero in the above equation. Integrating w.r.t. t with the change of variable $v = e^{-ct}$, one gets $\int_{e^{-\bar{c}t}}^{e^{-ct}} I_0(u,v)\,dv = 0$. Denoting $\theta = e^{-\bar{c}t}$, one has

$$\int_{\theta}^{\theta^{\bar{c}/c}} I_0(u,v)\,dv = 0,\ \forall\,\theta \in]e^{-\bar{c}T}, 1[,\ \forall\, u \in [0,A]. \tag{C.14}$$

Moreover, from the definition of $c^\star$ and the piecewise continuity of I_0, we deduce the existence of a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ and a sequence of open intervals $\{V_n\}_{n\in\mathbb{N}}$ verifying:

$$\forall\, n \in \mathbb{N}, \theta_n \in V_n \subset]0, e^{-c^\star \underline{m}}[, \tag{C.15}$$

$$\theta_n \xrightarrow[n\to+\infty]{} e^{-c^\star \underline{m}}, \tag{C.16}$$

$$\forall\, n \in \mathbb{N},\ \exists\, a_n \in]0, A[,\ \forall\, \theta \in V_n,\ I_0(a_n, \theta) > 0. \tag{C.17}$$

Since $\bar{c} > c^\star$ and $T \geq \underline{m}$, $]0, e^{-c^\star \underline{m}}[\cap]e^{-\bar{c}T}, 1[$ is nonempty and from (C.15, C.16), one can choose n_0 big enough such that $V_{n_0}\cap]e^{-\bar{c}T}, 1[$ is nonempty. From (C.14, C.17), we deduce that

$$\int_{\theta}^{\theta^{\bar{c}/c}} I_0(a_{n_0}, v)\,dv = 0,\ I_0(a_{n_0}, \theta) > 0,\ \forall\ \theta \in V_{n_0}\cap]e^{-\bar{c}T}, 1[,$$

which implies $c = \bar{c}$. It easily follows, since $X = \bar{X}$ on $\mathbb{R}^+$, that $F_\Theta = F_{\bar{\Theta}}$ and $\Theta = \bar{\Theta}$ on $[0,1]$, which ends the proof of the identifiability of $\mathcal{M}^u$ on $\mathbf{Q}^\star$.

Appendix D. Proof of Theorem 3.6.

Let us consider the vectors of parameters

$$p = \begin{pmatrix} c \\ \beta \\ \Theta \end{pmatrix} \in \mathbf{R}^\star,\ \text{ and } \bar{p} = \begin{pmatrix} \bar{c} \\ \beta \\ \bar{\Theta} \end{pmatrix} \in \mathbf{R}^\star,$$

where $\beta > 0$, $0 < \bar{c} < c < c^\star$, $\Theta \in \mathcal{A}_0(0,1)$ and $\bar{\Theta}$ is defined for $\theta \in [0,1]$ by

$$\bar{\Theta}(\theta) = \frac{c}{\bar{c}}\,\theta^{\frac{c-\bar{c}}{\bar{c}}}\,\Theta\left(\theta^{\frac{c}{\bar{c}}}\right).$$

We easily check that $\bar{\Theta}$ is an element of $\mathcal{A}_0(0,1)$ verifying

$$\bar{c}\, e^{\bar{c}(s-t)}\, \bar{\Theta}\left(e^{\bar{c}(s-t)}\right) = c\, e^{c(s-t)}\, \Theta\left(e^{c(s-t)}\right),\ \forall\ (s,t) \in [0,T]^2 \text{ with } s \leqslant t. \tag{D.1}$$

Let us prove that $K(I) = K(\bar{I})$ on $[0,T]$. From Corollary 2.3, $K(I)$ is the unique fixed point of L verifying (2.10) for all $t \in [0,T]$. We now check that $K(\bar{I})$ is also a fixed point of L to complete the proof.

Assume first that $t \leqslant T \leqslant A$, then $\underline{m} = T$ and, since $0 < c < \bar{c} < c^\star$, $e^{-ct} > e^{-c^\star \underline{m}}$ for $t \leqslant T$, and similarly for $\bar{c}$. According to Remark 1,

$$\int_0^{A-\min(t,A)} \int_0^{e^{-ct}} I_0(u,v)\,dv\,du = \int_0^{A-t} \int_0^{e^{-c^\star \underline{m}}} I_0(u,v)\,dv\,du = \int_0^{A-t} \int_0^{e^{-\bar{c}t}} I_0(u,v)\,dv\,du. \tag{D.2}$$

Then if $t \leqslant A < T$, then $\underline{m} = A$ and $e^{-ct} > e^{-c^\star \underline{m}}$, and similarly for $\bar{c}$ and Eq. (D.2) is still true.

Finally, if $A < t \leqslant T$, we also have

$$\int_0^{A-\min(t,A)} \int_0^{e^{-ct}} I_0(u,v)dvdu = 0 = \int_0^{A-\min(t,A)} \int_0^{e^{-\bar{c}t}} I_0(u,v)dvdu,$$

which shows that in all cases, Eq. (D.2) holds on $[0,T]$. Therefore, from Eq. (D.1) and Eq. (D.2), it follows that $K(\bar{I})$ is also a fixed point of L.

From Remark 1 and Eq. (2.11), the incidences when $t \leqslant a$ for p becomes

$$\mathfrak{i}(t,a) = S_0(a-t)e^{-\mu t} \int_0^t c\, e^{c(s-t)}\, \Theta\left(e^{c(s-t)}\right) \beta K(I)(s) e^{-\beta \int_0^s K(I)(u)du}\, ds,$$

and similarly for $\bar{p}$. Since $K(I) = K(\bar{I})$ on $[0,T]$ and (D.1) is satisfied, we obtain $\mathfrak{i}(t,a) = \bar{\mathfrak{i}}(t,a)$ when $t \leqslant a$. In the same way we can easily check that $\mathfrak{i}(t,a) = \bar{\mathfrak{i}}(t,a)$ when $t > a$.

We now prove that the populations $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are equal. From the differentiation of (2.4) w.r.t t and a and using (DS) we deduce that

$$\frac{\partial \mathfrak{p}}{\partial t}(t,a) + \frac{\partial \mathfrak{p}}{\partial a}(t,a) = -\mu\, \mathfrak{p}(t,a) - \mathfrak{i}(t,a),$$

Subtracting the resulting equation with identical computation for $\bar{\mathfrak{p}}$ we deduce that $\mathfrak{p} - \bar{\mathfrak{p}}$ satisfies the following PDE,

$$\frac{\partial(\mathfrak{p} - \bar{\mathfrak{p}})}{\partial t}(t,a) + \frac{\partial(\mathfrak{p} - \bar{\mathfrak{p}})}{\partial a}(t,a) = -\mu\, (\mathfrak{p} - \bar{\mathfrak{p}})(t,a).$$

with initial and boundary condition $(\mathfrak{p} - \bar{\mathfrak{p}})(0,a) = 0$ and $(\mathfrak{p} - \bar{\mathfrak{p}})(t,0) = 0$. Consequently, $\mathfrak{p} = \bar{\mathfrak{p}}$ on $[0,T]$, which ends the proof of the non identifiability of $\mathcal{M}^u$ on $\mathbf{R}^\star$.

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