Path planning on compact Lie groups using an homotopy method

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Abstract

In this paper, we address the issue of solving the motion planning problems (MPP for short) for a class of left-invariant control systems Σ whose state spaces are semi-simple compact Lie groups. The sub-Riemannian metrics induced by the dynamics of Σ admit nontrivial abnormal extremals. This fact a priori represents an obstruction for the procedure we use to tackle the MPP, which consists of an homotopy (or continuation) method. We are however able to provide complete answers for the MPP.

Keywords Homogeneous control systems, motion planning, sub-Riemannian metric, homotopy method.

1 Introduction

In [15], Sussmann proposed a homotopy method (also called continuation method) for non-holonomic path-finding problems. This method was successfully applied to driftless control-affine systems subject to the Strong Bracket Generating Condition (SBGC) (cf. [14] and [2]). The SBGC was taylored by Strichartz in [13] in order to guarantee that nontrivial abnormal extremals do not exist. Let $\dot{x} = \sum_{i=1}^{m} u_i f_i(x)$ be a driftless control-affine system where the state x belongs to a smooth Riemannian manifold M^n , the input $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ and $f_1, \cdots, f_m$ are smooth vector fields on M^n. To such a control system is associated a sub-Riemannian structure (cf. for instance [8]). Thanks to the Pontryagin maximum principle ([11]), every sub-Riemannian minimizer is an extremal, either normal or abnormal (for more details see [8] and [9]). As shown in [2, 3, 14], the abnormal extremals are the obstacles to a successful application of the continuation method to motion planning problems. In this paper, we investigate a class of left-invariant control systems Σ whose state spaces are semi-simple compact Lie groups and where nontrivial extremals occur. The distribution $\mathcal{D}$ defining the control system is given next: consider the foliation of the group by left-translates of its maximal torus. Then $\mathcal{D}$ is the (non-integrable) distribution which is orthogonal to the foliation with respect to a bi-invariant metric on the group. This class of systems was first considered by Lerman ([7]) in the context of symplectic geometry and later on by Montgomery ([10]). The last author was addressing the regularity issue for abnormal extremals and their corresponding singular curves. Our main result consists of showing for the class of systems Σ that the continuation method still completely solves the MPP.

Let us now recall the definitions of the motion planning problem and the continuation method. A control system S is a 4-tuple (M^n, U, D, H) where the Riemannian manifold M^n is the state space, the closed set $U \subset \mathbb{R}^m$ is the control space, H is the set of admissible controls and the distribution D is a C^∞, rank-m assignment $p \mapsto D(p)$ and such that the admissible trajectories of the system are parallel to D. (For more general definitions of a control system see [6].) For simplicity, we assume that $U = \mathbb{R}^m$, the distribution D admits a global basis of smooth vector fields $(f_i)_{1 \leq i \leq m}$ and $H = L^2([0,1], \mathbb{R}^m)$. The admissible trajectories of the system are then the absolutely continuous curves $c : [0,1] \to M$ for which there exists $u \in H$, such that for almost all $t \in [0,1]$, $\dot{c}(t) = \sum_{i=1}^m u_i(t) f_i(c(t))$. In addition, we assume that:

(NEC) for every $p \in M^n$ and every $u \in H$, there exists a trajectory $c : [0,1] \to M^n$ of S such that $c(0) = p$. In this case, such a trajectory is unique and we use $c_{p,u}$ to denote it;

(CC) S is completely controllable i.e. for every $p, q \in M^n$ there exists $u \in H$ such that $c_{p,u}(1) = q$. Note that such a $c_{p,u}$ is not necessarily unique for $p, q \in M^n$ given.

The MPP is the problem of finding a procedure that, for every $p, q \in M^n$, produces effectively a control $u_{p,q} \in H$ steering p to q. The continuation method (CM), (or deformation, homotopy method -cf. [1]-) is well known for solving nonlinear equations of the form $F(x) = y$. The advantages of the CM are twofold: theoretically it applies to a wide range of situations and numerically it is robust and easy to implement (cf. [1]). Let $F : X \to Y$ a (let say) smooth map. The CM proceeds by starting from a value x_0 of x and its corresponding image $y_0 = F(x_0)$, then by joining y_0 to the given y by a continuous path π and by trying to lift π to a path Π so that $F \circ \Pi = \pi$. To construct such a path Π, simply defined implicitely, we may differentiate $F(\Pi(s)) = \pi(s)$ to get $DF(\dot{\Pi}(s)) = \dot{\pi}(s)$, which is satisfied if we can solve

$$\dot{\Pi}(s) = P(\Pi(s))\dot{\pi}(s), \tag{1}$$

where $P(x)$ is a right inverse of $DF(x)$. Equation (1) is an ordinary differential equation (ODE) in X, called the Wazewski equation [17]. The link between the CM and the MPP is established by considering an end-point map ϕ_p. For $p \in M^n$ let $\phi_p : H \to M^n$ be the map given by $\phi_p(v) \overset{def}{=} c_{p,v}(1)$. Thanks to (NEC) and (CC), ϕ_p is well-defined for every $p \in M^n$ and is surjective. The MPP can then be reformulated as follows: for fixed $p \in M^n$, find a right inverse of ϕ_p, i.e. a map $i_p : M^n \to H$ such that $\phi_p \circ i_p =$ identity. It turns out that such a map exists in a neighborhood of any point $u \in H$ where ϕ_p is a submersion i.e. where $D\phi_p(u)$ is surjective. This follows easily from the constant rank theorem and a standard application of the implicit function theorem (cf. [2] for instance). Then, the delicate point consists of dealing with the singular points of ϕ_p, i.e. the controls $v \in H$ where rank $D\phi_p(v) < n$. These controls are exactly the abnormal extremals at p mentioned earlier and the singular curves are the trajectories of D starting at p corresponding to the abnormal extremals.

For simplicity, M^n is denoted M. By considering equation (1), the application of the CM to the MPP proceeds in two stages. The first one is identifying S_p, the set of abnormal extremals, and $\phi_p(S_p)$, the set of singular curves. The second stage consists of lifting paths $\pi : [0,1] \to M$ avoiding $\phi_p(S_p)$ to paths $\Pi : [0,1] \to H$ such that for every $s \in [0,1]$

$$\phi_p\big(\Pi(s)\big) = \pi(s). \tag{2}$$

We differentiate (2); then recalling that $D\phi_p\big(\Pi(s)$ has full rank, we take Π such that

$$\frac{d\Pi}{ds}(s) = P\big(\Pi(s)\big) \cdot \frac{d\pi}{ds}(s), \tag{3}$$

where $P(v)$ is a right inverse of $D\phi_p(v)$ when $v \in H/S_p$. (For instance, we can choose $P(v)$ to be the Moore-Penrose pseudoinverse of $D\phi_p(v)$.) We refer to the Wazewski equation defined in (3) as the Path Lifting Equation

(PLE) and it is an ODE in H. Therefore, two issues have to be solved in order to efficiently apply the CM to the MPP and each of them corresponds to each stage defined previously:

(a) Nondegeneracy: the path π has to be chosen so that, for every $s \in [0,1]$, $\pi(s) \notin \phi_p(S_p)$ and then $D\phi_p\big(\Pi(s)$ has always full rank;

(b) Nonexplosion: to solve (2), the PLE defined in (3) must have a global solution on $[0,1]$.

Local existence and uniqueness of the solution of the PLE hold as soon as ϕ_p is of class C^2. For more details and complete justifications regarding the CM, we refer to [2]. For the class of control systems under consideration, we exploit the results of [10] where S_p and $\phi_p(S_p)$ are characterized. Then the core of the paper will be resolving issue (b), which amounts to proving some estimates on line integrals along trajectories.

2 Notations and Definitions

Throughout this paragraph, the terminology concerning Lie groups and Lie algebras is borrowed from Humphreys [5] and Varadarajan [16]. Vector fields and maps act on the right of the elements of the group G, its tangent bundle TG and its tangent cobundle T^*G. Let G be a connected compact real semi-simple Lie group (c.c.r.s-s. for short). Under the adjoint representation, $G \subset GL_n(\mathbb{R})$, where $n = dim(G)$. We use e to denote Id_n. Let $\mathcal{G} \stackrel{def}{=} T_eG$, the tangent space to G at e. Then $\mathcal{G}$ has a structure of real semi-simple Lie algebra and is a vector subspace of $gl_n(\mathbb{R})$. Let $d \geq 1$ and $l \geq 1$ be respectively the dimension and the rank of $\mathcal{G}$.

Let $ad : \mathcal{G} \to L(\mathcal{G})$ be the adjoint operator, i.e. the application that associates to every $z \in \mathcal{G}$, the endomorphism ad_z of $\mathcal{G}$ given by $ad_z(x) = [z, x]$, for $x \in \mathcal{G}$. An element $z \in \mathcal{G}$ is said to be semi-simple if ad_z is diagonalizable. A toral subalgebra of $\mathcal{G}$ is a nonzero subalgebra of $\mathcal{G}$ consisting of semi-simple elements. Let $\mathcal{H}$ be a Cartan Subalgebra (CSA) of $\mathcal{G}$, i.e. a maximal toral subalgebra of $\mathcal{G}$. Since $\mathcal{H}$ is abelian, $ad_{\mathcal{G}}H$ is a commuting family of semi-simple endomorphisms of $\mathcal{G}$, which are then simultaneously diagonalizable. In other words, $\mathcal{G}$ is the direct sum of the subspaces

$$L_\alpha = \{x \in \mathcal{G}, [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\},$$

where α ranges over $\mathcal{H}^*$. A nonzero $\alpha \in \mathcal{H}^*$ for which L_α is not zero is said to be a root; the set of such

α's is denoted by Φ and called the root system relative to $\mathcal{H}$. We have then the corresponding root space decomposition $\mathcal{H}$ is

$$\mathcal{G} = \mathcal{H} \oplus \Big(\bigoplus\nolimits_{\alpha\in\Phi} L_\alpha\Big), \tag{4}$$

and, for $\alpha \in \Phi$, the root space L_α is equal to $\mathbb{R}e_\alpha$ where $e_\alpha \in \mathcal{G}$. Furthermore, $(e_\alpha, e_{-\alpha})$ generate a Lie subalgebra of $\mathcal{G}$ isomorphic to $so(3)$. We have,

$$\forall h \in \mathcal{H},\ \ [h, e_\alpha] = \alpha(h)e_{-\alpha} \text{ and } [h, e_{-\alpha}] = -\alpha(h)e_\alpha\,.$$

Let Δ be a *basis* of Φ, i.e. a subset of Φ of minimal cardinality among all the subsets of Φ such that each $\beta \in \Phi$ can be written as $\beta = \sum k_\alpha \alpha$ ($\alpha \in \Delta$), with integral coefficients k_α all nonnegative or all non positive. The roots in Δ are then called *simple*. The collections of positive and negative roots (relative to Δ) are usually denoted Φ^+ and Φ^-. We have that $\sharp\Delta = l$ and $\sharp\Phi^+ = \sharp\Phi^- = \frac{d-l}{2}$. (Here and henceforth, $\sharp A$ denotes the cardinality of the finite set A.) We define a left invariant distribution $\mathcal{D}$ by

$$\mathcal{D}(g) \overset{def}{=} g\Big(\bigoplus\nolimits_{\alpha\in\Phi} L_\alpha\Big),\ \text{for } g \in G. \tag{5}$$

Therefore, $\mathcal{G} = \mathcal{H} \oplus \mathcal{D}(e)$. Since the Killing form is negative definite on $\mathcal{G}$, we have a bi-invariant metric $< \cdot, \cdot >$, called the *normal metric*, on the tangent bundle such that $(e_\alpha)_{\alpha\in\Phi}$ is an orthonormal basis of $\mathcal{D}(e)$ and the previous direct sum is orthogonal with respect to the normal metric. Furthermore, this metric induces a distance function d_G on G that allows us to identify $\mathcal{G}^*$ and $\mathcal{G}$. Then, $\forall z \in \mathcal{G}$, $\forall x \in G$ $\|xz\| = \|z\| \overset{def}{=} < z, z >^{1/2}$.

Let $(h_i)_{i=1,\cdots,l}$ be an orthonormal basis of $\mathcal{H}$ and $A \subset B$ be two sets of finite cardinality in a real vector space. We say that A *integrally spans* B, if every element of B can be written as a linear combination with integer coefficients of elements of A. We say that a subset S of Φ is *symmetric* if $\forall \alpha \in S, -\alpha \in S$. We write $S = S^+ \cup S^-$, where $S^+ \subset \Phi^+$ and $S^- \subset \Phi^-$. Let S be a symmetric subset Φ and S' be the set of roots that belong to $\Phi \setminus S$ and that are linear combinations with integer coefficients of elements of S. Define

$$\bar{S} = S \cup S' \text{ and } \bar{S}^c = \Phi \setminus \bar{S}\,.$$

Clearly $\Delta = \Delta_S \cup \Delta_{S^c}$, with $\Delta_S \subset \bar{S}^+$ and $\Delta_{S^c} \subset \bar{S}^{c,+}$. The set $\bar{S}$ corresponds to the root system of some semisimple Lie subalgebra of $\mathcal{G}$ with Δ_S as a basis of $\bar{S}$. Therefore, every root α in $\Phi - \bar{S}$ can be written as

$$\alpha = \sum_{\beta\in\Delta_S} \lambda_\beta \beta + \sum_{\beta'\in\Delta_{S^c}} \lambda_{\beta'} \beta', \tag{6}$$

where the $\lambda_\beta, \lambda_{\beta'}$'s are integers and the $(\lambda_{\beta'})_{\beta' \in \Delta_{S^c}}$ are not all zero.

The purpose of this paper is to solve the MPP for the driftless control affine system $\Sigma = \Big(G, \mathbb{R}^{d-l}, E, H\Big)$ defined by

$$\dot{x} = x\Big(\sum_{\alpha \in \Phi} u_\alpha e_\alpha\Big), \tag{7}$$

where $E = (e_\alpha)_{\alpha \in \Phi}$ and $H = L^2\Big((0,1), \mathbb{R}^{d-l}\Big)$. Let ϕ be the end-point map corresponding to e. In order to describe the singular set S associated to ϕ, the following definiton is needed.

Definition 1 *A subset S of Φ is a good set if it is the root system of some semisimple Lie subalgebra $\mathcal{S}_S$ of $\mathcal{G}$.*

From this definiton, we will introduce several objects required in the sequel. If S is a good set, let $S^c \overset{def}{=} \Phi - S$. We also consider $\mathcal{S}_S$ the Lie subalgebra of $\mathcal{G}$ with root system S and we define $\mathcal{W}_S$ by

$$\mathcal{W}_S \overset{def}{=} \mathcal{H} + \mathcal{S}_S. \tag{8}$$

The previous equality only defines $\mathcal{W}_S$ as a sum of two vector spaces but it is easy to see that $\mathcal{W}_S$ is in fact a semisimple Lie subalgebra of $\mathcal{G}$. Note that

$$\mathcal{G} = \mathcal{W}_S \oplus \mathcal{W}_S^\perp \text{ and } \Big[\mathcal{W}_S, \mathcal{W}_S^\perp\Big] \subset \mathcal{W}_S^\perp, \tag{9}$$

where $\mathcal{W}_S^\perp \overset{def}{=} \bigoplus_{\alpha \in S^{c,+}} L_\alpha$. In (9), we have a direct sum of two vector spaces since in general $\mathcal{W}_S^\perp$ does not have the structure of a Lie subalgebra $\mathcal{S}_S$ of $\mathcal{G}$. There exist 2^l distinct such $\mathcal{S}_S$'s and hence $\mathcal{W}_S$'s. We can then write $(\mathcal{S}_k)_{k=1,\cdots,2^l}$ and $(\mathcal{W}_k)_{k=1,\cdots,2^l}$. Set $\mathcal{W}_1 \overset{def}{=} \mathcal{G}$ and note that

$$\forall k = 2, \cdots, 2^l \ \dim\mathcal{S}_k \leq \ \dim\mathcal{G} - 3 \text{ and } \ \dim\mathcal{W}_k \leq \ \dim\mathcal{G} - 2, \tag{10}$$

since removing a simple root from a basis of Φ implies that at least an $so(3)$-Lie subalgebra is removed from $\mathcal{G}$. For $y \in \mathcal{H}$, $\mathcal{C}_y \overset{def}{=} \{z \in \mathcal{G},\ [y,z] = 0\}$ is the *centralizer* of y and $S_y \overset{def}{=} \{\alpha \in \Phi,\ \alpha(y) = 0\}$ is a good set, since the roots of S that are linear of combinations with integer coefficients of elements of S_y belong to S_y. Then

$$\mathcal{C}_y = \mathcal{W}_{S_y}, \tag{11}$$

i.e. there exists $k \in \{1, \cdots, 2^l\}$ such that $\mathcal{W}_{S_y} = \mathcal{W}_k$. In this case, we use S_k to denote S_y. It is clear that S_k is the root system of $\mathcal{S}_k$ relative to the intersection of $\mathcal{H}$ and $\mathcal{S}_k$. This yields another characterization of $\mathcal{W}_{S_y}$, namely $\mathcal{W}_{S_y} = \mathcal{H} \oplus \Big(\bigoplus_{\alpha \in \Phi, \alpha(y)=0} L_\alpha\Big)$. For $k = 1, \cdots, 2^l$, let the subgroups H_k^o and H_k be respectively the maximal integral manifolds containing e and respectively the distributions $(x, x\mathcal{S}_k)$ and $(x, x\mathcal{W}_k)$ of TG.

3 Statement and Proof of the Main Theorem

The solution of the general motion planning problem follows from the next theorem, which is our main result:

Theorem 1 *Let $\Sigma = \Big(G, \mathbb{R}^{d-l}, E, H\Big)$ be the driftless control affine system defined in (7). Let ϕ be the end-point map corresponding to e. Then, for every $g \in G$, there exists a C^1 path $\pi : [0,1] \to G$, with $\pi(1) = g$, that can be lifted through ϕ using the Continuation Method.*

Recall that the end-point map $\phi : H \to M$ associates to $u \in H$, the point $c_{e,u}(1)$, where $c_{e,u}$ is the trajectory of Σ starting at e and corresponding to u. Theorem 1 says that, given any $g \in G$, there exists a C^1 path $\pi : [0,1] \to G$, with $\pi(1) = g$, such that, for every $s \in [0,1]$, the CM provides an input $u_s \in H$ with $\pi(s) = \phi(u_s) = c_{e,u_s}$. Moreover, the assignment $s \mapsto u_s$ is C^1. Note that the proof of the existence of such a u_s is constructive since it comes down to prove the existence of the solution of the PLE, which is an ODE explicitely defined in (3).

Remark 1 *Introducing a drift term xf in (7) leads to consider the control system*

$$\dot{x} = x\Big(f + \sum_{\alpha\in\Phi} u_\alpha e_\alpha\Big), \tag{12}$$

with $f \in \mathcal{G}$. It is not difficult to see that the present situation reduces to the driftless case. Indeed we have $f = h + \sum_{\alpha\in\Phi} \lambda_\alpha e_\alpha$ with $h \in \mathcal{H}$ and $\lambda_\alpha \in \mathbb{R}$. Therefore

$$f + \sum_{\alpha\in\Phi} u_\alpha e_\alpha = h + \sum_{\alpha\in\Phi} (u_\alpha + \lambda_\alpha) e_\alpha.$$

For $\alpha \in \Phi$, we change u_α to $u_\alpha + \lambda_\alpha$ and then we may assume that the drift term reduces to an element of $\mathcal{H}$. Consider now the change of variable $x = y\exp(th)$, $t \in \mathbb{R}$. In order to write the dynamics of y, we need to use ϕ_h^t, the linear application on $\mathcal{G}$ defined by $\phi_h^t(z) = \exp(th)z\exp(-th)$, $z \in \mathcal{G}$. Recall that

$$\phi_h^t(z) = \exp(th)z\exp(-th) = \sum_{n\geq 0} \frac{t^n}{n!} ad_h^n(z).$$

We have $\dot{y} = y\sum_{\alpha\in\Phi} u_\alpha \phi_h^t(e_\alpha)$. Using the fact that, for $\alpha \in \Phi$,

$$\phi_h^t(e_\alpha) = (1 - \alpha(h))e_\alpha + \alpha(h)(\cos(t)e_\alpha - \sin(t)e_{-\alpha}),$$

we get

$$\sum_{\alpha\in\Phi} u_\alpha \phi_h^t(e_\alpha) = \sum_{\alpha\in\Phi^+} (v_\alpha e_\alpha + v_{-\alpha}e_{-\alpha}),$$

where the inputs v_α *and* $v_{-\alpha}$ *are defined as*

$$(v_\alpha,\ v_{-\alpha})^T = M(t,\alpha(h))(u_\alpha,\ u_{-\alpha})^T.$$

The matrix $M(t,\alpha(h))$ *is the* 2×2 *matrix defined by* $(1-\alpha(h))Id_2 + \alpha(h)R_t$*, with* $R_t \in SO(2)$ *the rotation of angle* t*. Since* $M(t,\alpha(h))$ *is invertible with inverse*

$$M(t,\alpha(h))^{-1} = S(t,\alpha(h))M(-t,\alpha(h)),$$

with $S(t,\alpha(h)) = \Big((1-\alpha(h))^2 + 2(1-\alpha(h))\alpha(h)\cos(t) + \alpha(h)^2\Big)^{1/2} > 0$*, the control system (12) is transformed to*

$$\dot{y} = y\sum_{\alpha\in\Phi} v_\alpha e_\alpha,$$

which is the previous driftless control system (7).

Proof of Theorem 1: Recall that the end-point map $\phi : H \to M$ associates to $u \in H$ the point $c_{e,u}(1)$ where $c_{e,u}$ is the trajectory of Σ starting at e and corresponding to u. Since the $(e_\alpha)_{\alpha\in\Phi}$ are bounded, condition (NEC) holds for Σ. In addition, since the $(e_\alpha)_{\alpha\in\Phi}$ generate $\mathcal{G}$, Σ satisfies the Lie Algebraic Rank Condition and then is completely controllable.

The first step is to characterize the abnormal extremals. This was done by Montgomery, cf. [10], and using the notations, we have:

Proposition 1 *The singular set* S *of AE controls is the set of controls* u *for which there exist a nonzero covector* $y \in \mathcal{H}^*$ *and* $k \in \{2,\cdots,2^l\}$ *such that* $\mathcal{C}_y = \mathcal{W}_k$*, i.e.* $u_\alpha \equiv 0$ *for* $\alpha(y) \neq 0$*. Moreover, the singular value set* $\phi(S)$ *is given by*

$$\phi(S) = \bigcup_{k\in\{2,\cdots,2^l\}} H_k^o; \tag{13}$$

The second step consists of establishing the next proposition:

Proposition 2 *Every* C^1 *path* $\pi : [0,1] \to G \setminus \phi(S)$ *can be lifted through* ϕ *using the Continuation Method.*

We first finish the proof of Theorem 1 assuming Proposition 2 and then provide an argument for Proposition 2. It goes by induction on the *subrank* of $\mathcal{G}$ that is defined as the largest rank of a simple Lie subalgebra contained in $\mathcal{G}$. Without loss of generality, we may assume that $\mathcal{G}$ is simple. If rank $\mathcal{G} = 0$ or 1, the Strong Bracket Generating Condition (cf. [13]) holds for Σ and therefore, the theorem is true. Assume now that subrank $\mathcal{G} =$ rank $\mathcal{G} \geq 2$ and that the theorem is true for any (c.c.r.s-s.) Lie group of lower subrank. For $k \in \{2, \cdots, 2^l\}$, let $d_k = \sharp S_k$ and $E_k = (e_\alpha)_{\alpha \in S_k}$. By taking $u_\alpha \equiv 0$ for $\alpha \in \phi \setminus S_k$, we can reduce to Σ another control system $\Sigma_k = \left(H_k^o, \mathbb{R}^{d_k}, E_k, H\right)$. Since the subrank of $\mathcal{S}_k$ is smaller than rank $\mathcal{G}$, the CM solves the MPP for $\phi(S)$ by the induction hypothesis. In other words, for every $g \in \phi(S)$, we can find explicitly a control $u_g \in H$ such that the corresponding trajectory of Σ steers e to g. Therefore, it remains to solve the MPP for $G \setminus \phi(S)$. Fix a point $x_1 \in G \setminus \phi(S)$. Consider u_0 the constant $(d-l)$-tuple $(\nu)_\alpha$, where $\nu > 0$

$$x_0 \stackrel{def}{=} x_{u_0}(1) \notin \phi(S).$$

Then pick a C^1 path $\pi : [0,1] \to G \setminus \phi(S)$ such that $\pi(0) = x_0$ and $\pi(1) = x_1$. Such a path π exists since, by (10), $\phi(S)$ is included in a finite union of Lie subgroups of G, each of them of codimension at least 3 in G and such that their common intersection is not empty. By Theorem 1 part $(b1)$, we can lift π through ϕ and then get a control that steers e to x_1.

Proof of Proposition 2: The idea to prove the above proposition is technical but rather simple: it consists in showing, for (3), a linear growth of $\|P(u)\|$ with respect to $\|u\|$, i.e. an inequality of the type

$$\|P(u)\| \leq C_0 \|u\|. \tag{14}$$

Such an inequality cannot hold for general $u \in H/S_p$. However, we seek it for every $u \in H$ such that $\phi(u)$ belongs to an appropriate compact neighborhood K of $\pi([0,1])$. In particular, the constant C_0 must only depend on K. Once (14) is proved, the global existence of the solution of the PLE follows from Gronwall lemma.

For every $u \in H$, consider the solution ψ of the system

$$\dot{X}(t) = X(t) \sum_{\alpha \in \Phi} u_\alpha e_\alpha, \tag{15}$$

with the terminal condition $\psi(1) = e$. For every good set S, let $\psi_S(t)$ be the solution of the system

$$X(1) = e, \quad \dot{X}(t) = X(t) \sum_{\alpha \in S} u_\alpha e_\alpha. \tag{16}$$

If $\chi_S(t) \stackrel{def}{=} \psi(t)\psi_S(t)^{(-1)}$, then it satisfies the following differential equation:

$$\dot{\chi}_S(t) = \chi_S(t) \sum_{\alpha \in S^c} u_\alpha \psi_S(t) e_\alpha \psi_S(t)^{(-1)}. \tag{17}$$

Taking into account (9) and the fact that the metric is bi-invariant, we have, for a.e. $t \in [0,1]$, $\sum_{\alpha\in S^c} u_\alpha \psi_S(t) e_\alpha \psi_S(t)^{(-1)} = \sum_{\alpha\in S^c} v_\alpha e_\alpha$, with $\sum_{\alpha\in S^c} v_\alpha^2(t) = \sum_{\alpha\in S^c} u_\alpha^2(t)$. Then, we have

$$\dot\chi_S(t) = \chi_S(t) \sum_{\alpha\in S^c} v_\alpha e_\alpha. \tag{18}$$

We write χ_S in the canonical coordinates of the second kind (see Varadarajan [16]), i.e.

$$\chi_S(t) = \Big(\prod_{\alpha\in S^c} \exp\Big(a_\alpha(t) e_\alpha\Big)\Big)\Big(\prod_{\alpha\in S} \exp\Big(b_\alpha(t) e_\alpha\Big)\Big)\Big(\prod_{1\le i\le l} \exp\Big(c_i(t) h_i\Big)\Big), \tag{19}$$

when $t \in [t_1, 1]$, a neighborhood of 1 and the $a_\alpha(t)$'s, $b_\alpha(t)$'s and $c_i(t)$'s are "small" in a sense precised later. The existence of such a nontrivial interval $[t_1, 1]$ is shown in Wei and Norman [18].

Set $m \overset{def}{=} d - l - \sharp S$ and $a(t) = (a_\alpha(t))_{\alpha\in S^c}$ and $b(t) = \Big((b_\alpha(t))_{\alpha\in S}, (c_i(t))_{1\le i\le l}\Big)$. We proceed as in (18). Using (9), we rewrite the following expression

$$\Big(\prod_{\alpha\in S} \exp\Big(b_\alpha(t) e_\alpha\Big) \prod_{1\le i\le l} \exp\Big(c_i(t) h_i\Big)\Big) \cdot \Big(\sum_{\alpha\in S^c} v_\alpha(t) e_\alpha\Big) \cdot \Big(\prod_{\alpha\in S} \exp\Big(b_\alpha(t) e_\alpha\Big) \prod_{1\le i\le l} \exp\Big(c_i(t) h_i\Big)\Big)^{-1}$$

as $\sum_{\alpha\in S^c} w_\alpha(t) e_\alpha$. Since the metric $\langle\cdot,\cdot\rangle$ is bi-invariant, we have

$$\forall t\in[t_1,1] \ \sum_{\alpha\in S^c} w_\alpha^2(t) = \sum_{\alpha\in S^c} v_\alpha^2(t) = \sum_{\alpha\in S^c} u_\alpha^2(t). \tag{20}$$

We set $w(t) = (w_\alpha(t))_{\alpha\in S^c}$. Using (18), (19) and w, we get the dynamics of a and b: there exist analytic matrix-valued functions $A(a)$ and $B(a,b)$ such that:

$$\dot a = A(a) w \text{ and } A(0) = \mathrm{Id}_m, \tag{21}$$

$$\dot b = B(a,b)\dot a \text{ and } B_0 \overset{def}{=} B(0,0). \tag{22}$$

Set $M_0 = \sup_{\{S \text{ good set}\}} \|B_0\|$ and, for $\rho > 0$, consider

$$I^\rho = \left\{ (a,b) | \sup_{\alpha\in S^c} |a_\alpha| \le \rho \text{ and } \sup_{\alpha\in S} |b_\alpha|, \sup_{1\le i\le l} |c_i| \le 12 M_0 d\rho \right\}.$$

Then we can choose $\rho > 0$ small enough such that

(1') the application $f(a,b) = \prod_{\alpha\in S^c} \exp\Big(a_\alpha e_\alpha\Big) \prod_{\alpha\in S} \exp\Big(b_\alpha e_\alpha\Big) \prod_{1\le i\le l} \exp\Big(c_i h_i\Big)$ is an analytic diffeomorphism onto I^ρ(note that $f(I^\rho)$ contains e);

(2') $\sup_{(a,b)\in I^{2\rho}} \|A(a)\| \le 2\|\mathrm{Id}_m\|$ and $\sup_{(a,b)\in I^{2\rho}} \|B(a,b)\| \le 2M_0$;

(3') for $(a,b) \in I^{2\rho}$, we have the first order approximation of $f(a,b)$ given by

$$e + \sum_{\alpha \in S^c} \tilde{a}_\alpha e_\alpha + g(a,b), \tag{23}$$

with $g(a,b) \in \mathcal{W}_S$ and

$$\frac{1}{2} \sum_{\alpha \in S^c} a_\alpha^2 \leq \sum_{\alpha \in S^c} \tilde{a}_\alpha^2 \leq 2 \sum_{\alpha \in S^c} a_\alpha^2. \tag{24}$$

Set $M_\rho = \sup_{(a,b)\in I^{2\rho}} \Big(\|D(B(a,b)A(a))\| \Big)$ and note that

$$\sup_{(a,b)\in I^{2\rho}} \|B(a,b)A(a)\| \leq 4dM_0.$$

Since there is a finite number of good sets, we may choose the above $\rho > 0$ small enough so that it is independent of the good set S. Finally, t_1 is chosen so that

$(t_1 a)$ $\Big(a(1), b(1)\Big) = (0,0)$ and $\forall t \in (t_1, 1]$ $\Big(a(t), b(t)\Big) \in \overset{\circ}{I^{2\rho}}$;

$(t_1 b)$ and either $t_1 = 0$ or $\Big(a(t_1), b(t_1)\Big) \in \partial I^{2\rho}$.

Let $\pi : [0,1] \to G \setminus \phi(S)$ be a C^1-path, $\bar{\varepsilon} \overset{def}{=} \frac{d_G\Big(\pi([0,1]), \phi(S)\Big)}{3} > 0$ so that $f(I^\rho) \subset N_{\bar{\varepsilon}}(e)$. Define

$$J = \{\xi = (x,y) \in T^*G | x \in \pi\Big([0,1]\Big),\ \|y\| = 1\},$$

$$K(J) \overset{def}{=} \{\xi = (x,y) \in T^*G | x \in N_{2\bar{\varepsilon}}\Big(\pi\Big([0,1]\Big)\Big) \text{ and } 1/2 \leq \|y\| \leq 2\},$$

$$Z = \{\xi = (x,y) \in K(J) | \varphi_y \overset{def}{=} \sum_{\alpha \in \Phi} < y, e_\alpha > e_\alpha = 0\}.$$

For $u \in H$, let $Gr(u) \overset{def}{=} D\phi(u) D\phi(u)^T$. Recall that (see [2] or Sussmann [14]), for $y \in T^*_{\phi(u)}G$, $< y, Gr(u)y >= \|\varphi_y(t)\|^2_{L^2(0,1)}$, where

$$\forall [a,b] \subset [0,1]\ \|\varphi_y(t)\|^2_{L^2(a,b)} = \int_a^b \varphi_y(t)^2 dt.$$

Recall that, to prove global existence of the solution of the PLE, we try to establish for some control u a linear growth of the right-hand side of the PLE with respect to $\|u\|$. For regular values of u, to bound the right-hand side of equation (3) requires to control $\|P(u)\|$, where $P(u)$ is the Moore-Penrose pseudoinverse of $D\phi(u)$. Recalling that

$$\|P(u)\| = \big(\inf_{\|y\|=1} < y, Gr(u)y > \big)^{-1/2},$$

we then have to establish the following estimate (see [2] or [14]) :

$$\Big(\exists C>0\Big)\Big(\forall u\in H\Big)\Big(y\in T^*_{\phi(u)}G,\ \|y\|=1\Big)\ \ \Big((\phi(u),y)\in J\Rightarrow <y,Gr(u)y>\|u\|^2_{L^2(a,b)}\geq C\Big). \tag{25}$$

Define $\delta\overset{def}{=}\inf(\frac{\rho}{(3d+3)^{1/2}},\frac{\bar{\varepsilon}}{1+(3d)^{1/2}},\sqrt{2d})$. From now on, consider $u\in H$ and $y\in T^*_{\phi(u)}G$ and $\|y\|=1$ such that $\Big(\phi(u),y\Big)\in J$. We have $\phi(u)^{-1}y=h+z$, where $h\in\mathcal{H}$ and $z\in\mathcal{D}(e)$. We will consider two cases depending on whether $\|z\|\leq\delta$ or not.

First, assume that $\|z\|\leq\delta$ and consider the sets S^0_h and $S^{0,c}_h$ defined by

$$S^0_h\overset{def}{=}\{\alpha\in\Phi\ \ |\alpha(h)|<\delta\}\ ,\ \text{and}\ S^{0,c}_h\overset{def}{=}\{\alpha\in\Phi\ \ |\alpha(h)|\geq\delta\}\ .$$

Clearly these sets are symmetric and, if $S_h\overset{def}{=}\bar{S}^0_h$, it is a good set. Note that $S^c_h\subset S^{0,c}_h$. Choose a subset $S_{h,1}$ of S_h and a subset $S_{h,2}$ of S^c_h such that $S_{h,1}\cup S_{h,2}$ is a base of Φ. Then define $h_1\in\mathcal{H}$ as follows:

$$\forall\alpha\in S_{h,1}\ \alpha(h_1)=0\ \text{and}\ \forall\alpha\in S_{h,2}\ \alpha(h_1)=\alpha(h)\,.$$

Hence, using the results of Section 1, we have

(i) $1\geq\|h_1\|^2\geq(1-d\delta^2)\geq 1/2$;

(ii) $\forall\alpha\in S_h\ \ \alpha(h_1)=0$ and $\forall\alpha\in S^c_h\ \ |\alpha(h_1)|\geq\delta$.

Note that $\|\phi(u)^{-1}y-\frac{1}{\|h_1\|}h_1\|\leq(1+(3d)^{1/2})\delta\leq\bar{\varepsilon}$. We associate to the good set S_h, the Lie subalgebras $\mathcal{W}_{S_h}$ and the diffeomorphisms ψ_{S_h} given by (8) and (16). Since $\phi(u)^{-1}y\in N_{\bar{\varepsilon}}(\frac{1}{\|h_1\|}h_1)$, there exists $(a_h,b_h)\in I^\rho$ such that

$$\phi(u)^{-1}y=\frac{1}{\|h_1\|}h_1f(a_h,b_h)$$

and $g_u\overset{def}{=}\phi(u)\Big(f(a_h,b_h)\Big)^{-1}\in N_{\bar{\varepsilon}}(\phi(u))$. Therefore, $e\notin N_{\bar{\varepsilon}}(g_u)$.

Consider t_2 as the first time $t<1$ for which $\Big(x_u(t),c_y(t)\Big)$ reaches $\partial K(J)$. This time t_2 exists since $\|c_y(t)\|\equiv 1$ and $x_u(0)=e\notin K(J)$. Let $t_e\overset{def}{=}\sup(t_1,t_2)$ be *the first exit time* for (u,y). For $t\in[t_e,1]$, we can write $x_u(t)$ as

$$x_u(t)=g_u\chi_{S_h}(t)\psi_{S_h}(t). \tag{26}$$

From the definition of t_e, we have the following:

$$\forall t\in(t_e,1],\ \ x_u(t)\in\overset{\circ}{K}(J), \tag{27}$$

$$\forall \alpha \in S_h^c, \quad |a_\alpha(t)| < 2\rho, \tag{28}$$

$$\forall \alpha \in S_h, \quad |b_\alpha(t)| < 24 M_0 d\rho\,, \tag{29}$$

$$\forall k = 1, \cdots, l, \quad |c_k(t)| < 24 M_0 d\rho\,, \tag{30}$$

$$\text{and at } t = t_e \text{, one of these inequalities becomes an equality.} \tag{31}$$

These equations follow mostly from $(t_1 a)$ and $(t_1 b)$. Here $\Big(a(t), b(t)\Big)$ is the solution of the equations (21) and (22), with the terminal condition at $t = 1$, (a_h, b_h).

For $t \in [t_e, 1]$, we write $\varphi_y(t) = \varphi_1(t) + \varphi_2(t)$, where $\varphi_1(t)$ belongs to $\mathcal{W}_{S_h}^{\perp}$ and $\varphi_2(t)$ belongs to $\mathcal{W}_{S_h}$. We can also write $\varphi_y(t)$ as

$$\forall t \in [t_e, 1] \quad \varphi_y(t) = h_1 f\Big(a(t), b(t)\Big)\psi_{S_h}(t). \tag{32}$$

Then, by (23), (32) becomes

$$\forall t \in [t_e, 1] \quad \varphi_y(t) = h_1\Big(e + \sum_{\alpha \in S_h^c} \tilde{a}(t)_\alpha e_\alpha + g\Big(a(t), b(t)\Big)\Big)\psi_{S_h}(t). \tag{33}$$

Using (9) and the fact that the metric is bi-invariant, we get

$$\forall t \in [t_e, 1] \quad \|\varphi_1(t)\| = \| - \sum_{\alpha \in S_h^c} \tilde{a}(t)_\alpha \alpha(h_1) e_{-\alpha}\|, \tag{34}$$

and then

$$\forall t \in [t_e, 1] \quad \|\varphi_1(t)\|^2 \geq \delta^2 \sum_{\alpha \in S_h^c} \tilde{a}(t)^2. \tag{35}$$

Finally, we have from (24),

$$\forall t \in [t_e, 1] \quad \|\varphi_1(t)\|^2 \geq \frac{\delta^2}{2} \sum_{\alpha \in S_h^c} a(t)^2. \tag{36}$$

Consider now the subcase for which we have equality in (29) or (30). Then there exists a coordinate $b_j(t)$ of $b(t)$ for which we have

$$|b_j(t_e) - b_j(1)| \geq 24 M_0 d\delta - 12 M_0 \delta = 12 M_0 d\delta. \tag{37}$$

Integrating equation (21) between t_e and 1, then performing an integration by parts and using (28), (29) and (30), we obtain

$$12 M_0 d\delta \leq 8 d M_0 \delta + M_0 \sum_{\alpha, \alpha' \in S_h^c} \int_{t_e}^{1} |a_\alpha(t) w_{\alpha'}| dt. \tag{38}$$

Applying Cauchy-Schwarz inequality in (38) and combining it with (36), we conclude that

$$\|\varphi_y\|_{L^2(t_e,1)}\|u\|_{L^2(t_e,1)} \geq \frac{4M_0\delta^2}{\sqrt{2}dM_0},$$

which implies (25).

It remains the subcase for which we have equality in (28) and Case 2, where $\|z\| \geq \delta$. These two options are entirely similar: in the first one, we have $\|\varphi_y(t_e)\| \geq \rho$, while in the second one, we have $\|\varphi_y(1)\| \geq \delta$. In both cases, there exist $t_* < t^* \in [t_e, 1]$, such that

(a) $\forall t \in [t_*, t^*]$, $\|\varphi_y(t)\| \geq \frac{\inf(\rho,\delta)}{2} \stackrel{def}{=} \delta_1$;

(b) $|\int_{t_e}^1 \xi(t)dt| \geq \frac{\inf(\bar{\varepsilon},\delta)}{4} \stackrel{def}{=} \delta_2$,

where $\xi(t) = \sum_{\alpha\in\Phi} u_\alpha(t)e_\alpha$. Therefore, using Cauchy-Schwarz inequality, we obtain

$$\|\varphi_y\|_{L^2(t_*,t^*)}\|u\|_{L^2(t_*,t^*)} \geq \delta_1(t_* - t^*)^{1/2}\|u\|_{L^2(t_*,t^*)} \geq \frac{\delta_1}{d}\|\int_{t_*}^{t^*}\sum_{\alpha\in\Phi} u_\alpha(t)e_\alpha dt\| \geq \frac{\delta_1\delta_2}{d},$$

and (25) follows. The proof of the proposition is complete.

4 Conclusion

In this paper, we provided a constructive solution to the motion planning problem associated to the driftless control affine system $\Sigma = \left(G, \mathbb{R}^{d-l}, E, H\right)$ defined by (7). The compact character of the Lie group G is instrumental not only for describing the singular set of the end-point map (existence of the bi-invariant metric) but also for reducing several estimates in terms of the coordinates of the second kind. The removal of that hypothesis should require new techniques to handle the MPP. This also should be the case if one considers "smaller" nonholonomic distributions. It will result in a more complicated singular set S_p and the geometry of such an object is not yet well-understood and remains an open question in sub-Riemannian geometry.

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