Contents

Part I

Stability and stabilization

Stabilization and Finite-Gain Stabilizability of Delay Linear Systems Subject to Input Saturation

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Summary. This paper deals with two problems on stabilization of linear systems by static feedbacks which are bounded and time-delayed, namely global asymptotic stabilization and finite gain L^p-stabilization, $p \in [1, \infty]$. Regarding the first issue, we provide, under standard necessary conditions, two types of solutions for arbitrary small bound on the control and large (constant) delay. The first solution is based on the knowledge of a static stabilizing feedback in the zero-delay case and the second solution is of nested saturation type, which extends results of [2]. For the finite-gain L^p-stabilization issue, we assume that the system is neutrally stable. We show the existence of a linear feedback such that, for arbitrary small bound on the control and large (constant) delay, finite gain L^p-stability holds with respect to every L^p-norm, $p \in [1, \infty]$. Moreover, the corresponding L^p-gain is delay-independent.

Key words: Saturated feedback, Stabilization, Lyapunov functions, Time-delay systems, Linear continuous-time delay systems, Finite-gain stability.

1 Introduction

In this paper, we address two issues relative to the stabilization for continuous-time delay linear systems subject to input saturation, of the type

$$(S) : \dot{x}(t) = Ax(t) + Bu(t - h), \tag{1}$$

where (i) $A \in R^{n \times n}$ and $B \in R^{n \times m}$, with n the dimension of the system and m the number of inputs; (ii) the control u verifies $\|u\| \leq r$, where $r \in (0, 1]$ only depends on (S); (iii) there is an arbitrary constant delay $h \geq 0$ appears in the input.

We use $(S)^r_h$, $r \in (0, 1]$, $h > 0$, to denote the control system (S) with input bound r and input time delay h. We omit the index r if it is equal to one and, similarly for the index h if it is equal to zero.

The first problem is that of globally asymptotically stabilizing (S) to the origin by mean of a static feedback. We then seek u as

$$u(t-h) = -r\sigma\left(F_h^r(x(t-h))\right), \tag{2}$$

where the non-linearity σ is of "saturation" type (definitions are given in section (2)) and the function $F_h^r : R^n \longrightarrow R^m$ is at least locally Lipschitz (to obtain at least locally solutions).

In the zero–delay case, the stabilization of linear systems with saturating actuators has been widely investigated in the last years: static feedbacks of nested saturation type (see [11] and [12]) or based on maximal ellipsoid saturation (see [4]) can be used. It is well-known that such a global asymptotic stabilization is possible if and only if (S) satisfies

$$(C): \begin{cases} (i) A \text{ is neutrally stable }, \\ \\ (ii) \text{ the pair } (A,B) \text{ is stabilizable }. \end{cases}$$

It is trivial to see that condition (C) is also necessary in the case of non zero delay and it seems natural to expect condition (C) to be also sufficient. In that regard, partial results have been recently obtained by Mazenc, Mondie and Niculescu. To state the results, we define the unrestricted GAS property. We say that $(S)_h^r$ is unrestricted GAS if, for arbitrary delay $h > 0$ and any input rate $r \in (0,1]$ small enough, $(S)_h^r$ is global asymptotic stabilizable. The nested saturation construction is used to show that $(S)_h^r$ is unrestricted GAS if A is nilpotent ([2]) and for the two-dimensional oscillator ([3]). One of our main results is to complete that line of work, namely to show that condition (C) is sufficient for unrestricted GAS.

We will actually provide two different ways to solve the GAS problem. The first one is based on the knowledge of a globally Lipschitz static stabilizing feedback F in the zero–delay case. From it, one can build a static stabilizing feedback for $(S)_{h^*}$, with $h^* > 0$ only depending on A, B, σ and K_F, the Lipschitz constant of F. If, in addition, an extra hypothesis holds on stabilizing feedbacks of $(S)^r$, for r small enough, unrestricted GAS holds. It turns out that the nested saturated feedbacks of [11] verify these hypotheses, and thus we conclude, see [14].

The second solution for unrestricted GAS directly uses the nested saturated feedbacks of [11] and can be seen as a generalization of [2, 3]. However, the argument is an extension to the non-zero delay case of that of [11]. Recall that, at the heart of the argument of [11], lies a result on finite-gain L^∞-stability for one and two dimensional neutrally stable linear systems subject to input saturation. Such an argument was first introduced in [1], where was addressed the issue of finite-gain L^p-stability of neutrally stable linear systems subject to input saturation.

It is therefore natural to consider the L^p-stability question. We extend to the non-zero delay cases results of [1]. Our objective here consists in showing that the results of [1] carry over to continuous linear time-delay systems. More specifically, we show that, for neutrally stable continuous linear time-delay systems subject to input saturation, finite-gain L^p-stabilization can be achieved by the use of linear feedbacks, for every $p \in [1, \infty]$. While many of the arguments of the present paper are conceptually similar to those of [1], there are technical aspects that are different and not obvious. Indeed, as in [1], the proof to get finite gain L^p-stability relies on passivity techniques. We determine a suitable "storage" function V_p and establish for it a "dissipation inequality" of the form $\frac{dV_p(x_u(t))}{dt} \leq -\|x_u(t)\|^p + \lambda_p \|u(t)\|^p$, for some constant $\lambda_p > 0$ possibly depending on the input bound r and the delay h. For more discussion on passivity, see [13] for instance. Recall that the "storage function" in [1], V_p^0 is non-smooth. In the present situation, the "storage function" V_p will be the sum of a term similar to V_p^0 and a Lyapunov-Krasovskii functional, in order to take care of the delay. However, unlike in [1], the saturation in (1) needs to be multiplied by a small factor r dependent on the delay h in order to insure finite-gain L^p-stability. In addition, by choosing carefully the factor r and the linear feedback inside the saturation, we are able to provide upper bounds for the L^p-gains of $(S)_h^r$ which are *independent* of $r \in (0, 1]$ and $h > 0$. We refer to that property as the *unrestricted finite-gain L^p-stability.*

The argument corresponding to that uniformity result is specific to the non-zero delay case and constitutes the most technical part of [15]. To establish it, we first start with the single-input case where it amounts in estimating the behavior of the solution P_r of a parameterized Lyapunov equation (L_r), $r \in (0, 1]$, as the parameter r tends to zero. The multi-input case requires additional work. We first rewrite the original system as an appropriate cascade of single-input subsystems, all of them except one being perturbed by an external disturbance, appearing outside the saturation (see Theorem 5). We then proceed by an inductive argument on the number of distinct algebraic multiplicities of the eigenvalues of A.

Generally speaking, our treatment of the aforementioned issues on time-delay systems follows a common pattern. We always try to reformulate them as problems for perturbed *delay-free* systems and handle the perturbation by Lyapunov techniques. One of the reasons for which that strategy works well lies in the fact that the input saturation makes the perturbation uniformly bounded with respect to the delay.

The complete proofs of the results presented in this paper are contained in [14] for stabilization and [15] for finite-gain stabilizability.

2 Notations and statement of the main results

2.1 Notations

For $x \in R^n$, $\|x\|$ and x^T denote respectively the Euclidean norm of x and the transpose of x. Similarly, for any $n \times m$ matrix K, K^T and $\|K\|$ denote respectively the transpose of K and the induced $2-$norm of K. Moreover, $\lambda_{\min}(K)$ and $\lambda_{\max}(K)$ denote the minimal and the maximal singular values of the matrix K. If $f(.)$ and $g(.)$ are two real-valued functions, we mean by $f(r) \asymp_0 g(r)$, that there are positive constants ξ_1 and ξ_2 independent of r small enough, such that the inequalities

$$\xi_1 g(r) \leq f(r) \leq \xi_2 g(r),$$

are valid. Initial conditions for delayed systems are continuous vectors-valued functions defined on $[-h, 0]$ and taking values in R^n. For $h > 0$, let $C_h := C([-h, 0], R^n)$; $x_t(\theta) := x(t+\theta)$, for $-h \leq \theta \leq 0$ and $\|x_t\|_h := \sup_{-h\leq\theta\leq 0} \|x(t+\theta)\|$.

Definition 1. *(Saturation function) We call $\sigma : R \longrightarrow R$ a saturation function ("S-function" for short) if there exist two real numbers $0 < a \leq K_\sigma$ such that for all $t, t' \in R$*
(i) $|\sigma(t) - \sigma(t')| \leq K_\sigma \inf(1, |t - t'|)$,
(ii) $|\sigma(t) - at| \leq K_\sigma t\sigma(t)$.
(iii) $\sigma(t) = t$ when $|t| \leq a$.

It is assumed here that the function is normalized at the origin, i.e. $a = \sigma'(0) = 1$. The global lipschitzness of σ implies that for every real numbers x, y,

$$|x[\sigma(x + y) - \sigma(x)]| \leq K|y|.$$

For an $m-$tuple $k = (k_1, \ldots, k_m)$ of nonnegative integers, define $|k| = k_1 + \ldots + k_m$. We say that σ is an $R^{|k|}-$ valued S-function if

$$\sigma = (\sigma_1, \ldots, \sigma_{|k|}) = (\sigma_1^1, \ldots, \sigma_{k_1}^1, \ldots, \sigma_1^m, \ldots, \sigma_{k_m}^m)$$

$$= \left((\sigma_i^1)_{1\leq i\leq k_1}, (\sigma_i^2)_{1\leq i\leq k_2}, \ldots, (\sigma_i^m)_{1\leq i\leq k_m}\right),$$

where, for $1 \leq j \leq m$, $(\sigma_i^j)_{1\leq i\leq k_j}$ is an R^{k_j}-valued S-function ($i.e : (\sigma_i^j)_{1\leq i\leq k_j} = (\sigma_1^j, \ldots, \sigma_{k_j}^j)$ where each component $\sigma_i^j, 1 \leq i \leq k_j$ is an S-function and

$$(\sigma_i^j)_{1\leq i\leq k_j}(x) = \left(\sigma_1^j(x_1), \ldots, \sigma_{k_j}^j(x_{k_j})\right),$$

for $x = (x_1, \ldots, x_{k_j})^T \in R^{k_j}$. Here we use $(\ldots)^T$ to denote the transpose of the vector $(\ldots)$.)

Definition 2. *Consider the functional differential equation of retarded type*

$$(\Sigma)_h : \begin{cases} \dot{x}(t) = f(x_t), \text{ for } t \geq t_0; \\ \\ x_{t_0}(\theta) = \Psi(\theta), \forall\theta \in [-h, 0]. \end{cases}$$

It is assumed that $\Psi \in C_h$, the map f is continuous and Lipschitz in Ψ and $f(0) = 0$. We say that $(\Sigma)_h$ is globally asymptotically stable (GAS for short) if the following conditions hold:

(i) for every $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that, for any* $\Psi \in C_h$, *with* $\|\Psi\|_h \leq \delta$, *there exists* $t_0 \geq 0$, *such that the solution* $x(\Psi)$ *of* $(\Sigma)_h$ *satisfies* $\|x_t(\Psi)\|_h \leq \varepsilon$, *for all* $t \geq t_0$;

(ii) for all $\Psi \in C_h$, *the trajectory of* $(\Sigma)_h$ *with the initial condition* Ψ *and defined on* $[t_0, \infty)$ *converges to zero as* $t \to \infty$.

2.2 GAS using a stabilizing feedback in the zero delay case

Our objective is to relate the asymptotic stability properties of the system (S) with those of the delay-free system provided that it is globally asymptotically stable. The study is then extended to investigate conditions which ensure that the class of linear controllers, stabilizing the delay-free system, also stabilize (S) by stating the problem as an asymptotic stability problem. For this purpose, the delayed system (S) is considered as a perturbation of that of the delay-free system. We now state our first result.

Theorem 1. *Assume* $(\mathbf{H})_\mathbf{0}$: *There exists* $F : R^n \longrightarrow R^m$ *globally Lipschitz, with Lipschitz constant* K_F *such that the system*

$$(S)_0 : \quad \dot{x} = Ax - B\sigma(F(x)),$$

is globally asymptotically stable with respect to 0.
Then, there exists $h^* = h(A, B, \sigma, K_F) > 0$ *such that, for all* $h \in [0, h^*]$, *there exists* $F_h : R^n \longrightarrow R^m$ *that globally asymptotically stabilizes the system*

$$(S)_h : \dot{x} = Ax - B\sigma(F_h(x(t-h))),$$

with respect to zero.

Sketch of proof. Let $F_h(x(t)) = F(\Phi(t, t-h, x(t)))$, where Φ is the flow of the equation $(S)_0$. We rewrite $(S)_h$ as $\dot{x}(t) = Ax(t) - B\sigma(F(x(t))) - B\varepsilon(t)$, where $\varepsilon(t)$ as a perturbation of $(S)_0$. The perturbation ε may cause instability but we show that $\|\varepsilon(x(t))\| \leq \tilde{K}e^{-\lambda t}$ for some $\tilde{K}$ (that may depend on ε) and $t \geq 0$. Using Lemma 3.1 in [6], we are able to conclude.

The second result completes the stability result of Theorem 1 to get unrestricted global asymptotic stability (unrestricted GAS). It is stated as follows:

Theorem 2. *Assume* $(\mathbf{H})_\mathbf{0}^\mathbf{r}$: *For each* $r \in]0, 1]$, *there exists a globally Lipschitz function* $F^r : R^n \longrightarrow R^m$, *with Lipschitz constant* K_{F^r}, *such that*

$$(i) \quad (S)_0^r : \quad \dot{x} = Ax - rB\sigma(F^r(x)), \text{ is GAS with respect to zero },$$

$$(ii) \quad rK_{F^r} \to 0 \text{ if } r \to 0.$$

Then, for all $h \geq 0$, *there exists* $r^*(h) \in]0, 1]$, *such that for any* $r \in]0, r^*(h)]$, *a function* $F_h^r : R^n \longrightarrow R^m$ *exists for which the system*

$$(S)_h^r : \quad \dot{x} = Ax - rB\sigma(F_h^r(x(t-h))),$$

is globally asymptotically stable with respect to zero.

2.3 Feedbacks of nested saturation type

We next determine two explicit expressions of globally asymptotically stabilizing feedbacks for general time-delay linear systems, both of nested saturation type, according to the results of the stabilization of delay free-system. The above problem was first studied for delay-free continuous-time systems. It was shown in [11] that, under condition (C), there exists explicit expressions of globally asymptotically stabilizing feedbacks. Then, it is natural to investigate whether this technique can be extended to the case where there is a delay in the input. In this section, we will take for simplicity the initial state to be zero. We start by giving some definitions, first introduced in [11] and adapted here to the delay case.

Definition 3. *For a retarded system* $\dot{x}(t) = f(x(t), u(t-h)), x \in R^n, u \in R^m$*, we say that a feedback* $u(.) = k(x(.))$ *is stabilizing if zero is a globally asymptotically stable equilibrium of the system* $\dot{x}(t) = f(x(t), k(x((t-h)))$.

Definition 4. (*cf. [11]*) *For a square matrix* A*, let* $N(A) = s(A) + z(A)$*, where* $s(A)$ *is the number of conjugate pairs of nonzero purely imaginary eigenvalues of* A (*counting multiplicity*) *and* $z(A)$ *is the multiplicity of zero as an eigenvalue of* A.

Theorem 3. *Assume that condition* (C) *holds for* $(S)_h^r$. *Let* $N = N(A)$ *and* $\sigma = (\sigma_1, \ldots, \sigma_N)$ *be an arbitrary sequence of S-functions. Then, for all* $h > 0$*, there exist a number* $r^*(h) \in (0, 1]$*, an* $m-$*tuple* $k = (k_1, \ldots, k_m)$ *of non negative integers such that* $|k| = N$ *and for each* $1 \leq j \leq m$*, linear functions* $f_{h,i}^j, g_{h,i}^j : R^n \longrightarrow R, 1 \leq i \leq k_j$*, such that for all* $r \in (0, r^*(h)]$*, there are stabilizing feedbacks*

$$(*) \quad u_j(t-h) = -r\sigma_{k_j}^j\{f_{h,k_j}^j(x(t-h)) + \alpha_{k_j-1}^j\sigma_{k_j-1}^j[f_{h,k_j-1}^j(x(t-h)) + \ldots + \alpha_1^j\sigma_1^j(f_{h,1}^j(x(t-h)))\ldots]\}, \tag{3}$$

where $\alpha_i^j \geq 0$*, for all* $i \in [1, k_j - 1]$*, and*

$$(**) \quad u_j(t-h) = -r\Big[\beta_{k_j}^j\sigma_{k_j}^j\Big(g_{h,k_j}^j(x(t-h))\Big) + \beta_{k_j-1}^j\sigma_{k_j-1}^j\Big(g_{h,k_j-1}^j(x(t-h))\Big) + \ldots + \beta_1^j\sigma_1^j\Big(g_{h,1}^j(x(t-h))\Big)\Big], \tag{4}$$

where $\beta_1^j, \ldots, \beta_{k_j}^j$ *are nonnegative constants such that* $\beta_1^j + \ldots + \beta_{k_j}^j \leq 1$.

Sketch of proof. The argument of follows the strategy of proof of the principal result of [11]. We start therefore with the single-input case and prove the theorem by induction on the dimension of the system. In order to facilitate the analysis of the stabilizability properties by bounded feedback of $(S)_h^r$, a linear transformation is carried out in [11].

Lemma 1. (*cf. [11]*) *Let $(S_1)_h^r : \dot{x}(t) = Ax(t)+bu(t-h)$ be an n-dimensional linear single-input system. Suppose that (A, b) is a controllable pair and all eigenvalues of A are critical.*

(i) *If 0 is an eigenvalue of A, then there exists a linear coordinate transformation $y = Sx$ which transforms $(S_1)_h^r$ into*

$$\begin{cases} \dot{\bar{y}}(t) = A_1\bar{y}(t) + (y_n(t) + u(t-h))\, b_1, \\ \dot{y}_n(t) = u(t-h), \end{cases} \tag{5}$$

where the pair (A_1, b_1) is controllable, y_n is a scalar variable, and $\bar{y} = (y_1, \dots, y_{n-1})^T$.

(ii) *If A has an eigenvalue of the form $i\omega$, with $\omega > 0$, then there is a linear change of coordinates $Sx = (y_1, \dots, y_n)^T = (\bar{y}^T, y_{n-1}, y_n)^T$ of R^n that puts $(S_1)_h^r$ in the form:*

$$\begin{cases} \dot{\bar{y}}(t) = A_1\bar{y}(t) + (y_n(t) + u(t-h))\, b_1, \\ \dot{y}_{n-1}(t) = \omega y_n(t), \\ \dot{y}_n(t) = -\omega y_{n-1}(t) + u(t-h), \end{cases} \tag{6}$$

where the pair (A_1, b_1) is controllable and y_{n-1}, y_n are scalar variables.

The following lemma is the key technical point of the proof.

Lemma 2. *Let $\rho > 0$ and σ be an S-function. Then, for all $h > 0$ there exist $r^*(h) \in]0, 1]$ and an 2×1 matrix F_h such that, for any two bounded measurable functions $\alpha(t), \beta(t)$ converges both to zero as $t \longrightarrow \infty$ and for all $r \in]0, r^*(h)]$, the control system*

$$(S_2)_h^r : \begin{cases} \dot{x}_1(t) = \rho x_2(t) + r\alpha(t), \\ \dot{x}_2(t) = -\rho x_1(t) - r\sigma(F_h^T x(t-h) + u(t-h)) + rv(t-h) + r\beta(t), \\ x_0 = ((x_1)_0, (x_2)_0)^T = \bar{0}, \text{ on } [-h, 0], \end{cases}$$

with $\bar{0}$ the zero function in C_h, and $u, v \in L^\infty([-h, \infty), R)$, with $\|v\|_{L^\infty} \leq v^*$, ($v^*$ independent of r) verifies:

(i) There exists a finite constant $M_\infty > 0$ independent of r, such that

$$\limsup_{t\to\infty} \|x(t)\| \leq M_\infty \left(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|f\|_{L^\infty}\right), \tag{7}$$

where $x = (x_1, x_2)^T, f = (\alpha, \beta)^T$.

(ii) In the absence of u, v and f, the equilibrium $(x, y) = (0, 0)$ is globally asymptotically stable.

Sketch of proof. We consider the linear feedback $F_h = e^{-\rho A_0 h} b$, where $A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = (0,1)^T$. The argument here is the simplest case of the more general result given in Proposition 1 for the single input case, see the corresponding sketch of proof below. More precisely, it corresponds to $p = 2$, $A = A_0$ and b is defined above. Note that in this case, the matrix P_r can be computed explicitly as well as $\lambda_{\max}(P_r)$ and $\lambda_{\min}(P_r)$.

2.4 Finite gain stabilizability

Finite-gain stability results for various $p-$norms are presented. We start with definitions.

L^p-Stability. For $p \in [1, \infty]$ and $0 \leq h$, we use L^p to denote $L^p(-h, \infty)$ and we let $\|y\|_{L^p}$ denote the L^p-norm: $\|y\|_{L^p} = \left(\int_{-h}^{\infty} \|y(t)\|^p dt \right)^{\frac{1}{p}}$, if $p < \infty$ and $\|y\|_{L^\infty} = ess \sup_{-h \leq t < \infty} \|y(t)\|$.

Consider the control system with delay in the input given by

$$(\Sigma)_h : \dot{x}(t) = f(x(t), u(t-h)), \text{ for } t \geq 0,$$

where the state x and the control u take respectively values in R^n and R^m and $f : R^n \times R^m \to R^n$, is locally Lipschitz in (x, u), with $f(0,0) = 0$. Trajectories of $(\Sigma)_h$ starting at an initial condition $x_0 \in C_h$ and corresponding to an input $u \in L^p$ are defined for a time interval I of $^+$ (which may depend on x_0 and u) and verify the equation $(\Sigma)_h$ for almost every $t \in I$. Let $\bar{0}$ be the zero function in C_h.

Definition 5. *(L^p-stability): Given $p \in [1, \infty]$, the continuous-time delay system $(\Sigma)_h$ is said to be L^p-stable if, for every $u \in L^p$, we have $x_u \in L^p$, where x_u denotes the solution of $(\Sigma)_h$ corresponding to u with initial condition $x_0 = \bar{0}$.*

Definition 6. *(Finite-gain L^p- stability) : Given $p \in [1, \infty]$, the continuous-time delay system $(\Sigma)_h$ is said to be finite-gain L^p-stable if it is L^p-stable, and there exists a positive constant M_p such that, for every $u \in L^p$,*

$$\|x_u\|_{L^p} \leq M_p \|u\|_{L^p}.$$

Furthermore, the infimum of such numbers M_p will be called the L^p-gain of the system.

We next give our main results.

Theorem 4. *Let A, B be $n \times n, n \times m$ matrices respectively. Let σ be an R^m-valued $S-$function. Assume that A be neutrally stable and (A, B) controllable. Then, for every $h \geq 0$, there exists an $n \times m$ matrix F_h such that the system,*

$$(S)_h^r : \dot{x} = Ax - rB\sigma(F_h^T x(t-h) + u(t-h)), \ \textit{for } t \geq 0,$$

has the unrestricted finite gain L^p-stability property for every $p \in [1, \infty]$, i.e., for every $h > 0$, there exists $r^*(h) \in (0, 1]$ such that for every $p \in [1, \infty]$, $(S)_h^r$, $r \in (0, r^*(h)]$, is finite-gain L^p-stable.

Remark 1. In the absence of u, the equilibrium point $\bar{0}$ is globally asymptotically stable for the delayed system $\dot{x}(t) = Ax(t) - rB\sigma(F_h^T x(t-h))$.

Theorem 4 is a particular case of a stronger result given next.

Theorem 5. *With the same hypothesis on A, B and σ, consider the following delayed system (still denoted $(S)_h^r$)*

$$(S)_h^r : \dot{x}(t) = Ax(t) - rB\sigma\left(F_h^T x(t-h) + u_1(t-h)\right)$$

$$+ ru_2(t-h), \ \textit{for } t \geq 0,$$

where F_h is defined as in Theorem 4 and the input u_2 takes values in R^n. then, there exist a constant $C_0 > 0$ and, for every $1 \leq p \leq \infty$, a constant $M_p > 0$ such that, for every $h > 0$ there is an $r^*(h) \in (0, 1]$, for which the trajectories x_{u_1,u_2} of $(S)_h^r$, $r \in (0, r^*(h)]$, starting at $\bar{0}$ and corresponding to $u_1, u_2 \in L^p$ with $\|u_2\|_{L^\infty} \leq C_0$, verify

$$\|x_{u_1,u_2}\|_{L^p} \leq M_p \left(\|u_1\|_{L^p} + \|u_2\|_{L^p}\right). \tag{8}$$

Remark 2. It will be clear from our argument that we can in fact obtain the following stronger Input-To-State-Stable (ISS for short)-like property ([9] and references there):

$$\|x^{\psi}_{u_1,u_2}\|_{L^p} \leq \theta_p(\|\psi\|_h) + M_p(\|u_1\|_{L^p} + \|u_2\|_{L^p}), \tag{9}$$

where $\psi \in C_h$ is the initial condition for the trajectory $x^{\psi}_{u_1,u_2}$ corresponding to u_1, u_2 and θ_p is a $\mathcal{K}$-function (i.e. $\theta_p : R_+ \to R_+$, is continuous, strictly increasing and satisfies $\theta_p(0) = 0$.).

Sketch of proof of Theorem 5. From elementary linear algebra, a neutrally stable matrix A is similar to a matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 is an $q \times q$ Hurwitz matrix and A_2 is an $(n-q) \times (n-q)$ skew-symmetric matrix. So, up to a change of coordinates, we may assume that A is already in this form. In this coordinates, we write $B = (B_1^T \ \ B_2^T)^T$, where B_2 is an $(n-q) \times m$ matrix and we write vectors as $x = (x_1^T, x_2^T)^T$ and $u_2 = (u_{21}^T, u_{22}^T)^T$.

For $r \in (0, 1]$ and $h > 0$, consider the feedback law $(0, F_h^T)$. Then system $(S)_h^r$, with this choice of F_h^T, can be written as

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) - rB_1\sigma(F_h^T x_2(t-h) + u_1(t-h)) + ru_{21}(t-h), \\ \dot{x}_2(t) = A_2 x_2(t) - rB_2\sigma(F_h^T x_2(t-h) + u_1(t-h)) + ru_{22}(t-h). \end{cases}$$

Since A_1 is Hurwitz, it will be sufficient to show that there exists an $r^*(h) \in (0,1]$, such that the x_2- subsystem is finite gain L^p-stable, for all $r \in (0, r^*(h)]$.

The controllability assumption on (A, B) implies that the pair (A_2, B_2) is also controllable. Therefore, the theorem is a consequence of the following proposition.

Proposition 1. *Let σ, u_1, u_2 be as in Theorem 5. Let (A, B) a controllable pair with A skew-symmetric. Then, for every $h \geq 0$, there exist an $n \times m$ matrix F_h and $r^*(h) \in (0,1]$, such that, for every $r \in (0, r^*(h)]$, the system*

$$(S)_h^r : \dot{x}(t) = Ax(t) - rB\sigma[F_h^T x(t-h) + u_1(t-h)] + ru_2(t-h), \text{ for } t \geq 0,$$

verifies the conclusion of Theorem 5.

Sketch of proof. We start the proof by zooming on the single-input case. The general proof first starts with algebraic transformations and proceeds by induction on the number of distinct algebraic multiplicities of the eigenvalues of A.

1) The single-input case: The principal idea is to rephrase the delay systems as problems for perturbed delay-free systems and handle the perturbation by Lyapunov techniques. For this, Let $h > 0$ and consider y the solution of

$$\begin{cases} \dot{y}(t) = (A - rbb^T)y(t) + ru_2(t-h), \text{ for } t \geq 0, \\ y_0 = \bar{0}, \quad \text{on } [-h, 0]. \end{cases} \tag{10}$$

Since A is skew-symmetric, the matrix $A_r := A - rbb^T$ is Hurwitz for every $r > 0$. Then (10) is L^p-stable for any $1 \leq p \leq \infty$. Let γ_p be it is L^p-gain, so $\|y\|_{L^p} \leq \gamma_p \|u_2\|_{L^p}$.

Let x be the solution of $(S)_h^r$ starting at $\bar{0} \in C_h$ and corresponding to u_1, u_2. Set $z := x - y$. Then, z satisfies, for $t \geq 0$,

$$\begin{cases} \dot{z}(t) = Az(t) - rb\left[\sigma\left(F_h^T z(t-h) + \tilde{u}(t-h)\right) - \tilde{v}(t)\right], \\ z_0 = \bar{0}, \quad \text{on } [-h, 0]. \end{cases} \tag{11}$$

where $\tilde{u}(t-h) = F_h^T y(t-h) + u_1(t-h)$ and $\tilde{v}(t) = b^T y(t)$. From (11), we have

$$z(t) = e^{Ah} z(t-h) - r\int_{t-h}^{t} e^{A(t-\xi)} b[\sigma(F_h^T z(\xi - h) + \tilde{u}(\xi - h)) - \tilde{v}(\xi)]d\xi.$$

Then,

$$F_h^T z(t-h) + \tilde{u}(t-h) = b^T z(t) + \tilde{d}(t),$$

where

$$\tilde{d}(t) = \tilde{u}(t-h) + r\int_{t-h}^{t} b^T e^{A(t-\xi)} b[\sigma(F_h^T z(\xi - h) + \tilde{u}(\xi - h)) - \tilde{v}(\xi)]d\xi.$$

Consider the Lyapunov function defined by

$$V_{p,r}(t,z) := \lambda_{p,r} \frac{\|z(t)\|^{p+1}}{p+1} + (z^T(t)P_r z(t))^{\frac{p}{2}} + \mu_{p,r} \int_{t-2h}^{t} (\int_s^t \|z(l)\|^p dl) ds,$$

where P_r is the unique positive-definite solution to the Lyapunov equation

$$X(A - rbb^T) + (A - rbb^T)^T X = -Id_n.$$

The appropriate choice of $\lambda_{p,r}$ and $\mu_{p,r}$ requires careful estimates on P_r as r tends to zero.

We need the next lemma (in order to show ultimately the independence of M_p with respect the parameters r and h).

Lemma 3. *Let A and b be as in Proposition 1. Then, the following properties hold.*

There exists a $r^* \in (0,1]$ such that for all $t \geq 0$,

$$C_1' e^{-C_2' rt} \leq ||e^{(A-rbb^T)t}|| \leq C_1 e^{-C_2 rt}, \quad \forall r \in (0, r^*], \tag{12}$$

for some positive constants C_1, C_1', C_2 and C_2' independent of r, and

$$\lambda_{\max}(P_r) \asymp_0 \lambda_{\min}(P_r) \asymp_0 \frac{1}{r}. \tag{13}$$

We determine a dissipation inequality for $V_{p,r}$, i.e., we take the time derivative of $V_{p,r}(t, x(t))$ along trajectories of $(S)_h^r$. After some computation we get

$$\begin{aligned} \dot{V}_{p,r}(z(t)) \leq & -C_1(r)\|z(t)\|^p + C_2(r)\|z(t)\|^{p-1}[\|\tilde{u}(t-h)\| + \|\tilde{v}(t)\| + \\ & + rC_3 \textstyle\int_{t-h}^t (\|\tilde{u}(\xi - h)\| + \|\tilde{v}(\xi)\|)\, d\xi], \end{aligned} \tag{14}$$

where $C_1(r)$, $C_2(r)$ and C_3 denote constants that are dependent and independent of r. For every $t \geq 0$, integrating (14) from 0 to t and applying Hölder's inequality, we get

$$V_p(z(t)) + C_1(r)\|z\|^p_{L^p[0,t]} \leq (1 + rhC_3)C_2(r)\|z\|^{p-1}_{L^p[0,t]} \times (\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p}). \tag{15}$$

Since $V_{p,r} \geq 0$ and

$$\begin{cases} \|\tilde{v}\|_{L^p} \leq \|b\|\|y\|_{L^p} \leq \gamma_p \|b\|\|u_2\|_{L^p}, \\ \|\tilde{u}\|_{L^p} \leq \|u_1\|_{L^p} + \gamma_p\|b\|\|u_2\|_{L^p}, \\ \|z\|_{L^p} \geq \|x\|_{L^p} - \|y\|_{L^p} \geq \|x\|_{L^p} - \gamma_p\|u_2\|_{L^p}, \end{cases}$$

we get that $x \in L^p([0,\infty), R^n)$ and

$$\|x\|_{L^p} \leq M_p(\|u_1\|_{L^p} + \|u_2\|_{L^p}), \tag{16}$$

where

$$M_p = \max\{\frac{C_2(r)}{C_1(r)}(1 + rhC_3), \gamma_p \left[1 + 2\|b\|\frac{C_2(r)}{C_1(r)}(1 + rhC_3)\right]\}. \tag{17}$$

A careful computation shows that

$$\frac{C_2(r)}{C_1(r)} \asymp_0 (\frac{\lambda_{\max}(P_r)}{\lambda_{\min}(P_r)})^{|\frac{p}{2}-1|} \asymp_0 1, \tag{18}$$

thanks to Lemma 3 and by choosing $rh \leq 1$. In that way, the L^p-gain M_p is delay-independent.

2) The general case: Complete details of the argument of this case are given in [15] In the single input case ($m = 1$), F_h can be chosen as $e^{-Ah}B$, which corresponds, up to the delay h, to the linear feedback law suggested by the passivity approach and used in [1]. A simple adaptation of the proof to the multi-input case shows that such a feedback can also be used to get L^p-stability but the corresponding L^p-gain is delay-independent only for single-input systems. The difference between the single and the multi-input case shows up in (13). In the multi-input case, there are n eigenvalues of $A_r = A - rBB^T$, $\lambda_1(r), \ldots, \lambda_n(r)$, defining continuous functions, which are not analytic in general. These functions, though, can be written as Puiseux series (cf. [7]),

$$\lambda_i(r) = \lambda_i(0) + \sum_{j=1}^{\infty} \alpha_j^{(i)} r^{\frac{j}{p_i}},$$

where $\lambda_i(0)$ is a root of multiplicity ξ of A and p_i is positive integers eventually larger than one. It implies that

$$\lambda_{\max}(P_r) \asymp_0 (\frac{1}{r})^{s_{\max}} \text{ and } \lambda_{\min}(P_r) \asymp_0 (\frac{1}{r})^{s_{\min}},$$

for positive constants $1 \leq s_{\min} \leq s_{\max}$. Therefore, by equation (18) the L^p-gain M_p cannot be delay-independent.

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