Reachability and Steering of Rolling Polyhedra: A Case Study in Discrete Nonholonomy

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***Abstract*—Rolling a ball on a plane is a standard example of nonholonomy reported in many textbooks, and the problem is also well understood for any smooth deformation of the surfaces. For nonsmoothly deformed surfaces, however, much less is known. Although it may seem intuitive that nonholonomy is conserved (think e.g. to polyhedral approximations of smooth surfaces), current definitions of "nonholonomy" are inherently referred to systems described by ordinary differential equations, and are thus inapplicable to such systems. In this paper, we study the set of positions and orientations that a polyhedral part can reach by rolling on a plane through sequences of adjacent faces. We provide a description of such reachable set, discuss conditions under which the set is dense, or discrete, or has a compound structure, and provide a method for steering the system to a desired reachable configuration, robustly with respect to model uncertainties. Based on ideas and concepts encountered in this case study, and in some other examples we provide, we turn back to the most general aspects of the problem and investigate the possible generalization of the notion of (kinematic) nonholonomy to nonsmooth, discrete, and hybrid dynamical systems. To capture the essence of phenomena commonly regarded as "nonholonomic," at least two irreducible concepts are to be defined, of "internal" and "external" nonholonomy, which may coexist in the same system. These definitions are instantiated by examples.**

***Index Terms*—Hybrid systems, motion planning, nonholonomic systems, quantized control systems, reachability analysis.**

I. Introduction

Although nonholonomic mechanics has a long history, dating back at least to the work of Hertz and Hölder toward the end of the 19th century, it is still today a very active domain of research, both for its theoretical interest and its applications, e.g., in wheeled vehicles, robotics, and motion generation. In the past decade or so, a flurry of activity has concerned the study of nonholonomic systems as nonlinear dynamic systems to which control theory methods could be profitably applied. As a result, the control of classical nonholonomic mechanical systems such as cars, trucks with trailers, rolling three-dimensional objects, underactuated mechanisms, satellites, etc., has made a definite progress, and often met a satisfactory level.

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Systems considered in classical nonholonomic mechanics are smooth, continuous-time systems, i.e., they can be described by ordinary differential equations (ODEs) on a smooth (analytic) manifold of configurations, on which smooth (analytic) constraints apply. However, nonholonomic-like behaviors can be recognized in more general systems, including for instance discontinuities of the dynamics, discreteness of the time axis, and discreteness (e.g., quantization) of the input space.

More general systems with nonholonomic features may be used to represent some very general classes of systems and devices of great practical relevance. However, some very basic control problems such as the analysis of reachability and the synthesis of steering control sequences for such systems still pose quite challenging problems, to which, despite some deep analogies that can be shown to exist with continuous nonholonomic systems, known solution techniques from the continuous domain do not extend by any trivial means. For these problems are very hard in general, we focused our initial efforts, reported in this paper, on a practically relevant case study, from which some general insight can be inductively gained.

A. Nonholonomic Behaviors in Nonsmooth Systems

In general, classical nonholonomic constraints come in two varieties, kinematic constraints (often due to contact kinematics as, e.g., in rolling), and dynamic constraints (due to symmetries induced by conservation laws, for instance, of angular momentum) [1], [2]. In this paper, we focus on the former type. Recall the definition of a (smooth) nonholonomic constraint that is familiar from elementary mechanics textbooks: a mechanical system described by coordinates $q \in \mathcal{Q}$, with $\mathcal{Q}$ a smooth n-dimensional manifold, subject to m smooth constraints $A(q)\dot{q} = 0$, is nonholonomic if $A(\cdot)$ is not integrable.

An equivalent description of such systems is often useful, which uses a basis $G(q)$ of the distribution that annihilates $A(q)$ to describe allowable velocities $\dot{q} \in T_q\mathcal{Q}$ as

$$\dot{q} = G(q)u. \quad (1)$$

Thanks to Frobenius' theorem, nonholonomy can thus be investigated by studying the Lie algebra generated by the vector fields in $G(q)$, or, in other terms, by analyzing the geometry of the reachability set of (1). Such simple formulation of kinematic nonholonomic systems is sufficient to illustrate two fundamental aspects of nonholonomy.

1) Elements of $u \in \mathbb{R}^{n-m}$ in (1) play the role of control inputs in a nonlinear, affine-in-control, driftless dynamic system. If the original constraint is nonholonomic, the dimension of the reachable manifold is larger than the number of inputs. This has motivated purposeful intro-

duction of nonholonomy in the design of mechanical devices, to spare actuator hardware while maintaining steerability (see, e.g., [3] and [4]). Notice explicitly that for driftless systems, reachability on a manifold with dimension larger than the dimension of the input space is an essentially nonlinear phenomenon, which is altogether destroyed by linearization, and can be considered as a synonym of nonholonomy.

2) The effects of different consecutive inputs in nonholonomic systems do not commute. In other words, periodic inputs may produce net motions of the system in directions not belonging to the input distribution evaluated at the starting point. This observation is crucial in the interpretation of the role of Lie-brackets in deciding integrability of the system [5].

Behaviors that, by similarity, could well be termed "nonholonomic," may actually occur in a much wider class of systems than mechanical systems with smooth contact constraints or symmetries. Let us refer to general time-invariant dynamic systems as a quintuple $\Sigma = (\mathcal{Q}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$, with $\mathcal{Q}$ denoting the configuration set, $\mathcal{T}$ an ordered time set, $\mathcal{U}$ a set of admissible input symbols, Ω a set of admissible input streams (continuous functions, or discrete sequences) formed by elements in $\mathcal{U}$, and $\mathcal{A}$ a state-transition map $\mathcal{A} : \mathcal{Q} \times \Omega \to \mathcal{Q}$.

It has been observed that in piecewise smooth (p.s.) systems (where time is continuous, $\mathcal{Q}$ is a p.s. manifold, and $\mathcal{A}$ is a p.s. map) with holonomic dynamics within each smooth region, nonholonomic behaviors can arise when the system evolves passing through different smooth regions of the configuration space. Piecewise holonomic systems have been studied rather extensively (related work is reported, e.g., in [6]–[10]). A prominent role in the study of p.s. nonholonomic systems is played by tools from differential geometric control theory (cf. [1] and [2]) and from the theory of stratified manifolds [11].

Nonholonomic behaviors may also be exhibited by discrete-time systems ($\mathcal{T} = \mathbb{N}$). Consider that, if $\mathcal{Q}$ and $\mathcal{U}$ in the system quintuple represent continuous sets, a classical discrete-time control system is described. For such systems, the reachability problem has been already clarified in the literature (see, e.g., [12]–[15]). On the other hand, if $\mathcal{Q}$ and $\mathcal{U}$ are assumed to be discrete sets, then the system essentially represents a sequential machine (automaton). Reachability questions for such systems are an extensively studied topic.

A particularly stimulating problem arises when $\mathcal{Q}$ has the cardinality of a continuum, but $\mathcal{U}$ is quantized (i.e., finite or discrete with values on a regular mesh). Such systems, which will be referred to as quantized control systems (QCS), are encountered in many applications due, e.g., to the need of using finite-capacity digital channels to convey information through an embedded control loop, or to abstract symbolic information from too complex sensorial sources (such as video images in visual serving applications). As a consequence, several researchers devoted their attention to this type of systems (see, e.g., [6] and [16]–[18]). It is important to notice that, while inputs are quantized, the system configurations are not *a priori* restricted to any finite or discrete set: Thus, it may happen that the reachable set has accumulation points, or is dense in the whole space, or in some subsets, or nowhere [19].

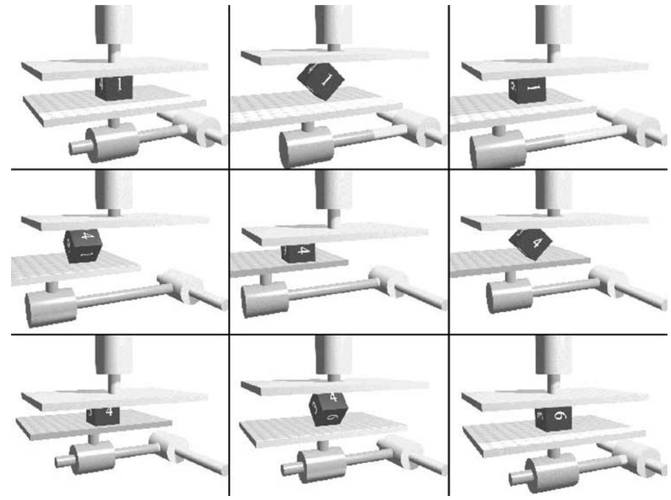

Fig. 1. Die being rolled between two movable parallel plates. The plates can be thought of as the jaws of a robotic gripper, manipulating the polyhedron for reorientation purposes. The sequence illustrates a behavior which could be qualitatively described as nonholonomic.

Chitour and Piccoli [20] have studied a quantized control synthesis problem for the linear case $x^+ = Ax + Bu$, providing sufficient conditions and a constructive technique to find a finite input set $\mathcal{U}$ to achieve a reachability set which is dense in $\mathcal{Q}$. The analysis of the reachability set of a QCS with a given quantized input set $\mathcal{U}$, has been considered in [19] and [21]. In these papers, a complete analysis is achieved for driftless linear systems (while it is pointed out that the problem for general linear systems is as though as some reputedly hard problems in number theory [22]), and for a particular class of driftless nonlinear systems, namely the exact sampled models of n-dimensional chained-form systems [23], which can be considered as the simplest nonholonomic system model.

In this paper, we study and solve the reachability and steering problems for another class of quantized nonholonomic systems, consisting of a polyhedral body rolling on a planar surface. The problem is representative of a more general, and considerably more complex, class of nonholonomic systems than chained form systems, and is thus believed to offer, besides its own interest in applications such as manipulation of industrial parts, further illustration of the nature of the problems and of possible solution techniques.

B. Rolling Polyhedra

Manipulation of polyhedra through rolling by means of robotic end-effectors (see, e.g., Fig. 1), was proposed in [24], in an endeavor to generalize to industrial parts with edges and vertices the manipulation-by-rolling idea that proved effective with regular bodies ([4], [25]). The goal of manipulation is to bring the part from a given initial configuration to another desired one: it is desired to know whether this will be possible for a given pair of configurations, and if so, to provide a method to steer the part. The example of a rolling polyhedron, already mentioned in [19], can be considered as the discrete counterpart of the well known plate-ball system (see, e.g., [4], [26]–[30]).

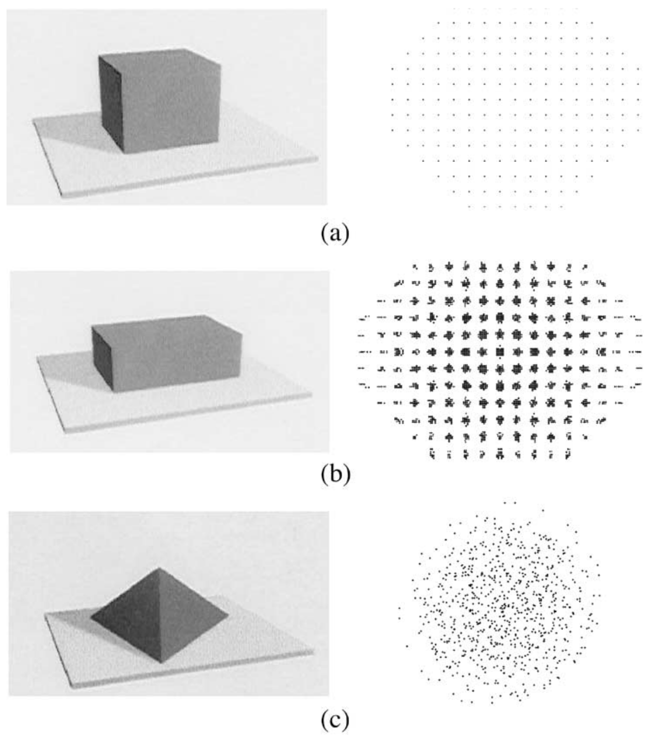

Fig. 2. Sets of positions reached by the centroid of different polyhedra by rolling on a plane in all possible sequences of N turns. Only points lying on a regular grid can be reached by rolling the cube (a), while points reached by rolling the parallelepiped (b) or the polyhedron (c) tend to fill the plane as N grows. Also, consider a line fixed with the polyhedron (not perpendicular to any face), and the angle formed by its projection on the plane with a fixed axis: angles obtained by rolling (a) and (b) only differ by multiples of $\pi/2$, while in case (c) they tend to fill the unit circle as N grows.

The operation of rolling a polyhedron on a planar surface is illustrated in Fig. 1. For this system (to be defined in more detail later), consider input actions as rotations about one of the edges of the face lying on the plate, by exactly the amount that brings an adjacent face on the plate. A first important aspect of the reachability analysis for rolling polyhedra is illustrated in Fig. 2, showing the reachable set in a large number of steps as obtained by direct computation. The fact that for some polyhedra the reachable set has a lattice structure, while for others the set gets denser and denser as manipulation proceeds, is apparent from simulation results. This phenomenon is akin to the one studied in detail for a simpler class of systems in [19]. Rolling polyhedra also exhibit a second interesting phenomenon which clearly bears some resemblance with the nonholonomic behavior of the plate-ball system. Indeed, consider applying first (through suitable forces applied by the upper plate, possibly resorting to compliance and friction) a rotation on the right, hence forward, left and backward (see Fig. 1). While the center of the die after the four actions returns to its initial position, the orientation has changed: input actions do not commute. However, the fact that at each configuration of a polyhedron, only a finite set of actions is available, makes classical definitions of nonholonomy and differential geometric approaches to reachability analysis (such as, e.g., those proposed for discrete-time, continuous control systems in [12]–[15]) altogether unapplicable.

C. *Paper Outline*

In this paper, we consider the reachability problem for rolling polyhedra as a case study for understanding some fundamental nonlinear dynamical effects in discrete and hybrid control systems. The paper is accordingly conceptually organized in two parts: A first part, dealing with the case study, and a second part, dedicated to wider system-theoretic definitions and discussion. In a first reading, some rather technical details of the first part might be skipped to better grasp the overall contribution.

The case-study consists of three sections: A mathematical model of the system is provided in Section II, while Section III presents our results on a classification of the structure of the reachable set in relation with the geometry of the polyhedron. In Section IV, the constructive proofs of these results are exploited to provide a method to steer the polyhedron to any reachable configuration. Of particular interest here is the discussion of robustness of structural results to tolerances in the system description.

The second part consists in Section V, where we turn our attention to the generalization of problems and ideas encountered in the case study, and consider nonholonomic behaviors that may originate in more general (nonsmooth, discrete, or hybrid) systems. Definitions of nonholonomy that generalize the classical notion to discrete and hybrid systems is proposed, along with some related concepts and illustrative examples. A short conclusion section completes the paper.

II. Rolling Polyhedra: Modeling and Main Notations

We consider manipulation of parts that have a piecewise flat, closed surface, comprised of a finite number of faces, edges, and vertices. Observe that actual parts need not be convex, in general. However, the finger plates being assumed to be large w.r.t. the diameter of parts, we need only be concerned with the convex hull of parts themselves.

Several kinds of motions for a polyhedron on a plane are possible such as, e.g., sliding on a face [8], pivoting about a vertex [31], or tumbling about an edge [32], [33]. In this paper, we will consider only the latter possibility, i.e., sequences of rotations about one of the edges in contact, by the amount that exactly brings another face to ground. Such actions will be referred to as *elementary turns*.

The reason for restricting to this kind of rolling actions on the polyhedra is illustrated in Fig. 3. Recall from standard differential geometry [34] that the nonholonomic phase associated with a closed curve on a regular surface is equal to the total curvature of the enclosed region (the total curvature being the integral of the gaussian curvature, which in turn is the product of the principal curvatures). Such phase also represents the net effect on the object orientation of a rolling operation, conducted in such a way that the contact point traces the given closed curve on the object's surface [4]. The same applies to polyhedral surfaces, provided that the gaussian curvature function is replaced by a distribution which is zero everywhere (all planar faces and edges having zero gaussian curvature) except at the vertices, where Dirac's δ-functions of curvature are concentrated. Consider now pivoting (i.e., have the contact point pass through a vertex) with

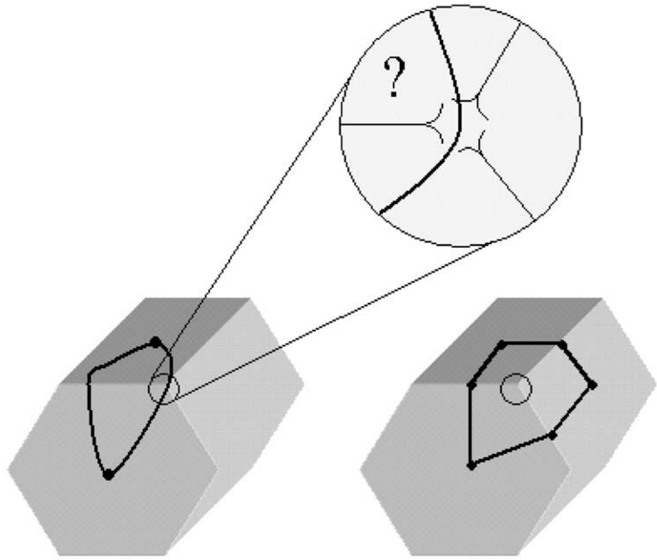

Fig. 3. Illustrating pivoting and turning operations.

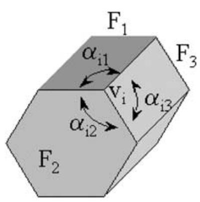

Fig. 4. Defect angle at a vertex equals its total curvature.

a "practical" polyhedron with somewhat smoothed (and imprecisely defined) edges and vertices [see Fig. 3(left)]. The total curvature of the region enclosed within the path of the contact point will depend very sensitively on the particular path and on the uncertain geometry near the vertex, where a large amount of curvature is concentrated. On the other hand, a sequence of turns through all the faces adjacent to the vertex will achieve a net effect equal to the total curvature *at* (ideally) or *near* (practically) the vertex [see Fig. 3(right)], irrespective of those details. It can be easily seen that such vertex curvature is equal to the so-called *defect* angle at the vertex, i.e., the difference between 2π and the sum of all angles between pairs of coplanar edges adjacent to the vertex (see Fig. 4).

In the rest of this section, we will provide a detailed description of the elements of the quintuple $\Sigma = (\mathcal{Q}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$ that models the rolling polyhedra dynamics.

Let us first consider the configuration set $\mathcal{Q}$. Let $\mathcal{P}$ denote a polyhedron rolling on a plane Π, and

- $\mathcal{V} = \{v_1, \ldots, v_h\}$ the set of its vertices;
- $\mathcal{E} = \{e_1, \ldots, e_k\}$ the set of its edges;
- $\mathcal{F} = \{F_1, \ldots, F_r\}$ the set of its faces.

For a general polyhedron, it holds

$$h - k + r = \chi \tag{2}$$

where χ is the Euler–Poincaré characteristic of the surface to which the polyhedron is homeomorphic. We assume that $\mathcal{P}$ is convex and simple, i.e., continuously deformable into a sphere, hence $\chi = 2$ and $h \geq 4$.

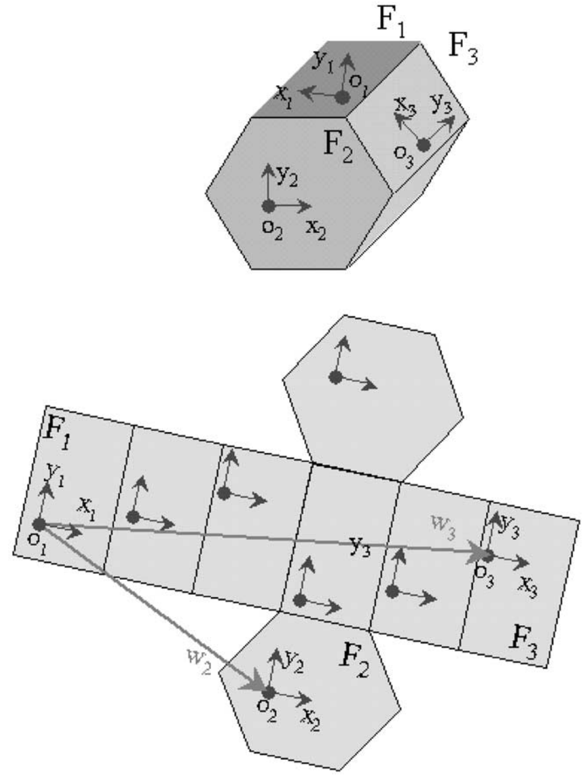

Fig. 5. Development $\mathcal{P}_D$ of a polyhedron $\mathcal{P}$ on the plane Π.

A generic configuration of $\mathcal{P}$ could be identified by giving the index of the face lying on the plane, the position of the projection on the plane of an arbitrarily fixed point in $\mathcal{P}$, and the orientation of the projection of an arbitrarily fixed line in $\mathcal{P}$ (provided the line is not perpendicular to any face). Hence, the configuration set can be identified with the stratified manifold $\mathcal{Q} = \mathbb{R}^2 \times S^1 \times \mathcal{F}$. Although such a description of the configuration set is very direct, it does not produce a convenient set of coordinates to describe the dynamic evolution of a rolling polyhedron, which motivates the introduction of a different description of $\mathcal{Q}$.

A two-dimensional (2-D) Cartesian frame (o_i, x_i, y_i) (o_i denoting the origin) is affixed to each face F_i by the following procedure. Choose a 2-D cartesian frame Oxy fixed on Π. Fix, once for all, a planar development, or "unfolding," of $\mathcal{P}$ on Π (denoted $\mathcal{P}_D$), consisting of a simply connected union of r closed polygons each corresponding to a different face (see Fig. 5), such that two polygons are adjacent in $\mathcal{P}_D$ only if the corresponding faces are adjacent in $\mathcal{P}$ (such a development is always possible, though not unique). Affix to all polygons in $\mathcal{P}_D$ a 2-D cartesian frame (o_i, x_i, y_i) obtained by translation of the frame Oxy of Π to a point o_i of the polygon. This choice gives a unique frame fixed on each face of $\mathcal{P}$ when $\mathcal{P}_D$ is folded back into the original polyhedron. It will be useful to define, for all $j = 2, \ldots, r$, the planar vectors $w_j := o_j - o_1 \in \Pi$ relative to (o_1, x_1, y_1) (see Fig. 5).

A configuration of $\mathcal{P}$ will henceforth be described by a triple $q = (z, \theta, F_i) \in \mathcal{Q} = \mathbb{R}^2 \times S^1 \times \mathcal{F}$, where F_i indicates the face currently on Π, $z \in \mathbb{R}^2$ the coordinates of the point o_i with respect to the frame Oxy fixed on Π, and θ the orientation of (o_i, x_i, y_i) w.r.t. Oxy. On this manifold, a distance can be defined as

$$d((x_1, y_1, \theta_1, F_i) - (x_2, y_2, \theta_2, F_j)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \|\theta_1 - \theta_2\|_{S^1} + \delta(F_i, F_j)$$

where $\|\theta_1 - \theta_2\|_{S^1} = \min\{|\theta_1 - \theta_2 \pmod{2\pi}|, |\theta_2 - \theta_1 \pmod{2\pi}|\}$ is the distance induced by the Riemannian metric on S^1 (inherited from $\mathbb{R}^2$) and $\delta(F_i, F_j) = 0$ if $i = j$, $\delta(F_i, F_j) = \infty$ if $i \neq j$.

As for the time set $\mathcal{T}$ in Σ, given the discrete nature of input actions for the polyhedron, it is natural to consider $\mathcal{T} = \mathbb{N}_+$. Regarding admissible inputs, let us indicate by $(F_i F_j) \in \mathcal{U}$ the elementary turn between two adjacent faces F_i and F_j. If, for $n \geq 2$ and $k = 2, \ldots, n$, F_{j_k} is adjacent to (or coincident with) $F_{j_{k-1}}$, the concatenation of the elementary turns $(F_{j_1} F_{j_2}), \ldots, (F_{j_{k-1}} F_{j_k}) \ldots (F_{j_{n-1}} F_{j_n})$ is denoted by $\omega := (F_{j_1} \cdots F_{j_n})$, and referred to as a *word of length* n.

If F_i is on Π, (F_i) denotes the lack of turns, i.e., F_i remains on Π. The set of all admissible words Ω is clearly a language on the alphabet of the F_i's, such that any two consecutive F_i's in an admissible word correspond to adjacent faces of $\mathcal{P}$. For ω, $\omega' \in \Omega$ such that the last face of ω coincides with the first face of ω', the word $\omega\omega'$ is defined as the concatenation of ω and ω'. The relation $(F_i F_j F_i) = (F_i F_i) = (F_i)$ can be used to reduce words in Ω, i.e., to replace a word with a shorter one which has the same net effect on the polyhedron. For each word $\omega = (F_{j_1} \cdots F_{j_n})$, the word $(F_{j_n} \cdots F_{j_1})$ is clearly admissible and will be denoted by ω^{-1}. Using the relations in Ω we have that $\omega.\omega^{-1} = (F_{j_1})$. Furthermore, for $i, j = 1, \cdots, r$, let

a) Ω_{ij} denote the subset of Ω consisting of words that start at F_i and finish at F_j. If $i = j$, we simply write $\Omega_{ii} = \Omega_i$;

b) $\omega_{ij} \in \Omega_{ij}$ denote a particular word from F_i to F_j, called "transit," which is uniquely defined as follows: if $i = j$ then $\omega_{ii} = (F_i)$; for $1 < j \leq r$, ω_{1j} contains the ordered sequence of faces encountered when moving from F_1 to F_j on $\mathcal{P}_D$, without repetitions; $\omega_{ij} = \omega_{1i}^{-1}.\omega_{1j}$ for $i, j = 1, \ldots, r$.

It follows that $\omega_{ij}^{-1} = \omega_{ji}$ and, for all $k = 1, \ldots, r$, $\omega_{ij} = \omega_{ik}.\omega_{kj}$.

As a consequence of these definitions, each Ω_i is a group for the concatenation with identity element (F_i) and inverse ω^{-1} for each $\omega \in \Omega_i$. Moreover, recalling that equality among words is defined modulo the above relations, one can write $\Omega = \bigcup_{1 \leq i,j \leq r} \Omega_i \omega_{ij}$ (where, by a common slight abuse of notation, the action of a word on a group replaces the action on all the elements of the group). Indeed, any word $\omega = (F_i \ldots F_j)$ can be rewritten $\omega.\omega_{ij}^{-1}.\omega_{ij}$, and $\omega.\omega_{ij}^{-1} \in \Omega_i$. Moreover, we have $\Omega_i = \omega_{i1} \Omega_1 \omega_{1i}$, i.e., every Ω_i is conjugate to Ω_1. We then get that

$$\Omega = \bigcup_{1 \leq i,j \leq r} \omega_{i1} \Omega_1 \omega_{1j}. \tag{3}$$

Let $[F_i]$ denote the set of configurations with face F_i in contact, which can be identified with the manifold $\mathbb{R}^2 \times S^1$. For all $q = (z, \theta, F_i) \in [F_i]$, the same set of admissible inputs is available, namely $\mathcal{U}_q = \{(F_i F_j) : F_j \text{ is a face adjacent to } F_i\}$. The set of admissible input words at $q = (z, \theta, F_i)$ is then $\Omega_q = \bigcup_{1 \leq j \leq r} \Omega_{ij}$.

The description of the quintuple Σ for a rolling polyhedron will be now completed by describing the state-transition map, i.e., the state $\mathcal{A}_q(\omega)$ that the system reaches from q under $\omega \in \Omega_q$.

Let $q = (z, \theta, F_i)$ and $\omega = (F_i \cdots F_k) \in \Omega_q$. Rewrite first ω as the composition of the transit from F_i to F_1 with a word in Ω_1, followed by the transit from F_1 to F_k, i.e. $\omega = \omega_{1i}^{-1} \tilde{\omega} \omega_{1k}$ with $\tilde{\omega} = \omega_{1i} \omega \omega_{1k}^{-1} \in \Omega_1$. Recalling the construction of the plane development of the polyhedron $\mathcal{P}_D$ (see Fig. 5), and the definition of transits, we directly get

$$\mathcal{A}_{(z,\theta,F_i)}\left(\omega_{1i}^{-1}\right) = (z - e^{\jmath\theta} w_i, \theta, F_1) \tag{4}$$

and

$$\mathcal{A}_{(z',\theta',F_1)}(\omega_{1k}) = \left(z' + e^{\jmath\theta'} w_k, \theta', F_k\right). \tag{5}$$

Next, observe that the action of the group Ω_1 of words that start and end with face F_1 on the plane, is clearly a subgroup of $SE(2)$, the Lie group of rigid planar motions (indeed, the same holds for Ω_j, $j = 1, \ldots, r$). Usual rules for composition of two elements g_1, g_2 in $SE(2)$ apply: denoting $g_j = (t_j, \theta_j)$, $t_j \in \mathbb{R}^2$, $\theta \in S^1$, one has

$$g_1 g_2 = (t_1 + e^{\jmath\theta_1} t_2, \theta_1 + \theta_2). \tag{6}$$

Each element $\tilde{\omega} \in \Omega_1$ corresponds then to a unique pair $(\tilde{t}, \tilde{\theta}) \in \mathbb{R}^2 \times S^1$, depending on the polyhedron geometrical parameters, and its action on $[F_1]$ is

$$\mathcal{A}_{(z,\theta,F_1)}(\tilde{\omega}) = (z + e^{\jmath\theta} \tilde{t}, \theta + \tilde{\theta}, F_1). \tag{7}$$

In conclusion, using (4), (5), and (7), we can write

$$\begin{aligned} \mathcal{A}_q(\omega) &= \mathcal{A}_{(z,\theta,F_i)}\left(\omega_{1i}^{-1} \tilde{\omega} \omega_{1k}\right) \\ &= \left(z + e^{\jmath\theta}(\tilde{t} - w_i + e^{\jmath\tilde{\theta}} w_k), \theta + \tilde{\theta}, F_k\right). \end{aligned} \tag{8}$$

III. Reachability Analysis

Consider the reachable set (or *orbit*) from a configuration $q = (z, \theta, F_i)$, defined as

$$\mathcal{R}_q = \{\mathcal{A}_q(\omega) : \omega = (F_i \cdots F_k) \in \Omega_q\}. \tag{9}$$

Thanks to (3) and (8), and with a little abuse of notation, we can write

$$\mathcal{R}_q = \bigcup_{1 \leq j \leq r} \mathcal{A}_q(\omega_{i1}.\Omega_1.\omega_{1j}) \tag{10}$$

hence the reachable set from q can be regarded as the union of r copies of the set

$$\mathcal{R}_q^1 = \mathcal{A}_q(\omega_{i1}.\Omega_1) \tag{11}$$

each copy being translated, rotated and taken to $[F_j]$ by the set of fixed transits ω_{1j}, $1 \leq j \leq r$. Therefore, regarding $\mathcal{A}_q(\omega_{i1})$ as a given element of $[F_1] = \mathbb{R}^2 \times S^1$ on which Ω_1 acts as a Lie subgroup of $SE(2)$, the reachability analysis of the rolling polyhedron system reduce to the following algebraic problem: Study Ω_1 as a subgroup of $(SE(2), \cdot)$, find a set of generators for Ω_1, hence decide whether Ω_1 is dense in $SE(2)$ or not, and if not, investigate its structure.

In this section, we first show that Ω_1 is indeed a finitely generated free group, and provide explicitly a finite set of generators along with their actions on $\mathcal{Q}$ (Section III-A). Next, by analyzing the action of Ω_1 on S^1, we reduce the study of Ω_1 to that of its normal subgroup $\mathcal{H}_1$, which is the subgroup of translations, and give a general result regarding all possible structures of the reachable set (Section III-B). Finally, we end up the section by carefully studying the reachable set when it turns out to be discrete (Section III-C).

A. Study of Ω_1

1) Description of Ω_1 as a Finitely Generated Free Group: We use in this paragraph standard definitions and results of graph theory (cf. [35]) and algebraic topology (cf. [36]), which are reported in the Appendix for the reader's convenience.

To a polyhedron $\mathcal{P}$, we associate a graph $G_{\mathcal{P}} = (N_{\mathcal{P}}, E_{\mathcal{P}})$ such that

i) $N_{\mathcal{P}} = \mathcal{F}$ (each node in $G_{\mathcal{P}}$ corresponds uniquely to a face of the polyhedron $\mathcal{P}$);
ii) $E_{\mathcal{P}} = \{(F_i, F_j) | F_i \text{ adjacent to } F_j\}$ (an edge exists only between nodes corresponding to adjacent faces in $\mathcal{P}$). It can be shown (see the Appendix) that, for a convex polyhedron, the associated graph $G_{\mathcal{P}}$ is a simple planar connected graph (see Fig. 6).

As a consequence of this esult, there is a one-to-one correspondence between faces of $G_{\mathcal{P}}$ and vertices of $\mathcal{P}$. Being the number of nodes in $G_{\mathcal{P}}$ equal to r and the number of faces in $G_{\mathcal{P}}$ equal to h, by using the Euler relation (2) we also get that the number of edges in $G_{\mathcal{P}}$ is equal to k.

The group of words Ω_1 discussed previously can, hence, be identified with the *fundamental group* of the graph $G_{\mathcal{P}}$ with base node F_1. The classical result reported in the Appendix, Proposition 8, can be rephrased in this context as follows: any element of Ω_1 can be rewritten as an integer combination (by concatenation) of a finite number of generator words in Ω_1.

Let us apply Proposition 9 (in the Appendix) to the dual graph of $S(\mathcal{P})$ (see Fig. 6), namely $G_{\mathcal{P}} = (S(\mathcal{P}))_d$, observing first that the number f of faces of the planar graph is equal to the number of vertices of $\mathcal{P}$, i.e., $f = h$. A generator set $A_{G_{\mathcal{P}}} = \{\alpha_\lambda : \lambda = 1, \cdots, f-1\}$, is given by $\alpha_\lambda = \alpha_\lambda^{n_\lambda} C_\lambda (\alpha_\lambda^{n_\lambda})^{-1}$, where C_λ is a cycle encompassing exactly one bounded face of $G_{\mathcal{P}}$.

In terms of our previous notation of input words, such a generator α_λ corresponds to a word of type $R_\lambda = \omega_{1j_\lambda} \hat{R}_\lambda \omega_{j_\lambda 1}$, $1 \leq \lambda \leq h-1$. Here, $\hat{R}_\lambda$ is a word starting and finishing at some face F_{j_λ} of the polyhedron adjacent to the vertex v_λ, and including all faces which are adjacent to the vertex v_λ, in the order in which they are encountered turning around the vertex. Note also that ω_{1j_λ} is the transit word from F_1 to F_{j_λ} in $\mathcal{P}_D$. We finally obtain the following equivalent: Characterizations of Ω_1

i) Ω_1 is a free group generated by $h-1$ generators $R_{\lambda_1}, \cdots, R_{\lambda_{h-1}}$ corresponding to $h-1$ distinct vertices of $\mathcal{P}$;

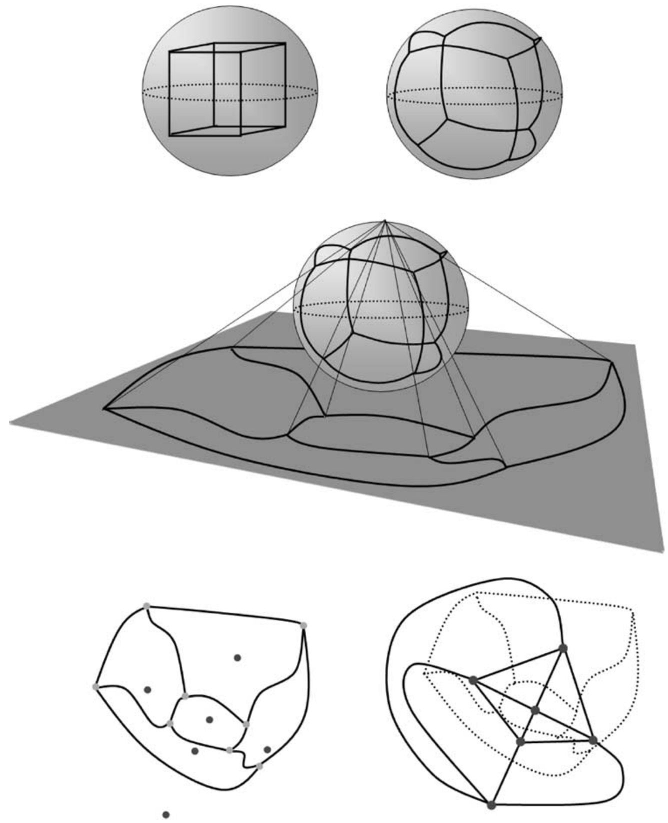

Fig. 6. Geometric construction underlying the association of a planar graph $G_{\mathcal{P}}$ to a convex polyhedron $\mathcal{P}$. Edges of $\mathcal{P}$ are projected from an inner point onto a circumscribed sphere (top), hence, stereographically onto a plane (middle). A partition $S(\mathcal{P})$ is thus induced in the plane (bottom left), with the same number of faces, edges, and vertices as $\mathcal{P}$ (including the unbounded, outer face). $S(\mathcal{P})$ can be regarded as a simple planar graph by identifying its vertices and edges with graph nodes and edges, respectively. The dual graph $G_{\mathcal{P}}$ (bottom right) is also simple and planar.

ii) for all $\omega \in \Omega_1$, there exists $N \in \mathbb{N}$ such that

$$\omega = \prod_{k=1}^{N} R_{j_k}^{\varepsilon_k} \tag{12}$$

with $j_k \in \{\lambda_1, \cdots, \lambda_{h-1}\}$ and $\varepsilon_k = \pm 1$.

2) Action of the Generators of Ω_1 on the Polyhedron: From the previous definition of $\hat{R}_\lambda$, it follows that, if $\mathcal{P}$ is initially lying on F_{j_λ}, then the effect of $\hat{R}_\lambda$ on $\mathcal{Q}$ is a rotation about an axis perpendicular to Π through v_λ (which lies on Π throughout the action of $\hat{R}_\lambda$), by an angle, denoted by β_λ, equal to the defect angle at $v_\lambda \in \mathcal{P}$, that is $\beta_\lambda = 2\pi - \sum_{j_\lambda} \alpha_{\lambda j_\lambda}$, where the sum runs over all j_λ such that F_{j_λ} is adjacent to v_λ and $\alpha_{\lambda j_\lambda}$ is the angle between two edges adjacent to the vertex v_λ and belonging to the same face F_{j_λ} (see Fig. 4). The following proposition highlights a useful property of the defect angles of a polyhedron.

Proposition 1: Let β_λ, $\lambda = 1, \ldots, h$ be the defect angles of $\mathcal{P}$. They satisfy

$$\sum_{\lambda=1}^{h} \beta_\lambda = 4\pi. \tag{13}$$

Proof: The previous equation can be deduced from the definition of the defect angle and the Euler relation (2). We have

$$\sum_{\lambda=1}^{h} \beta_\lambda = 2\pi h - \sum_{\lambda,j} \alpha_{\lambda j}.$$

We have $\sum \alpha_{\lambda j} = \sum_{j=1}^{r} \gamma_j$ where $\gamma_j = \sum_\lambda \alpha_{\lambda j}$ is the sum of inner angles of face F_j. Denoting by $e_j^{\#}$ the number of edges of face F_j, and recalling that $\gamma_j = (e_j^{\#} - 2)\pi$, we get $\sum \alpha_{\lambda j} = \pi \sum_{j=1}^{r} e_j^{\#} - 2\pi r$.

Observe now that in $\sum_{j=1}^{r} e_j^{\#}$, each edge of $\mathcal{P}$ is counted twice, as it belongs to two different faces. Therefore, $\sum_{j=1}^{r} e_j^{\#} = 2k$. We conclude that

$$\sum_{\lambda=1}^{h} \beta_\lambda = 2\pi(h - k + r) = 4\pi.$$

■

Let the 2-vector ${}^{j_\lambda}v_\lambda$ denote the position of the vertex v_λ with respect to the reference frame $(o_{j_\lambda}, x_{j_\lambda}, y_{j_\lambda})$ affixed onto face F_{j_λ}. Then, simple geometric calculations show that the action of $\hat{R}_\lambda$ is described as an element of $SE(2)$ by $((1 - e^{\jmath\beta_\lambda})^{j_\lambda}v_\lambda, \beta_\lambda)$, or, equivalently, that $\mathcal{A}_{(z,\theta,F_{j_\lambda})}(\hat{R}_\lambda) = (z + e^{\jmath\theta}(1 - e^{\jmath\beta_\lambda})^{j_\lambda}v_\lambda, \theta + \beta_\lambda, F_{j_\lambda})$.

More generally, the action of words of type $R_\lambda = \omega_{1j_\lambda}\hat{R}_\lambda\omega_{j_\lambda 1}$, is described by $((1 - e^{\jmath\beta_\lambda})^{1}v_\lambda, \beta_\lambda) \in SE(2)$, or

$$\mathcal{A}_{(z,\theta,F_1)}(R_\lambda) = \left(z + e^{\jmath\theta}(1 - e^{\jmath\beta_\lambda})^{1}v_\lambda, \theta + \beta_\lambda, F_1\right) \quad (14)$$

where ${}^1v_\lambda$ is the 2-vector from the origin o_1 of the frame affixed to face F_1 to the image of v_λ as a point of F_{j_λ} on the planar development $\mathcal{P}_D$, in coordinates (o_1, x_1, y_1).

It should be pointed out explicitly that the actions of both $\hat{R}_\lambda$ and R_λ are dependent on which face F_{j_λ} is considered. However, without any loss of generality, we will henceforth regard every vertex v_λ as associated to one of its adjacent faces, or, which is equivalent, all copies of each vertex will be removed in the planar development of the polyhedron except for one. Such an arbitrary choice is tantamount to taking a particular set of generators of the free group Ω_1, which is not going to alter the ensuing study of the group orbit.

B. Structure for Reachable Sets of a Rolling Polyhedron

1) Dense Structure and Virtual Vertex: Recall that to each element $\tilde{\omega}$ of Ω_1 there corresponds a unique element $(\tilde{t}, \tilde{\theta})$ of $SE(2)$. Let then $a : \tilde{\omega} \in \Omega_1 \mapsto (\tilde{t}, \tilde{\theta}) \in SE(2)$ be the group homomorphism defined by

$$a(\tilde{\omega}) : \mathbb{R}^2 \times S^1 \times \{F_1\} \to \mathbb{R}^2 \times S^1 \times \{F_1\}$$
$$a(\tilde{\omega})(z, \theta, F_1) = \mathcal{A}_{(z,\theta,F_1)}(\tilde{\omega}) = (z + e^{\jmath\theta}\tilde{t}, \theta + \tilde{\theta}, F_1).$$

Let $\pi_2 : SE(2) \to S^1$ be the projection on the second factor. Recall that $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is an Abelian Lie group. Then, π_2 is a Lie group homomorphism, i.e., for every $g, g' \in G$, we have $\pi_2(g.g') = \pi_2(g)\pi_2(g')$ and π_2 is continuous. Then $\pi_2(a(\Omega_1))$ is the subgroup of S^1, generated by the β_λ's, $1 \leq \lambda \leq h - 1$. Thanks to (13), it is evident that β_h is generated by β_λ, $\lambda = 1, \ldots, h - 1$. Subgroups of S^1 are well studied (cf. [37]), and some useful definitions are recalled here.

Let G be a group and i_G its identity element. We use $\langle g_1, \cdots, g_s \rangle$ to denote the subgroup of G generated by $g_1, \ldots, g_s \in G$. The order of an element g of G is the smallest integer $n \in \mathbb{N}$ such that $g^n = i_G$. We write $o_G(g) = n$. If no finite integer exists such that $g^n = i_G$, we let $o_G(g) = +\infty$. The order of a group G is the smallest positive integer $\bar{n}$ such that $g^{\bar{n}} = i_G$, $\forall g \in$ and we write $o(G) = \bar{n}$. If there exists some $g \in G$ such that $o_G(g) = +\infty$, then we let $o(G) = +\infty$. Otherwise, $o(G) = \mathrm{l.c.m.}_{g \in G}(o_G(g)) < +\infty$, where l.c.m. stands for least common multiple.

All possible structures of $\pi_2(a(\Omega_1))$ are captured by the following proposition, which is a direct consequence of a classical result from the theory of Diophantine approximation (cf. [37]).

Proposition 2: Let $\pi_2(a(\Omega_1))$ be the subgroup of S^1 defined previously. Then, one of the two following cases occurs.

1) If $(\beta_\lambda/\pi) \notin \mathbb{Q}$ for at least one defect angle, then $\pi_2(a(\Omega_1))$ is dense in S^1;
2) If $(\beta_\lambda/\pi) \in \mathbb{Q}$ for all the defect angles, then there exists a positive integer p such that

$$\pi_2(a(\Omega_1)) = \left\langle \frac{2\pi}{p} \right\rangle.$$

In case 1), the following result also holds.

Proposition 3: Assume that $(\beta_\lambda/\pi) \notin \mathbb{Q}$ for at least one defect angle. Then, for every $q \in \mathcal{Q}$, the reachable set from q, $\mathcal{R}_q$ is dense in $\mathcal{Q}$.

Proof: If $(\beta_\lambda/\pi) \notin \mathbb{Q}$, $\pi_2(a(\langle R_\lambda \rangle))$ is dense in S^1. This implies that the polyhedron can be turned about an axis perpendicular to Π through v_λ (the vertex whose defect is β_λ) so as to reach as close as desired to any given orientation. On the other hand, (13) guarantees that if $(\beta_\lambda/\pi) \notin \mathbb{Q}$ for some λ, then $(\beta_{\lambda'}/\pi) \notin \mathbb{Q}$ for some $\lambda' \neq \lambda$. Therefore, the polyhedron can pivot about two different vertices v_λ and $v_{\lambda'}$, thus achieving arbitrary motions in the plane. Proposition 3 readily follows. ■

For the rest of this section, we study case 2). For $1 \leq \lambda \leq h$, we can write $\beta_\lambda = 2\pi(m_\lambda/p_\lambda)$ with $1 \leq m_\lambda < p_\lambda$ two coprime positive integers. Then, each $\beta_\lambda \in \pi_2(a(\Omega_1))$ is of order p_λ and

$$o(\pi_2(a(\Omega_1))) = \mathrm{l.c.m.}_{1 \leq \lambda \leq h}(p_\lambda).$$

Let $p = \mathrm{l.c.m.}(p_\lambda)(p \geq 2)$ and denote $d_\lambda = (p/p_\lambda)$ for $1 \leq \lambda \leq h$, then we have that any element $\theta \in \pi_2(a(\Omega_1))$ can be written

$$\theta = \beta \left(\sum_{1 \leq \lambda \leq h-1} n_\lambda m_\lambda d_\lambda \right)$$

for arbitrary $n_\lambda \in \mathbb{Z}$ and $\beta = 2\pi/p$. Since the d_λ, m_λ's are coprime, we get that $\pi_2(a(\Omega_1)) = \langle \beta \rangle$, i.e.,

$$\pi_2(a(\Omega_1)) = \{\theta = k\beta \pmod{2\pi}, k \in \mathbb{Z}\}. \quad (15)$$

We call β the quantization angle and denote $\mathcal{Q}_{k\beta}$ the set of configurations $(z, \theta, F_1) \in [F_1]$ such that $\theta = k\beta$.

Fix $n_\lambda \in \mathbb{Z}$, $1 \leq \lambda \leq h - 1$, such that

$$1 = \sum_{1 \leq \lambda \leq h-1} n_\lambda m_\lambda d_\lambda$$

and define $R_0 = \prod_{\lambda=1}^{h-1} R_\lambda^{n_\lambda}$. Then it holds $R_0 = (t_0, \beta)$, for some $t_0 \in \mathbb{R}^2$. Note also that the n_λ's do not depend on the choice of the reference point on F_1. Let $v_0 \in \Pi$ be defined by

$$v_0 = \left(1 - e^{\jmath\beta}\right)^{-1} t_0. \tag{16}$$

We can thus write $R_0 = ((1 - e^{\jmath\beta})v_0, \beta)$. Notice that R_0 acts as if it were a rotation about a point whose projection on $\mathcal{P}_D$ would be $v_0 \in F_1$, in coordinates (o_1, x_1, y_1). We will refer to such a point v_0 as to the *virtual vertex*. Moreover, denoting

$$\mathcal{H}_1 = \{(t, 0) \in a(\Omega_1)\}$$

the set of translations, we get the following.

Corollary 1: For every $l \in \Omega_1$, there exist $k_l \in \mathbb{Z}$ and $T_l \in \mathcal{H}_1$ such that

$$l = R_0^{k_l}.T_l. \tag{17}$$

Proof: Let $l \in \Omega_1$. We have $l = (t_l, \theta_l)$ with $\theta_l = k_l\beta$, $k_l \in \mathbb{Z}$. Then, $R_0^{-k_l}.l \in \mathcal{H}_1$. Setting $T_l = R_0^{-k_l}.l$, we get the conclusion. ■

2) Structure of the Translation Group $\mathcal{H}_1$: In order to fully determine the structure of the reachable set of a rolling polyhedron in case 2) holds, following from Corollary 1 it remains to investigate if the projection on the first factor of $\mathcal{H}_1$ is dense in $\mathbb{R}^2$. For such purpose, we introduce the symmetry angle $\alpha = \pi/p' \in S^1$ with $p' = p$ if p is odd and $p' = p/2$ if p is even. Such a definition is motivated by the next proposition.

Proposition 4: The translation group $\mathcal{H}_1$ is invariant by a rotation of angle α, i.e., if $(t, 0) \in \mathcal{H}_1$, then $(e^{\jmath\alpha}t, 0) \in \mathcal{H}_1$.

Proof: To simplify the notation, we assume here that the reference point on F_1 coincides with the virtual vertex v_0, hence that $R_0 = (0, \beta)$. Let $T = (t, 0) \in \mathcal{H}_1$. Since $\mathcal{H}_1$ is a group, $-T = (-t, 0) \in \mathcal{H}_1$. Moreover, for every $l \in \mathbb{Z}$, $\pm R_0^l T R_0^{-l}$ belongs to $\mathcal{H}_1$. Let

$$(t_{l,-}, 0) = -R_0^l T R_0^{-l} \tag{18}$$

with

$$l = \begin{cases} \frac{p'+1}{2}, & \text{if } p \text{ is odd} \\ p'+1, & \text{if } p \text{ is even.} \end{cases}$$

An easy computation shows that $t_{l,-} = e^{\jmath\alpha}t$. ■

In the sequel, we identify $\mathcal{H}_1$ with its projection on the first factor i.e. with a subset of $\mathbb{R}^2$. We will now give a simple set of generators of $\mathcal{H}_1$. Let G_{n_1} be defined by

$$G_{n_1} = \left\{ e^{\jmath u\alpha} R_0^{-m_\lambda d_\lambda} R_\lambda : 1 \le \lambda \le h-1,\ 0 \le u \le p'-1 \right\}. \tag{19}$$

Proposition 5: The group of translations $\mathcal{H}_1$ is an Abelian subgroup of $\mathbb{R}^2$ generated by the elements of G_{n_1}.

Proof: Let $G_1 = \langle G_{n_1} \rangle \subset \mathcal{H}_1$. First note that, for $1 \le \lambda \le h-1$, we have $R_\lambda R_0^{-m_\lambda d_\lambda}$ is a translation and

$$R_\lambda R_0^{-m_\lambda d_\lambda} = e^{\jmath u\alpha} R_0^{-m_\lambda d_\lambda} R_\lambda$$

where $u = -2m_\lambda d_\lambda$, if p is odd or $u = -m_\lambda d_\lambda$, if p is even and then $R_\lambda R_0^{-m_\lambda d_\lambda} \in G_{n_1}$.

According to (12), we can write every $T \in \mathcal{H}_1$ as

$$T = \prod_{k=1}^{N} R_{j_k}^{\varepsilon_k}.$$

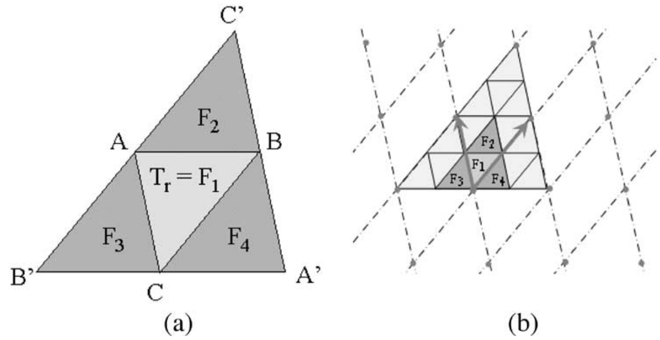

Fig. 7. Polyhedra whose defect angles are multiples of $\beta = \pi$ are tetrahedra with isometric faces, whose plane development is a triangle similar to each face (a). For such polyhedra, the reachable set is a lattice, irrespective of the edge lengths. G_{n_2} reduces to two elements, and $\mathcal{H}_1$ is a nondegenerate lattice (b).

We rewrite the previous equation as

$$T = \prod_{k=1}^{N} R_0^{\varepsilon_k m_{j_k} d_{j_k}} \left(R_0^{-\varepsilon_k m_{j_k} d_{j_k}} R_{j_k}^{\varepsilon_k} \right).$$

Notice that if $T \in G_1$, then $R_0^u T R_0^{-u} = e^{\jmath u\beta}T \in G_1$ for all $u \in \mathbb{Z}$. Using this fact, we have that T is equal to the product of $R_0^{N_T}$ with $N_T = \sum_{k=1}^{N} \varepsilon_k \beta_{j_k} \in \mathbb{Z}$ and a finite number of elements of G_1. We hence get that T is congruent, modulo G_1, to $R_0^{N_T}$. Since $T \in \mathcal{H}_1$, we have $\sum_{k=1}^{N} \varepsilon_k \beta_{j_k} = 0$ and $R_0^{N_T} = (0, 0) \in \mathcal{H}_1$, hence, we have that T is congruent, modulo G_1, to 0 i.e. $T \in G_1$. Then $\mathcal{H}_1 \subset G_1$, completing the proof. ■

For $1 \le \lambda \le h-1$, let z_λ be the translation vector corresponding to $R_0^{-m_\lambda d_\lambda} R_\lambda$. We have

$$z_\lambda = (1 - e^{-\jmath\beta_\lambda})(v_0 -^1 v_\lambda). \tag{20}$$

Then, (the projection on the first factor of) $\mathcal{H}_1$ is generated by

$$G_{n_2} = \{e^{\jmath u\alpha} z_\lambda, \quad 1 \le \lambda \le h-1, \quad 0 \le u \le p'-1\}. \tag{21}$$

A standard result on the classification of Abelian subgroups G of $\mathbb{R}^2$ asserts that one of the three possibilities can occur (cf. [37])

a) G is a lattice i.e. $G = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where e_1 and e_2 are two linearly independent vectors of $\mathbb{R}^2$;
b) $G = \widetilde{G} \oplus \mathbb{Z}e_2$ where $\widetilde{G}$ is a dense subgroup of $\mathbb{R}e_1$ with e_1 and e_2 two linearly independent vectors of $\mathbb{R}^2$;
c) G is dense in $\mathbb{R}^2$.

More generally, we use $\mathcal{L}(a, b)$ to denote the lattice of $\mathbb{R}^2$ generated by the pair of vectors a, b. We say that $\mathcal{L}(a, b)$ is nondegenerate if a, b are linearly independent.

We will now show that case b) cannot actually occur. We first show that if $p' = 1$, i.e., $\alpha = \beta = \pi$, then case a) occurs. Indeed, since $h \ge 4$, then $\alpha = \pi$, by (13) implies that $h = 4$, $\mathcal{P}$ is a tetrahedron and all the β_λ's are equal to π. We deduce that the virtual vertex can be actually taken to be any existing vertex and G_{n_2} reduces to two elements. Then $\mathcal{H}_1$ is a nondegenerate lattice (see Fig. 7). Assume next that $p' > 1$ and then $0 < \alpha < \pi$. If case (b) holds, then $\mathcal{H}_1 = \widetilde{\mathcal{H}}_1 \oplus \mathbb{Z}e_2$ with $\widetilde{\mathcal{H}}_1$ a dense subgroup of $\mathbb{R}e_1$, $e_1 \neq 0$. By Proposition 4, $\mathcal{H}_1$ contains $e^{\jmath\alpha}\widetilde{\mathcal{H}}_1$. Since e_1 and $e^{\jmath\alpha}e_1$ are linearly independent, $\mathcal{H}_1$ is dense in $\mathbb{R}^2$ and we obtain a contradiction. Therefore, we have proved that

Lemma 1: Let $\mathcal{H}_1$ be the group of translations of Ω_1. Then, either $\mathcal{H}_1$ is dense in $\mathbb{R}^2$ or it is a nondegenerate lattice.

Remark 1: Notice that it is now easy to check whether $\mathcal{H}_1$ is a lattice or not. Indeed, when $\beta = \pi$, this is the case. If $\beta < \pi$, one of the z_λ's is not zero, let say z_1. Then z_1 and $e^{\jmath\alpha}z_1$ are linearly independent, i.e., they define a basis B of $\mathbb{R}^2$. We can therefore decompose every element of G_{n_2} in B. By a classical result of Diophantine approximation, we get that $\mathcal{H}_1$ is a lattice if and only if every element of G_{n_2} is written in B with rational coordinates.

A classical result on lattices (cf. [37]) says that, given a nondegenerate lattice $\mathcal{L}(a,b)$, we have

$$\mu = \inf_{t\in\mathcal{H}_1, t\neq 0} \|t\| > 0$$

and there exists $t_{\min} \in \mathcal{H}_1$ so that $\|t_{\min}\| = \mu$. We call such $t_{\min}$ a shortest element of $\mathcal{L}(a,b)$. We thus have the following result.

Lemma 2: Assume that $\mathcal{H}_1$ is a lattice with quantization angle $\beta = 2\pi/p$, $p \geq 2$. Then

$$\beta \in \mathcal{D} = \left\{\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}\right\}. \tag{22}$$

Proof: Recall that the symmetry angle $\alpha = \pi/p'$ is smaller than π. Let $t_{\min}$ be a shortest element of $\mathcal{H}_1$. By Proposition 4, $e^{\jmath\alpha}t_{\min} \in \mathcal{H}_1$ and then $t = (e^{\jmath\alpha} - 1)t_{\min} \in \mathcal{H}_1$. Since $\|t\| \geq \mu = \|t_{\min}\| > 0$, we must have $|e^{\jmath\alpha} - 1| \geq 1$. Then, p' can only take the values 1, 2, or 3. Going back to the definition of α, we get (2). ■

We deduce from the previous lemma that

Lemma 3: Assume that $\mathcal{H}_1$ is a lattice with quantization angle $\beta \in \mathcal{D}$ where $\mathcal{D}$ was defined in (22). Let $t_{\min}$ be a shortest element of $\mathcal{H}_1$. Then either $\beta = \pi$ or

$$\mathcal{H}_1 = \mathcal{L}(t_{\min}, e^{\jmath\alpha}t_{\min}) \tag{23}$$

and α is equal to $\pi/2$ or $\pi/3$.

Proof: We assume that $\beta < \pi$. Let $t_{\min}$ be a shortest element of $\mathcal{H}_1$. We use $\mathcal{L}_0$ to denote $\mathcal{L}(t_{\min}, e^{\jmath\alpha}t_{\min})$. We have of course $\mathcal{L}_0 \subset \mathcal{H}_1$ and, by Remark 1, every element t of $\mathcal{H}_1$ can be written

$$t = \frac{r_1}{s_1}t_{\min} + \frac{r_2}{s_2}e^{\jmath\alpha}t_{\min}$$

where r_1/s_1 and r_2/s_2 are rational. Let $t' \in \mathcal{L}_0$ such that $t' = n_1 t_{\min} + n_2 e^{\jmath\alpha}t_{\min}$ where n_1 and n_2 are the nearest integers to r_1/s_1 and r_2/s_2, respectively. Then, $t - t' \in \mathcal{H}_1$ and verifies

$$t - t' = \frac{r_1'}{s_1'}t_{\min} + \frac{r_2'}{s_2'}e^{\jmath\alpha}t_{\min}$$

with $|r_1'/s_1'w|$,$|r_2'/s_2'| \leq 1/2$. By taking norms, we obtain

$$\|t - t'\|^2 \leq \left(\frac{1}{4} + \frac{1}{4} + 2\cos(\alpha)\frac{1}{4}\right)\|t\|^2 \leq \frac{3}{4}\mu^2.$$

By definition of μ, we must have $\|t - t'\| = 0$, i.e., $t \in \mathcal{L}_0$. Since t is an arbitrary element of $\mathcal{H}_1$, we conclude. ■

Thanks to (10), (11), (21), Proposition 3, and the two previous lemmas, we are now in a position to state our main result concerning reachability of rolling polyhedra. Let us recall from [19] few useful definitions for quantized control systems (see also Fig. 2): We say that a QCS is *approachable* if closure$(\mathcal{R}_q) = \mathcal{Q}$, $\forall q \in \mathcal{Q}$. On the other hand, the reachable set $\mathcal{R}_q$ is *discrete* if it is nowhere dense, and *dense in a subset* $\mathcal{Q}' \subset \mathcal{Q}$ if closure$(\mathcal{R}_q) \cap \mathcal{Q}' = \mathcal{Q}'$, $\forall q \in \mathcal{Q}$. To describe the coarseness of discrete reachable sets, we talk of *ϵ-approachability* of a configuration q' from q whenever $\exists\omega \in \Omega_q$, such that $d(\mathcal{A}_q(\omega), q') < \epsilon$. The set of configurations that are ϵ-approachable from q is denoted by $\mathcal{R}_q^\epsilon$. The system is said to be ϵ-approachable if $\mathcal{R}_q^\epsilon = \mathcal{Q}$, $\forall q \in \mathcal{Q}$.

Theorem 1: Let $q = (z_q, \theta_q, F_{j_q}) \in \mathcal{Q}$. The possible structures for the reachable set from q, $\mathcal{R}_q \subset \mathcal{Q}$ are the following.

a) If at least one defect angle is irrational with π, then $\mathcal{R}_q$ is dense in $\mathcal{Q}$ and the system is approachable, i.e. closure$(\mathcal{R}_q) \cap [F_i] = [F_i]$, $\forall q \in \mathcal{Q}$ and $\forall i = 1, \ldots, r$.

b) If all defect angles are rational with π and $\beta = 2\pi/p$ is the quantization angle, then either

(b_1) $\mathcal{H}_1$ is dense in $\mathbb{R}^2$, hence $\mathcal{R}_q$ is dense in $\mathcal{Q}_{k\beta} \subset \mathcal{Q}$, i.e. closure$(\mathcal{R}_q) \cap \mathcal{Q}_{k\beta} = \mathcal{Q}_{k\beta} \forall q \in \mathcal{Q}$ and $\forall k = 1, \ldots, p$, or

(b_2) $\mathcal{H}_1$ is a nondegenerate lattice and, hence, $\mathcal{R}_q = \bigcup_{j=1}^r \mathcal{R}_q^j$, where each $\mathcal{R}_q^j$ is isometric to

$$\bigcup_{k=0}^{p-1} (t_0(k) + \mathcal{H}_1, k\beta, F_1) \tag{24}$$

where $t_0(0) = 0$ and $t_0(k) = t_0(1 + e^{\jmath\beta} + \cdots + e^{\jmath(k-1)\beta})$, $k = 1, \ldots, p-1$, is the $\mathbb{R}^2$ component of R_0^k.

Moreover, within case (b_2), we necessarily have that either

(b_{21}) $p = 2$ and, hence, all the β_λ's are equal to π and $\mathcal{P}$ is a tetrahedron; in this case, the system is ϵ-approachable with $\epsilon = \epsilon_{S^1} + \epsilon_{\mathbb{R}^2}$ where $\epsilon_{S^1} = \pi/2$ and $\epsilon_{\mathbb{R}^2} = \max\{\|z_1 + z_2\|/2, \|z_1 - z_2\|/2\} = \max\{\|(v_2 - v_1) + (v_3 - v_1)\|/2, \|(v_2 - v_3)\|/2\}$, where the z_λ's and the v_λ's are defined in (20) and (14), respectively, or

(b_{22}) $p = 3, 4, 6$ and, hence, there exists $t_p \neq 0$ such that $\mathcal{H}_1 = \mathcal{L}(t_p, e^{\jmath(\pi/3)}t_p)$ if $p = 3, 6$ or $\mathcal{H}_1 = \mathcal{L}(t_p, e^{\jmath(\pi/2)}t_p)$ if $p = 4$. In this case, the system is ϵ-approachable with $\epsilon = \epsilon_{S^1} + \epsilon_{\mathbb{R}^2}$ where $\epsilon_{S^1} = \pi/p$ and $\epsilon_{\mathbb{R}^2} = (\sqrt{3}/3)\|t_p\|$, if $p = 3, 6$, or $\epsilon_{\mathbb{R}^2} = (\sqrt{2}/2)\|t_p\|$, if $p = 4$.

Proof: It only remains to prove (24). We start with an arbitrary point $q \in \mathcal{Q}$. Using (10) and (11), we let $\omega_{j_q 1}$ act on q. It is then enough to consider points $q \in [F_1]$ given by $q = (z_q, \theta_q, F_1)$. Since we have

$$\mathcal{R}_q^1 = \bigcup_{k=0}^{p-1} \left(z_q + e^{\jmath\theta_q}\left(\mathcal{H}_1 + t_0(k)\right), \theta_q + k\beta, F_1\right)$$

we get an exact expression $\mathcal{R}_q^j$ by concatenating with ω_{1j}. ■

C. Discrete Case

It is clear that Theorem 1 is not as precise for the discrete structures as it is for the dense ones. Indeed, to get density in $\mathcal{Q}$, Theorem 1 provides a necessary and sufficient condition in terms of a geometric quantity directly related to the polyhedron itself. On the other hand, for the discrete case, the discussion relies on quantities defined on $\mathcal{P}_D$, a development of $\mathcal{P}$ (cf. Remark 1). In this section, we describe the relationship between

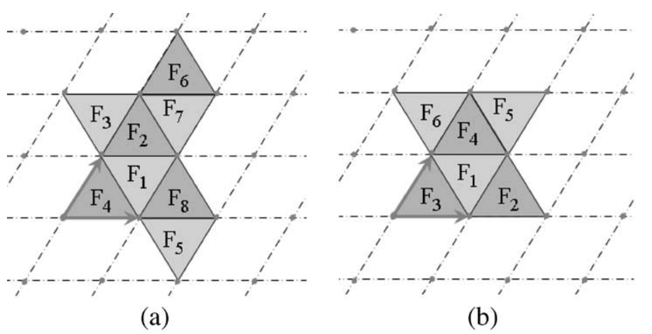

Fig. 8. For polyhedra whose edge lengths satisfy (25), and whose defect angles are multiples of $\beta = 2\pi/3$ [as, e.g., the regular octahedron developed on the in (a)], $\mathcal{H}_1$ is a rhomboidal lattice with small angle $\pi/3$. The same holds for polyhedra satisfying (25) and with defect angles multiples of $\beta = \pi/3$ [as, e.g., the esahedron with equilateral faces developed in (b)].

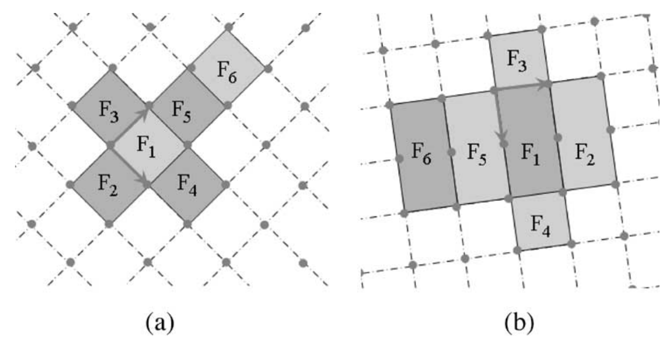

Fig. 9. Polyhedra satisfying (25) and with defect angles multiples of $\beta = \pi/2$ are cubes (a) or convex assemblies of identical cubes (b). For these, $\mathcal{H}_1$ is a square lattice.

the structure of the reachable set and such geometric quantities associated to $\mathcal{P}$ as lengths of edges, angles at vertices, etc.

For the rest of the paragraph, we consider a nondegenerate polyhedron $\mathcal{P}$ with quantization angle $\beta = 2\pi/p, p \geq 2$ and we identify $\mathcal{H}_1$ with $\langle G_{n_2} \rangle \subset \mathbb{R}^2$. We will also denote by $\mathbf{v}_\lambda$ the ith vertex as a point on the polyhedron $\mathcal{P}$, while v_λ denotes its image on the plane development $\mathcal{P}_D$. For all vertices $\mathbf{v}_{\lambda_1}$, $\mathbf{v}_{\lambda_2}$ and $\mathbf{v}_{\lambda_3}$ such that $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ and $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_3})$ are edges of $\mathcal{P}$, let $D_{\lambda_1\lambda_2}$ and $\delta_{\lambda_1\lambda_2\lambda_3}$ denote the length of $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ and the angle between the $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ and $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_3})$, respectively. Also, let Tr denote a nondegenerate triangle (i.e. a triangle of nonzero area), and use $d(Tr)$ to denote the triangle whose vertices are the middle points of the edges of Tr.

We start by giving more details on the case where $\beta = \pi$. Recall that $\mathcal{P}$ is a tetrahedron and every β_λ is equal to π. Let F_1 be the face on which $\mathcal{P}$ is lying on Π and let $\widetilde{\mathcal{P}_D}$ be the development obtained by unfolding $\mathcal{P}$ along the words F_1F_i, $i = 2, 3, 4$.

Proposition 6: Assume that $\mathcal{P}$ is a nondegenerate polyhedron with quantization angle $\beta = \pi$. Then, all faces of $\mathcal{P}$ are isometric to the nondegenerate triangle Tr defined by F_1 and $\widetilde{\mathcal{P}_D}$ is a triangle so that $d(\widetilde{\mathcal{P}_D}) = Tr$ (see Fig. 7).

Proof: The face F_1 on $\widetilde{\mathcal{P}_D}$ is represented by a triangle ABC. Since every $\beta_\lambda = \pi$, we get $\widetilde{\mathcal{P}_D}$ is a triangle $A'B'C'$ such that A belongs to the segment $B'C'$, etc. Since $B'A$ and $C'A$ represent the same edge of $\mathcal{P}$, we get that $d(\widetilde{\mathcal{P}_D}) = Tr$. By Thales theorem, we then obtain that all the four triangles defined by the faces of $\mathcal{P}$ inside $\widetilde{\mathcal{P}_D}$ are isometric. ■

Remark 2: Conversely, if a nondegenerate triangle Tr_0 is given, one can build a tetrahedron $\mathcal{P}_0$ with quantization angle β_0 equal to π and all faces isometric to Tr_0. Indeed, consider the triangle Tr'_0 such that $d(Tr'_0) = Tr_0$. By drawing Tr_0 inside Tr'_0, we define three other triangles inscribed inside Tr'_0 all isometric to Tr_0. By folding these three triangles along the edges of Tr_0, we get, by using elementary geometric arguments, the polyhedron $\mathcal{P}_0$.

Notice explicitly that for such polyhedra the reachable set is discrete, irrespective of the lengths of their edges. The remaining cases are covered by the next proposition which is a translation of the results of Remark 1 in terms of geometric quantities only involving $\mathcal{P}$; see Figs. 8 and 9.

Proposition 7: Assume that $\beta = \pi/2$, $\pi/3$ or $2\pi/3$. Then, $\mathcal{H}_1$ is a nondegenerate lattice if and only if it holds the following "edge-angle rationality" condition

$$\frac{D_{\lambda_1\lambda_3}}{D_{\lambda_1\lambda_2}} \frac{\sin(l\alpha + \delta_{\lambda_1\lambda_2\lambda_3})}{\sin\alpha} \in \mathbb{Q} \tag{25}$$

for $l = 0, 1$ and for all triples of vertices $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}, \mathbf{v}_{\lambda_3})$ such that $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ and $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_3})$ are adjacent edges in $\mathcal{P}$.

Proof: Define for all distinct vertices v_{λ_1} and v_{λ_2} such that $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ is an edge, $w_{\lambda_1\lambda_2} = (1 - e^{-\jmath\beta_{\lambda_2}})(v_{\lambda_1} - v_{\lambda_2})$. Let G_{n_3} be the set given by

$$G_{n_3} = \left\{ e^{\jmath l\alpha} w_{\lambda_1\lambda_2}, \qquad 0 \leq l \leq p' - 1 \quad (\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}) \in \mathcal{E} \right\}$$

and $G_3 = \langle G_{n_3} \rangle \subset \mathbb{R}^2$ (recall that α is the symmetry angle defined in Proposition 4). We first show that

Lemma 4: With the above hypothesis, $\mathcal{H}_1$ is a nondegenerate lattice if and only if G_3 is as follows.

Proof: It is enough to show that every element of G_{n_2} is written as a linear combination of elements of G_{n_3} with rational coefficients and viceversa. We can clearly restrict ourselves to the elements of G_{n_2} and G_{n_3}. This simply follows from the three next facts.

aa) For every $1 \leq \lambda_1, \lambda_2 \leq h - 1$, $(1 - e^{-\jmath\beta_{\lambda_2}})/(1 - e^{-\jmath\beta_{\lambda_1}})$ can be written as a sum of terms of type $re^{\jmath l\alpha}$ where $r \in \mathbb{Q}$ and $l \in \mathbb{Z}$. To see that, notice that

$$\frac{1 - e^{-\jmath\beta_{\lambda_2}}}{1 - e^{-\jmath\beta_{\lambda_1}}} = \frac{(1 - e^{-\jmath\beta_{\lambda_2}})(1 - e^{-\jmath\beta_{\lambda_1}})}{\left|1 - e^{\jmath\beta_{\lambda_1}}\right|^2}$$

and the denominator of the last expression is always a positive integer with the considered values of β.

bb) For every $1 \leq \lambda_1, \lambda_2 \leq h - 1$, let $(\mathbf{v}_{k_l})$, $l = 1, \cdots, N$ with $\mathbf{v}_{k_1} = \mathbf{v}_{\lambda_2}$ and $\mathbf{v}_{k_N} = \mathbf{v}_{\lambda_1}$ a sequence of vertices such that three consecutive vertices in the sequence define adjacent edges on $\mathcal{P}$. Then

$$v_{\lambda_1} - v_{\lambda_2} = \sum_{l=1}^{N-1} v_{k_{l+1}} - v_{k_l}.$$

cc) The virtual vertex v_0 is either an existing vertex of $\mathcal{P}$ or more generally it is equal to an integral linear combination of vertices and rotated of angles $k\beta$ in the sense of (16).

We first have for $1 \leq \lambda_1, \lambda_2 \leq h - 1$

$$w_{\lambda_1\lambda_2} = z_{\lambda_2} - \frac{1 - e^{-\jmath\beta_{\lambda_2}}}{1 - e^{-\jmath\beta_{\lambda_1}}} z_{\lambda_1}.$$

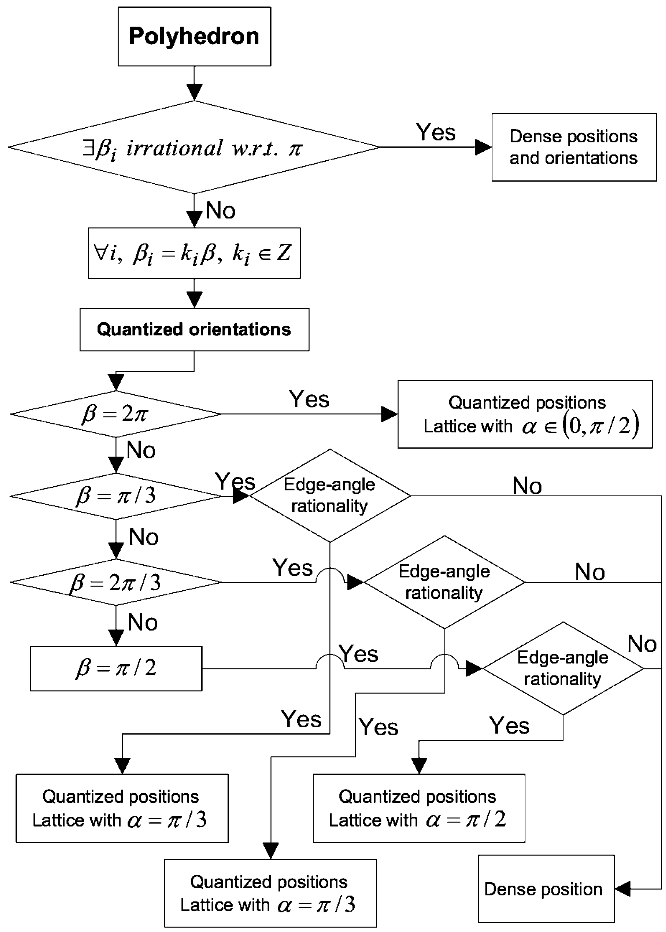

Fig. 10. Flow-chart summarizing the reachability analysis for rolling polyhedra.

From aa), $w_{\lambda_1\lambda_2}$ can be written as a rational combination of elements of type $e^{\jmath u\alpha}z_\lambda$ of G_{n_2}. Analogously for $e^{\jmath l\alpha}w_{\lambda_1\lambda_2}$. For the converse, from the definition of $w_{\lambda_1\lambda_2}$ we get

$$v_{\lambda_1} - v_{\lambda_2} = \frac{1 - e^{-\jmath\beta_{\lambda_2}}}{\left|1 - e^{-\jmath\beta_{\lambda_2}}\right|^2} w_{\lambda_1\lambda_2}.$$

Thanks to the definition of z_λ, aa), bb), and cc), we conclude. ■

Recall Remark 1. Because of the structure of G_{n_3}, it is clear that G_3 is a nondegenerate lattice if and only if for every $1 \le \lambda_1$, λ_2, $\lambda_3 \le h-1$ so that $\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}, \mathbf{v}_{\lambda_3}$ define two adjacent edges at $\mathbf{v}_{\lambda_2}$, and for every $0 \le l \le p'-1$, one has

$$e^{\jmath l\alpha}w_{\lambda_1\lambda_3} = a^l_{\lambda_1\lambda_2\lambda_3}w_{\lambda_1\lambda_2} + b^l_{\lambda_1\lambda_2\lambda_3}e^{\jmath\alpha}w_{\lambda_1\lambda_2}$$

for some rational numbers $a^l_{\lambda_1\lambda_2\lambda_3}$ and $b^l_{\lambda_1\lambda_2\lambda_3}$. Simplifying by $w_{\lambda_1\lambda_2}$, we get

$$a^l_{\lambda_1\lambda_2\lambda_3} + b^l_{\lambda_1\lambda_2\lambda_3}e^{\jmath\alpha} = e^{\jmath\left(l\alpha - \delta_{\lambda_1\lambda_2\lambda_3}\right)}\frac{1-e^{-\jmath\beta_{\lambda_3}}}{1-e^{-\jmath\beta_{\lambda_2}}}\frac{D_{\lambda_1\lambda_3}}{D_{\lambda_1\lambda_2}} \quad (26)$$

(recall that $\delta_{\lambda_1\lambda_2\lambda_3}$ is the angle between the edges $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2})$ and $(\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_3})$). From aa) and (26), it is easy to obtain (25). ■

The classification of reachable sets for rolling polyhedra thus far obtained is summarized in Fig. 10.

IV. Steering Motions of Rolling Polyhedra

It follows from previous results (see Fig. 10) that conditions upon which density or discreteness of reachable sets depend are given in terms of rationality of certain parameters. This entails that two very similar polyhedra may have qualitatively different reachable sets: indeed, for any polyhedron whose reachable set has a discrete structure, there exists a polyhedron with arbitrarily close geometric parameters that gives density. Lattice structures appear to be nongeneric in this sense. On the other hand, considering that in practical applications lengths and angles of physical parts are only known with a limited accuracy, one is led to question the meaning and practical applicability of the foregoing analysis. In this section we will show that indeed discrete structures and tools are instrumental to deal with questions regarding robustness of the reachable set analysis and planning.

In the study of reachability for smooth dynamical systems, the problem of constructive reachability, also referred to as "steering" or "planning" problem, is usually defined as to find, given an initial and a final configuration, a finite-length word of inputs that takes the system from the former to the latter. For a rolling polyhedron system (and more generally for quantized control systems) our previous analysis clearly shows that the problem should rather be posed as follows: Given the initial configuration (0,0,0,F_1), a final configuration $C_f = (x_f, y_f, \gamma_f, F_f)$, and a number η, determine if there exists a finite sequence of turns that brings the part from the former to an η-neighborhood of the latter configuration (in the metric defined in Section II), and, if so, provide one such sequence.

We will first discuss planning for the nominal case of polyhedra whose reachable set is a lattice. Secondly, we describe how one could plan manipulation of exactly modeled polyhedra with a dense reachable set. Finally, we discuss extensions of these results to the general case of polyhedra described with limited accuracy. Some of these ideas and the corresponding algorithms were first reported in [38], where more details and proofs can be found.

A. Planning in Discrete Reachable Sets

Assume that (15) and (25) hold. Hence, there exists a quantization angle β, and $\mathcal{H}_1$ is a 2D lattice generated over $\mathbb{Z}$ by a set of $\bar{N} = q'(h-1)$ generators (see Section III-B2, (19)). Let $T = [t_1, t_2, \ldots, t_{\bar{N}}] \in \mathbb{Q}^{2\times\bar{N}}$ denote a matrix collecting such a set of generators. By computing the *Hermite normal form* for the 2-D lattice (see, e.g., [39] and [40]) as

$$X = [X_1 \quad X_2 \quad \mathbf{0}] = TU$$

with U a unimodular integral matrix, two vectors X_1, X_2 generating the same lattice are obtained. Denoting by U_{ij} the element of U in the i, j position, one has

$$X_j = \sum_{i=1}^{\bar{N}} t_i U_{ij}.$$

Let $\Delta := \max\{\|(X_1+X_2)/2\|, \|(X_1-X_2)/2\|\}$ denote the half-length of the longest diagonal of the lattice mesh. If the required accuracy η is such that $\beta > 2\eta$, or $\Delta > \eta$, the steering problem is unfeasible for an arbitrary C_f. Otherwise, proceed as follows.

1) Compute (x_1, y_1, γ_1) such that $\omega_{1f} : (x_1, y_1, \gamma_1, F_1) \mapsto (x_f, y_f, \gamma_f, F_f)$.
2) Let $k = \arg\min_{\kappa\in\mathbb{Z}} \|\kappa\beta - \gamma_1\|_{S^1}$. Let $\|k\beta - \gamma_1\|_{S^1} = \epsilon \leq \eta$ and compute (x_2, y_2) such that $\bar{R}^k$: $(x_2, y_2, 0, F_1) \mapsto (x_1, y_1, \gamma_2, F_1)$, where $\|\gamma_2 - \gamma_1\|_{S^1} = \epsilon$.
3) Let $(k_1, k_2) = \arg\min_{\kappa_1,\kappa_2} \|\kappa_1 X_1 + \kappa_2 X_2 - (x_2, y_2)\|$. If $\|k_1 X_1 + k_2 X_2 - (x_2, y_2)\| > \eta - \epsilon$, the planning problem has no solution; otherwise, apply the original generators of the lattice, $v_1, \ldots, v_{\bar{N}}, U_i = U_{i1}k_1 + U_{i2}k_2$ times each.

A manipulating sequence is thus obtained which consists in applying the word corresponding to $v_i^{U_i}$, $i = 1, \ldots, N$, $\bar{R}^k$, and ω_{1f}, in this order. A configuration $\tilde{C}_f = (x, y, \gamma, F_f)$ is thus reached such that $d(\tilde{C}_f - C_f) \leq \eta$.

B. Planning in Dense Reachable Sets

If (15) and (25) do not hold, and if a perfect model of the polyhedron is available, it is possible to obtain a solution to the planning problem with arbitrary accuracy η. To do so, it would suffice to find a rotation $\hat{R} \in \Omega_1$ of angle $\hat{\beta}$ with $\hat{\beta}/\pi \notin \mathbb{Q}$, and an approximation $\hat{\beta} \approx 2\pi/\hat{p}$, with $\hat{p}$ large enough so that, for $k = \arg\min_{\kappa\in\mathbb{Z}} \|\kappa\hat{\beta} - \theta_f\|_{S^1}$, it holds $\|k\hat{\beta} - \theta_f\|_{S^1} = \eta_0 \leq \eta$. Furthermore, consider any rotation $R \in \Omega_1$ with $R \neq \hat{R}$, and the set of generators

$$\hat{\mathcal{H}}_1 = \left\{ \hat{R}^k.(R\hat{R}^m).(\hat{R}^m R)^{-1}.\hat{R}^{-k};\ k, m \in \mathbb{Z} \right\} \subset \mathcal{H}_1.$$

The $\hat{N}$ elements of $\hat{\mathcal{H}}_1$ and their projections on the first factor, $\{\hat{t}_1, \ldots, \hat{t}_{\hat{N}}\}$, are irrationally related and thus generate a dense set over the integers. To find a possible solution of finite length, proceed to approximate the dense set with a lattice, obtained with the rational representations $\hat{t}_i'$ of the components of $\hat{t}_i$, $i = 1, \ldots, \bar{N}$. The number $\bar{N}$ of generators and their representation accuracy can be chosen (in the ideal case) so that the lattice resolution Δ is arbitrarily small. Hence, a feasible solution would be obtained by solving a planning problem on this (arbitrarily fine) lattice by the same techniques used in the previous paragraph. The case in which the reachable set is dense in positions, but discrete in orientations can be worked out simply based on the same considerations.

C. Planning With Limited Accuracy Models

To provide a correct model of the phenomenon of rolling real polyhedral parts, it is necessary to describe how uncertain quantities are represented in the computer. It can be assumed that a geometric length or angle (the latter measured in $\pi\ rad$ units) of nominal value a with tolerance $\pm\tau_a$ is represented by a truncated continued fraction expansion $\bar{a} = p_a/q_a$ with $q_a = \lceil \tau_a{}^{-1/2} \rceil$, so as to match representation accuracy to tolerance. Tolerances on geometric parameters also reflect directly in a limited meaningful representation accuracy for the quantization angle β and for the generator set t_i, $i = 1, \ldots, \bar{N}$. The reachable set will be thus described approximately by the discrete set generated by those representations, and planning will be addressed again through the solution of the one-dimensional and two-dimensional Diophantine equations encountered previously. The real reachable set is actually an uncertain distribution about this discrete approximation. Bounds on the maximum discrepancy between a point reached with a word of given length, and the nominal point on the approximated polyhedron based on description tolerances were given in [38], along with a discussion of the computational complexity and bounds on the length of manipulating words.

V. Discrete Nonholonomy

In this section, we should like to generalize some of the particular features encountered in the case study to systems $\Sigma = (\mathcal{Q}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$ of a rather general class, and in particular to address a definition of nonholonomy which may apply to nonsmooth and quantized systems as well as to classical systems.

As a first lesson from the case-study, we recognize that it is important that the input set in the system quintuple Σ is considered in general as state-dependent. In other words, different sets of inputs may be available at different states, as it is clearly the case for the polyhedron when lying with different faces on the plane. To deal with this problem, let us be more specific on the definition of the input set $\mathcal{U}$, and assume that there exists a multivalued function $\phi : \mathcal{Q} \to \mathcal{U}$ where $\phi(q) = \mathcal{U}_q \subset \mathcal{U}$ is the set of admissible inputs at q. Consider the equivalence relation on $\mathcal{Q}$ given by $q_1 \overset{\mathcal{U}}{\equiv} q_2$ if $\phi(q_1) = \phi(q_2)$, and denote $\mathcal{Q}/\phi$ the set of equivalence classes, $[q]_\mathcal{U}$ the equivalence class of q.

Further, let Ω_q be the language over $\mathcal{U}$ consisting of admissible input streams for the system being currently in configuration q.[1] For each $q \in \mathcal{Q}$ and $\omega \in \Omega_q$, let the end-point map, i.e. the state that the system reaches from q under $\omega \in \Omega_q$, be denoted as $\mathcal{A}(q, \cdot) : \Omega_q \to \mathcal{Q}$, or simply as $\mathcal{A}_q(\omega)$.

Two configurations q_1, q_2 are stream equivalent (denoted $q_1 \overset{\Omega}{\equiv} q_2$) iff $\Omega_{q_1} = \Omega_{q_2}$. Accordingly, $\mathcal{Q}/\Omega$ denotes the set of stream equivalence classes, and $[q]_\Omega$ is the stream equivalence class of q. Clearly, input and stream equivalence classes coincide if the following compatibility condition of the map $\mathcal{A}$ with the equivalence relation $\overset{\mathcal{U}}{\equiv}$ holds:

$$[\mathbf{H1}]\ \forall q_1 \overset{\mathcal{U}}{\equiv} q_2 \text{ and } \forall u \in \mathcal{U}_{q_1} (= \mathcal{U}_{q_2}) \quad \mathcal{A}_{q_1}(u) \overset{\mathcal{U}}{\equiv} \mathcal{A}_{q_2}(u).$$

We assume in the following that $\mathcal{Q}$ is a manifold and that each input and stream equivalence classes are connected submanifolds of $\mathcal{Q}$.

Denote by $\widetilde{\Omega}_q = \{\omega \in \Omega_q : \mathcal{A}_q(\omega) \in [q]\}$ the sublanguage consisting of those input streams which steer the system eventually back to the same equivalence class of the initial point. For $\omega_1, \omega_2 \in \widetilde{\Omega}_q$, the stream concatenation $\omega_1\omega_2$ is well defined. The notion of kinematic (i.e., driftless) systems of the form (1) can be extended in this context by the assumption that $\widetilde{\Omega}_q$ contains an identity element, $0 \in \widetilde{\Omega}_q$, such that $\mathcal{A}_q(0) = q$, for all $q \in [q]$. In general, the language $\widetilde{\Omega}_q$ is not prefix closed. However, we will also consider the *equivalence orbit* of $q \in [q]$ under $\widetilde{\Omega}_q$ (denoted as $\mathcal{R}_q(\overline{\widetilde{\Omega}}_q)$) as the reachable set from q under words in the prefix-closure $\overline{\widetilde{\Omega}}_q$ of $\widetilde{\Omega}_q$, in other

[1]By the use of the term "stream" instead of "word" we want to underscore here that, in a general setting including continuous time systems, inputs can also be more general, e.g. piecewise continuous functions obtained by concatenating finite-support functions in $\mathcal{U}$.

words $\mathcal{R}_q(\overline{\overline{\Omega}}_q) := \{p \in \mathcal{Q} : p = \mathcal{A}_q(\omega_s), \omega_s \in \overline{\overline{\Omega}}_q\}$ with $\overline{\overline{\Omega}}_q := \{\omega_s \in \mathcal{U}^* : \exists \omega_t \in \mathcal{U}^*, (\omega_s\omega_t \in \widetilde{\Omega}_q)\}$.

The introduction of input equivalence classes and orbits induces us to consider two different types of behaviors which may be termed "nonholonomic" by analogy with observations made in paragraph I-A about the increased reachability afforded by cyclic controls. Loosely speaking, we will refer to the case where cyclic switchings that temporarily "get out" of an equivalence class add to reachability more than what availed by paths "staying in," as to an "external" type of nonholonomy. On the other hand, when there exist cycles that, while staying within an equivalence orbit, afford reaching states which could not be reached by noncyclic paths, then we will speak of an "internal" type of nonholonomy.

More precisely, consider the maximal sublanguage $\widehat{\Omega}_q \subseteq \widetilde{\Omega}_q$ of words that always keep the configuration within the same equivalence class, and compare the corresponding orbit $\mathcal{R}_q(\overline{\overline{\Omega}}_q) = \mathcal{R}_q(\widehat{\Omega}_q) \subseteq [q]$ with the set reachable from q under $\widetilde{\Omega}_q$, $\mathcal{R}_q(\widetilde{\Omega}_q) = \{\mathcal{A}_q(\omega) : \omega \in \widetilde{\Omega}_q\}$.

Definition 1: A system $(\mathcal{Q}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$ is said to be externally nonholonomic at $q \in \mathcal{Q}$ if $\mathcal{R}_q(\widetilde{\Omega}_q) \supsetneq \mathcal{R}_q(\widehat{\Omega}_q)$.

Describing the second type of nonholonomy requires more work. We need first to give more structure to the set $\widetilde{\Omega}_q$ and to its action on $[q]$. A system is said to be invertible if for every $q \in \mathcal{Q}$ and $\omega \in \widetilde{\Omega}_q$, there exists $\bar{\omega} \in \widetilde{\Omega}_q$ such that $\mathcal{A}_q(\omega\bar{\omega})$ and $\mathcal{A}_q(\bar{\omega}\omega)$ are both equal to q. Consider the following relation in $\widetilde{\Omega}_q : \omega_1 \equiv \omega_2$ if, for all $q \in [q]$, $\mathcal{A}_q(\omega_1) = \mathcal{A}_q(\omega_2)$. Then, on $\widetilde{\Omega}_q/\equiv$, the inverse of each element is defined uniquely. Indeed, if $\bar{\omega}_1, \bar{\omega}_2$ are two inverses for $\omega \in \widetilde{\Omega}_q$ (hence $\mathcal{A}_q(\omega\bar{\omega}_i) = \mathcal{A}_q(\bar{\omega}_i\omega) = q$, $i = 1, 2$) then

$$\mathcal{A}_q(\bar{\omega}_1) = \mathcal{A}_q(\bar{\omega}_1\omega\bar{\omega}_2) = \mathcal{A}_q(\bar{\omega}_2)$$

i.e., $\bar{\omega}_1 \equiv \bar{\omega}_2$. In the following, up to taking the quotient $\widetilde{\Omega}_q = \widetilde{\Omega}_q/\equiv$, we will restrict to consider driftless invertible systems where the inverse is defined uniquely, which is tantamount to assuming that $\widetilde{\Omega}_q$ is a group. We assume that $\widetilde{\Omega}_q$ is finitely generated and denote by $S = \{s_1, \ldots, s_n\}$ a set of generators.

Consider now the subset Ω_q^S of *simple* input words over S, i.e. those strings that either include a generator, or its inverse, but not both. More precisely, let

$$\widetilde{\Omega}_q^S = \Big\{ s_{\sigma(1)}^{k_{\sigma(1)}} s_{\sigma(2)}^{k_{\sigma(2)}} \cdots s_{\sigma(n)}^{k_{\sigma(n)}} : \sigma \in \mathcal{P}(n), \quad k_{\sigma(j)} \in \mathbb{Z}, \quad j = 1, \ldots, n \Big\}$$

where $k_{\sigma(i)}$ is the number of times the symbol $s_{\sigma(i)}$ is used (negative values meaning that $\bar{s}_{\sigma(i)}$ is used instead), and $\mathcal{P}(n)$ is the set of permutations of $(1, 2, \ldots, n)$. Let $\mathcal{R}_q(\widetilde{\Omega}_q)$ and $\mathcal{R}_q(\widetilde{\Omega}_q^S)$ denote the reachable set from q under input streams in $\widetilde{\Omega}_q$ and in $\widetilde{\Omega}_q^S$, respectively. Definitions we propose to capture the second type of nonholonomy are then as follows.

Definition 2: A system $(\mathcal{Q}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$ is said to be noncommutative at $q \in \mathcal{Q}$ if $\widetilde{\Omega}_q$ contains at least two elements ω_1 and ω_2 such that for their *commutator* $[\omega_1, \omega_2] := \omega_1\omega_2\bar{\omega}_1\bar{\omega}_2$, it holds $\mathcal{A}_q([\omega_1, \omega_2]) \neq q$.

A system is *internally nonholonomic* at q if there exists a set of generators S and $\omega_1, \omega_2 \in \widetilde{\Omega}_q^S$ such that $\mathcal{A}_q([\omega_1, \omega_2]) \notin \mathcal{R}_q(\widetilde{\Omega}_q^S)$.

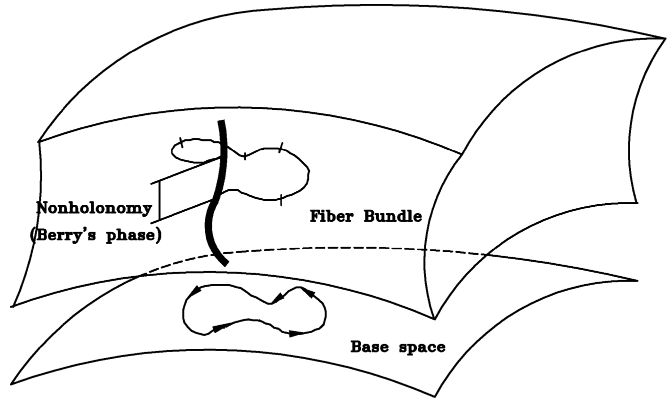

Fig. 11. Illustrating the definition of nonholonomic systems.

Remark 3: The term internal refers to the fact that such nonholonomic phenomena can be observed on trajectories that always remain within the same equivalence orbit, or even class (see examples later).

A suggestive geometric interpretation can be given of these definitions (see Fig. 11), which is reminiscent of Berry's phase in quantum mechanics [41]. Berry noticed that if a quantum system evolves in a closed path in its parameter space, after one period the system would return to its initial state, however with a multiplicative phase containing a term depending only upon the geometry of the path the system traced out, or Berry's phase. The concept has been influential in modern mechanics, where it is often referred to as *geometric phase* (cf. [1] and [2]). In our setting, consider a local decomposition of $\mathcal{Q}$ in a *base* space $\mathcal{B}$ and a *fiber* space $\mathcal{F}$, with $\mathcal{B} \times \mathcal{F} = \mathcal{Q}$. Choosing coordinates $q = (q_B, q_F)$ and denoting the canonical projections $\Pi_{\mathcal{B}}(q) = q_B$, $\Pi_{\mathcal{F}}(q) = q_F$, let $\mathcal{B}$ be a maximal codimension set such that $\Pi_F(\mathcal{R}_q(\widetilde{\Omega}_q^{[q]})$ (for external nonholonomy), or $\Pi_F(\mathcal{R}_q(\widetilde{\Omega}_q^S)$ (for internal nonholonomy), are constant. If there exists an input stream which would steer the system from q to $q^\star$ with $\Pi_B(q) = \Pi_B(q^\star)$ but $q \neq q^\star$, then the system is nonholonomic at q, and the difference between $\Pi_F(q^\star)$ and $\Pi_F(q)$ is the corresponding holonomy phase.

Example 1: A first set of elementary examples can be obtained considering the classical Heisenberg–Brockett nonholonomic integrator [6]

$$Dq = \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} u_2 \tag{27}$$

with $q \in \mathcal{Q} = \mathbb{R}^3$, and $[q] = \mathcal{Q}$. Only internal nonholonomy can obviously apply.

i) Consider first the example in the classical setting, i.e., in continuous time ($t \in \mathcal{T} = \mathbb{R}^+$, $Dq := (d)/(dt)q(t)$) and with a continuous control set ($u \in \mathcal{U} = \mathbb{R}^2$). We assume, without loss of generality, that Ω is comprised of piecewise constant functions $\mathbb{R}^+ \mapsto \mathcal{U}$ [42]. Internal nonholonomy of this system according to (2) can be shown by taking the input construction commonly used in textbooks to illustrate "lie-bracket motions" (see, e.g., [5]). Namely, let $S = (s_1, s_2)$ with $s_1(t) = (\delta_1\ 0)$, $t \in [t_1, t_1 + \tau_1]$ and $s_2(t) = (0\ \delta_2)$, $t \in [t_2, t_2 + \tau_2]$ (hence $\bar{s}_i = -s_i$, $i = 1, 2$). One easily gets $\mathcal{R}_{q_0}(\widetilde{\Omega}^S) = (x_0 + \alpha, y_0 + \beta, z_0 -$

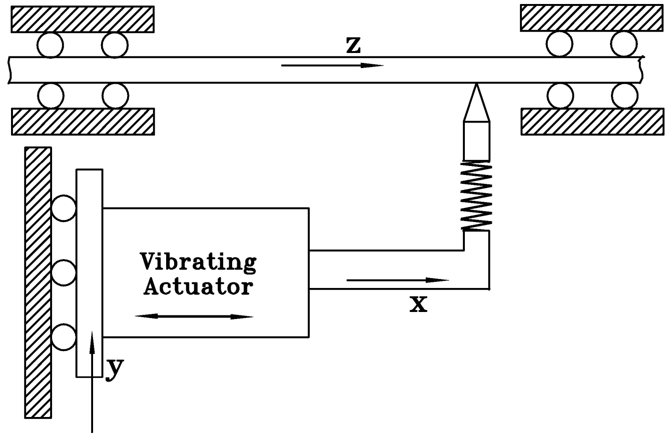

Fig. 12. Microelectromechanical motion rectifier illustrating the definition of external nonholonomy in a piecewise holonomic system.

$y_0\alpha + x_0\beta + \alpha\beta)$, α, $\beta \in \mathbb{R}$, while $\mathcal{A}_{q_0}(s_1 s_2 \bar{s}_1 \bar{s}_2) = (x_0, y_0, z_0 + 2\delta_1\delta_2\tau_1\tau_2)$. Hence $\mathcal{A}_{q_0}([s_1, s_2]) \notin \mathcal{R}_{q_0}(\widetilde{\Omega}^S)$. This example [which could be easily generalized to systems as in (1)] shows that the classical notion of small-time, local nonholonomy related to the Lie algebra rank condition, is a particular case of internal nonholonomy.

ii) Definition (2) equally applies to (27) when considered in discrete time, i.e., $t \in \mathcal{T} = \mathbb{N}$, $Dq := q(t+1) - q(t)$. This can be shown by taking, e.g., $s_1 = (\delta_1\ 0)$, $s_2 = (0\ \delta_2)$, so that $\mathcal{A}_{q_0}([s_1, s_2]) = (x_0, y_0, z_0 + 2\delta_1\delta_2)$, while $\mathcal{R}_{q_0}$ is as before. The continuity of the control set guarantees complete reachability for this system in both the continuous and discrete time cases.

iii) Consider now a finite input set such as

$$\mathcal{U} = \{(u_1, u_2) | u_1 \in \{0, a, -a\}, u_2 \in \{0, b, -b\}, a, b \in \mathbb{R}\}$$

and $\Omega = \{\text{strings of symbols in } \mathcal{U}\}$. The restriction on controls does not substantially change the analysis under continuous time. Indeed, considering $s_1(t) = (a\ 0)$, $t \in [t_1, t_1 + \tau_1]$, $s_2(t) = (0\ b)$, $t \in [t_2, t_2 + \tau_2]$, one gets $\mathcal{A}_{q_0}([s_1, s_2]) = (x_0, y_0, z_0 + 2ab\tau_1\tau_2)$, and both nonholonomy and complete reachability easily follow from arbitrarity of τ_1, τ_2.

iv) In the discrete input, discrete-time case, the input commutator $[s_1, s_2]$ with $s_1 = (a,\ 0)$, $s_2 = (0\ b)$, produces $\mathcal{A}_{q_0}([s_1, s_2]) = (x_0, y_0, z_0 + 2ab)$. Internal nonholonomy is maintained. However, the reachable set from the origin is only comprised of configurations in a discrete set, $\mathcal{R}_0 = \{q : x = \ell a,\ y = mb,\ z = nab,\ \ell, m, n \in \mathbb{Z}\}$. The situation is completely different, and density of the reachable set is guaranteed, if, e.g., $\mathcal{U} = \{(u_1, u_2) |\ u_1 \in \{0, a, -a, c, -c\}, u_2 \in \{0, b, -b, d, -d\},\ a, b, c, d \in \mathbb{R}\}$ with a/c, $(b/d) \notin \mathbb{Q}$.

The interpretation of nonholonomy given in Fig. 11 applies to all cases above, using coordinates x, y to describe the base space, while z parameterizes the fiber.

Example 2: As an example of a piecewise (hybrid) holonomic system, consider the simplified version of one of Brockett's rectifiers [43] in Fig. 12. The tip of a piezoelectric or electrostrictive element oscillates in the x-direction, while an actuator drives the oscillator support along the y-direction. When y reaches a threshold y_0, dry friction is sufficient to push the rod in the z-direction. Disregarding dynamics, the rectifier can be modeled by a continuous-time system with configurations $q = (x, y, z) \in \mathcal{Q} = \mathbb{R}^3$. Assuming that the velocity of the support $(\dot{y})$, and of the oscillator tip $(\dot{x})$ can be freely chosen, a model for this system congruent with the aforementioned definitions would be

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_3$$

with the input restrictions

$$\begin{cases} u_3 = 0 & y < y_0 \\ u_2 = 0 & y \geq 0 \end{cases}.$$

Two input equivalence classes are thus defined in $\mathcal{Q}$ as $[q]_{free} = \{q \in \mathcal{Q} : y < y_0\}$ and $[q]_{engaged} = \{q \in \mathcal{Q} : y \geq y_0\}$. Clearly, $\mathcal{R}_{q_0}^{[q]_{\text{free}}} = \{(x, y, z) \in \mathcal{Q} : z = z_0\}$, for all $q_0 = (x_0, y_0, z_0) \in [q]_{free}$, while $\mathcal{R}_{q_0} = \mathbb{R}^3$. The system is thus externally nonholonomic according to definition (1).

Interestingly enough, however, the system is not internally nonholonomic as per definition (2). Indeed, to generate the set $\widetilde{\Omega}_{q_0}$, at least two types of streams must be considered: an internal type, e.g., $s_i : (x_0, y_0, z_0) \mapsto (x, y, z_0)$, and an external type (taking the state out of $[q]_{free}$ temporarily), e.g. $s_e : (x_0, y_0, z_0) \mapsto (x', y', z')$. Clearly, simple streams over this set of generators are sufficient to reach any configuration of the system ($\mathcal{R}_q(\widetilde{\Omega}_q^S) = \mathbb{R}^3$), hence, internal nonholonomy does not apply.

Base variables for this example would be x and y, while z represents the fiber variable. Rectification of motion is obtained by holonomic phase accumulation in successive cycles. By changing frequency and phase of the inputs, different directions, and velocities of the rod motion can be achieved. Note in particular that input u_2 need not actually to be finely tuned, as long as it is periodic, and it could be chosen as a resonant mode of the vibrating actuator: tuning only u_2 still guarantees in this case the (nonlocal) reachability of the system (cf. [11] and [44]).

Example 3: Rolling polyhedra are both externally and internally nonholonomic systems. External nonholonomy holds trivially since the set of controls in $\mathcal{U}_q$ that leave the system in the same configuration class $[q]$ is the identity element, and represents the behavior illustrated in Fig. 1.

Internal nonholonomy according to definition 2 also holds: Indeed, $\widetilde{\Omega}_q$, the set of words that bring back the polyhedron on the same face lying on the plane, is generated by the finite set $S = \{R_\lambda,\ \lambda = 1, \ldots, h-1\}$. If $\beta_\lambda/\pi \in \mathbb{Q}$ for all $\lambda = 1, \ldots, h-1$ then $\widetilde{\Omega}_q^S$ is a finite set because, if $\beta_\lambda = 2\pi(m_\lambda/p_\lambda)$, $R_\lambda^{p_\lambda} = (0, 0)$. Therefore, $\mathcal{R}_q(\widetilde{\Omega}_q^S)$ is a finite set. Since $\mathcal{R}_q(\widetilde{\Omega}_q)$ is an infinitely countable set, nonholonomy immediately follows. If, otherwise, there exists λ such that $\beta_\lambda/\pi \notin \mathbb{Q}$ then, by (13), there exists another index λ', $\lambda' \neq \lambda$ for which it also holds $\beta_{\lambda'}/\pi \notin \mathbb{Q}$. Without loss of generality, we can assume $\lambda = 1$ and $\lambda' = 2$ and choose the set of $h-1$ generators given by $\beta_2, \ldots, \beta_h$. In order to prove nonholonomy, we have to compare commutators with translations in $\widetilde{\Omega}_q^S$. Translations in $\widetilde{\Omega}_q^S$ are written as $R_{\sigma(2)}^{k_{\sigma(2)}} R_{\sigma(3)}^{k_{\sigma(3)}} \ldots R_{\sigma(h)}^{k_{\sigma(h)}}$, with $k_{\sigma(j)} = 0$ if

$\beta_{\sigma(j)}/\pi \notin \mathbb{Q}$. In other words translations in $\widetilde{\Omega}_q^S$ have to be generated only by those generators with λ such that β_λ is irrational with π. Now, let t be any translation in $\widetilde{\Omega}_q^S$. Then, the commutator $[R_2, t]$ gives a translation of $t(e^{-j\beta_2} - 1)$ which cannot be generated by simple words. Notice that in this case, the nonholonomic phenomenon is internal to the orbit of $[q]$ under $\widetilde{\Omega}_q$, rather than strictly to the input equivalence class $[q]$ (which contains no trajectories).

VI. Conclusion

The notions of nonholonomy and reachability are conventionally related to differentiable control systems, and are defined in terms of their differential geometric properties. However, these notions apply also to more general systems, including discrete and hybrid systems. In this paper, we have attacked, as a concrete case study, the problem of describing the structure of the reachability set for a rolling polyhedron, and of steering the system to desired configurations. The problem is important in its own right, e.g., in robotics applications. Moreover, notwithstanding its specificity, some lessons can be learned.

It so turns out that, while the powerful tools of differential geometric control theory have to be abandoned, their role in many respects is taken by the theory of groups, and Lie groups in particular. A second important fact is that, in many cases, the reachable set of systems with quantized inputs has a lattice structure, or at least can be thus approximated. For systems on lattices, problems of steering and planning can be efficiently solved by using linear integer programming techniques.

The combination of such techniques produced a planning algorithm for rolling polyhedra which was in fact more efficient than existing methods for rolling regular surfaces (the problem from which our interest in rolling polyhedra actually started). As a practical fallout of research on rolling polyhedra, a better algorithm for regular surfaces inspired to quantized control techniques was generated in [45].

Consideration of the case study led to new insight into the general phenomenon of nonholonomy, and to proposing extensions thereof to the discrete and hybrid case, introducing a distinction between internal and external nonholonomy. Further work will be devoted to establishing tests for these properties. Many problems remain open with discrete nonholonomic systems and quantized control, including for instance the implications on stabilization of different structures of the reachable set. It is our belief that the complete solution of this case study will be of help in addressing more complex and general problems in this field.

Appendix

A graph $G = (N, E)$ is a structure consisting of a finite set N of nodes and a finite set E of edges. A graph $G' = (N', E')$ is a subgraph of $G = (N, E)$ if $N' \subset N$ and $E' \subset E$. We denote an edge that joins two nodes n_i and $n_j \in N$ by $(n_i, n_j) \in E$ or, equivalently, by e_{ij}. An edge joining a node to itself is a loop. If two or more edges join the same pair of nodes, these edges are called multiple edges. A graph is simple if it has no loops or multiple edges. A path between two nodes n_1 and n_s is a finite sequence of nodes and edges of the type $n_1, e_{12}, n_2, e_{23}, \ldots, e_{s-1,s}, n_s$. Note that if G is simple, a path is defined by a sequence of nodes. A path between a node and itself is a closed path. A closed path in which all the edges and nodes (except the first and the last one) are distinct is a cycle. A graph is connected if there is a path between every pair of nodes. A graph is said to be planar if it can be drawn on a plane so that no two edges intersect except at a node. The two-dimensional regions defined by the edges in a planar graph are referred to as the faces of the planar graph. In a planar graph, all faces are bounded by edges, except for exactly one unbounded face. Denoting by F the set of faces (including the unbounded one), and by $S^{\#}$ the cardinality of a finite set S, the Euler relation (2) for any connected planar graph is written as $N^{\#} - E^{\#} + F^{\#} = 2$.

Let $G = (N, E)$ be a simple planar connected graph and F the set of its faces. The dual graph of G is defined as $G_d = (N_d, E_d)$ where $N_d = F$ and $e \in E_d$ if $e = (F_1, F_2)$ for any two adjacent faces $F_1, F_2 \in F$ of G. Clearly, G_d is also a simple planar connected graph. A connected graph with no cycles is a tree. A tree with $N^{\#}$ nodes has $N^{\#} - 1$ edges. A subgraph T of a graph G is a maximal tree if T is a tree and it contains all the nodes of G.

Fix a maximal tree T of $G = (N, E)$ and let $\{\tau_\lambda | \lambda = 1, \ldots, f - 1\}$ be the set of edges of G which are not in T. It obviously holds $f - 1 = E^{\#} - (N^{\#} - 1)$, hence, by the Euler relation (2), $f = 2 + E^{\#} - N^{\#} = F^{\#}$. Also, for $\lambda = 1, \cdots, f - 1$, let n_λ^a and n_λ^b denote nodes connected by τ_λ. The *fundamental group* of G at a node $n_0 \in N$ is the set of all closed paths starting and ending in n_0 with the composition law given by concatenation. With this notation fixed, the following classical proposition allows one to describe a finite set of generators for the fundamental group of a graph (for a proof, see [46]).

Proposition 8: Fix a maximal tree T of a graph G. The fundamental group of G at a node n_0 is a free group generated by $f - 1$ elements of the generator set $A_G = \{\alpha_\lambda | \lambda = 1, \cdots, f - 1\}$, where for $\lambda = 1, \ldots, f - 1$, α_λ is a closed path on G described by three subpaths $\alpha_\lambda = \alpha_\lambda^a \tau_\lambda \alpha_\lambda^b$, with

— α_λ^a any path on T from n_0 to n_λ^a;

— α_λ^b any path on T from n_λ^b to n_0.

We now give another set of generators for the fundamental group. We define a bijective map

$$\tau_\lambda \mapsto F_{\tau_\lambda} \tag{28}$$

from the set of edges outside T to the set of faces of G such that τ_λ is adjacent to F_{τ_λ}, in the following way. There exists τ_{λ_1} such that $T \cup \tau_{\lambda_1}$ contains a loop enclosing a single face $F_{\tau_{\lambda_1}}$ of G. Indeed if $T \cup \tau_{\lambda_1}$ contains a loop enclosing more than one face, then we can choose $\tau_{\lambda'}$ enclosed inside the loop. Now, $T \cup \tau_{\lambda'}$ contains a smaller loop and in a finite number of steps we conclude. Let G_1 be the graph obtained by removing τ_{λ_1} from G. Then, G_1 has one edge and one face less than G. Moreover, $T_1 = G_1 \cap T$ is a maximal tree of G_1. We can now choose τ_{λ_2} such that $T_1 \cup \tau_{\lambda_2}$ contains a loop enclosing a single face $\tilde{F}_{\tau_{\lambda_2}}$. Now, either $\tilde{F}_{\tau_{\lambda_2}}$ is a face of G or $\tilde{F}_{\tau_{\lambda_2}} = F_{\tau_{\lambda_1}} \cup F_{\tau_{\lambda_2}}$ for some face $F_{\tau_{\lambda_2}} \neq F_{\tau_{\lambda_1}}$ of G. Then, we proceed recursively removing τ_{λ_2} from G_1 and so on.

By choosing suitably the paths α_λ^a and α_λ^b in Proposition 8 we can set

$$\alpha_\lambda = \alpha_\lambda^{n_\lambda} C_\lambda \left(\alpha_\lambda^{n_\lambda}\right)^{-1} \tag{29}$$

where $\alpha_\lambda^{n_\lambda}$ is a path from n_0 to a node n_λ adjacent to the face F_λ of (28), and C_λ is a single rotation around F_λ starting from n_λ.

Proposition 9: Let T be a maximal tree of a graph G. Then, the fundamental group of G at a node n_0 is generated by $f-1$ elements of the type (29)

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