Available online at www.sciencedirect.com
SCIENCE DIRECT®

Systems & Control Letters 54 (2005) 999–1013

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

# Quantization of the rolling-body problem with applications to motion planning

Y. Chitour[a], A. Marigo[b], B. Piccoli[b,*]

[a]Département de Mathématiques, Université Paris-Sud, Orsay, France
[b]I.A.C., C.N.R., Viale del Policlinico 137, 00161 Rome, Italy

Received 2 January 2004; received in revised form 29 November 2004; accepted 21 February 2005
Available online 7 April 2005

**Abstract**

The problem of manipulation by low-complexity robot hands is a key issue since many years. The performance of simplified hardware manipulators relies on the exploitation of nonholonomic effects that occur in rolling. Beside this issue, more recently, the attention of the scientific community has been devoted to the problems of finite capacity communication channels and of constraints on the complexity of computation. Quantization of controls proved to be efficient for dealing with such kinds of limitations. With this in mind, we consider the rolling of a pair of smooth convex objects, one on top of the other, under quantized control. The analysis of the reachable set is performed by exploiting the geometric nature of the system which helps in reducing to the case of a group acting on a manifold. The cases of a plane, a sphere and a body of revolution rolling on an arbitrary surface are treated in detail.


*Keywords:* Rolling bodies; Quantized systems; Motion planning

## 1. Introduction

The control of robotic end-effectors that are designed to achieve high operational versatility with limited constructive complexity is a key issue in the control literature since the last decade. The design of such end-effectors is based on the intentional exploitation of nonholonomic effects that occur in rolling. Such hardware simplifications imply the need of more complex control algorithms for keeping high-performance levels, thus requiring deepen analysis and programming.

A first prototype of a hand implementing purposefully rolling manipulation was presented in [22] along with a numeric algorithm for planning. Controllability of rolling for smooth, strictly convex, axial-symmetric surfaces rolling on planes was shown in [6]. The class of rolling surfaces on planes was generalized in [16,3], where the object to be manipulated had a polyhedral description. A general result was presented in [15], showing the generic controllability of rolling pairs (i.e., any two surfaces, with the only exception of

* Corresponding author. Tel.: +39 06 8847 0244; fax: +39 06 440 4306.
*E-mail address:* b.piccoli@iac.cnr.it (B. Piccoli).

0167-6911/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2005.02.012

surfaces that are mirror images of each other, can be arbitrarily reoriented and relocated by rolling).

In this paper, we want to extend this general result on controllability of rolling pairs to the case where quantization of controls is introduced.

Quantized control systems arise in a number of applications because of many physical phenomena or technological constraints. In the control literature, quantization of inputs has been mostly regarded as an approximation-induced disturbance to be rejected [2,21]. We take here a different point of view and introduce quantization on purpose in order to reduce the complexity of control [14] or the technological complexity of the control systems. As a consequence of taking such a viewpoint, the focal point of research is to understand how to design the quantization of controls. Particular phenomena may appear in quantized control systems, which have no counterpart in classical systems theory, and deeply influence the qualitative properties and performance of the system. These concern the structure of the set of points that are reachable, and particularly its density. Some understanding of the structure of the reachable set has been obtained recently for quantized linear systems [8] and for quantized chained form systems [4]. A deeper insight in quantized control systems is given in [17], where the action of sequences of controls is formalized, under suitable assumptions, as a group action of a set of words. In this formalism, orbits for the group action are precisely the reachable sets for the system; therefore, the analysis of its structure reduces to the analysis of the acting group.

The idea of encoding control actions with symbols is present in many other recent papers, see [5,9,11]. The main difference is in the way this encoding is operated: in [9] the author uses motion description languages; in [11] a set of trim trajectories is defined; finally in [5] a general framework is proposed based on control quanta.

In our work, we consider the quantization of a rolling pair. This system does not belong to the class of systems for which the effect of quantization is known. However, geometric tools help in reducing the problem to the analysis of a group acting on a manifold. More precisely, for two bodies one rolling over the other, a quantization is obtained assigning on, say, the first body: a finite set of base points, a finite set of closed paths for each point and a transit path for each couple of base points. The set of controls is obtained by concatenating closed and transit paths. Since the bodies are always in contact, the rolling is described by the motion of their boundaries; thus we write it as the dynamics of one surface over another. We are able to show that the corresponding orbits can be studied by group actions over the unit tangent bundle of the second surface. In particular, in Section 5.1, we prove that, when the second body is a plane, the problem is reduced to the analysis of the action of a subgroup $H$ of SE(2) over SE(2) itself. Similarly in Section 5.2, we treat the case of a sphere (as second surface) and get the action of a subgroup $H$ of SO(3) over SO(3) itself. In all the previous instances, the point-to-point controllability issue reduces to the algebraic question of determining the topological closure of the subgroup $H$ in SE(2) (resp. in SO(3)) when the second surface is a Euclidean plane (a sphere of radius one).

## 2. Basic definitions

If $P$ is a matrix, we use $P^{\mathrm{T}}$ and $\mathrm{tr}(P)$ to denote the transpose of $P$ and the trace of $P$, respectively.

Let $(\mathscr{M}, \langle\cdot,\cdot\rangle)$ be a two-dimensional, connected, oriented $C^\infty$ complete Riemannian manifold for the Riemannian metric $\langle\cdot,\cdot\rangle$. We use $T\mathscr{M}$ to denote the tangent bundle over $\mathscr{M}$ and $U\mathscr{M}$ the unit tangent bundle, i.e. the subset of $T\mathscr{M}$ of points $(x, v)$ such that $x \in \mathscr{M}$ and $v \in T_x\mathscr{M}$, $\langle v, v\rangle = 1$. The unit tangent bundle $U\mathscr{M}$ is endowed with the Sasaki metric (cf. [20]) and the volume form of $T\mathscr{M}$ restricts to a volume form on $U\mathscr{M}$.

Let $\{U_\alpha, \alpha\}_{\alpha\in\mathscr{A}}$ be an atlas on $\mathscr{M}$. For $\alpha, \beta \in \mathscr{A}$ such that $U_\alpha \cap U_\beta$ is not empty, we denote by $J_{\beta\alpha}$ the jacobian matrix of $\varphi^\beta \circ (\varphi^\alpha)^{-1}$ the coordinate transformation on $\varphi^\alpha(U_\alpha \cap U_\beta)$. For $\alpha \in \mathscr{A}$, the Riemannian metric is represented by the symmetric definite positive matrix $\mathscr{I}^\alpha$ and set $M^\alpha := \sqrt{\mathscr{I}^\alpha}$. For our purpose, it is natural to consider surfaces embedded in $\mathbb{R}^3$. In that case, charts are of the type $\alpha = (f, U)$, where $f : U \to \mathbb{R}^3$ is a (local) parameterization of the manifold $\mathscr{M}$. The Riemannian metric $\langle\cdot,\cdot\rangle$ is the one induced on $\mathscr{M}$ by the Euclidean metric of $\mathbb{R}^3$. The matrix $\mathscr{I}^\alpha$ is then called the first fundamental form of $\mathscr{M}$ in the chart domain and is given by

$$\mathscr{I}^{\alpha} = (\mathrm{d}f)^2 = \|f_u\|^2\,\mathrm{d}u^2 + 2\langle f_u f_v\rangle\,\mathrm{d}u\,\mathrm{d}v + \|f_v\|\,\mathrm{d}v^2.$$

At each point $x \in \mathscr{M}$, $T_x\mathscr{M}$ denotes the tangent plane at $x$ and a positively oriented basis for $\mathbb{R}^3$ is given by $f_u,\ f_v$ and $n = (f_u \times f_v)/(|f_u \times f_v|)$, where $f_u,\ f_v \in T_x\mathscr{M}$ and $n$ is the unit tangent vector perpendicular to $T_x\mathscr{M}$. Then, the second fundamental form $\mathscr{I}\mathscr{I}^{\alpha}$ is given by

$$\mathscr{I}\mathscr{I}^{\alpha} = \mathrm{d}^2 f = \langle f_{uu}, n\rangle\,\mathrm{d}u^2 + 2\langle f_{uv}, n\rangle\,\mathrm{d}u\,\mathrm{d}v + \langle f_{vv}, n\rangle\,\mathrm{d}v^2.$$

A simply connected manifold $\mathscr{M}$ such that the determinant of $\mathscr{I}\mathscr{I}^{\alpha}$ is positive at every point of $\mathscr{M}$ is called convex.

For $x \in \mathscr{M}$, a frame $F$ at $x$ is an ordered basis for $T_x\mathscr{M}$ and, for $\alpha, \beta \in \mathscr{A}$, $f^{\beta} = J_{\beta\alpha} f^{\alpha}$. The frame $f$ is orthonormal if, in addition, $M^{\alpha} f^{\alpha}$ is an orthogonal matrix. An orthonormal moving frame (OMF) defined on an open subset $U$ of $\mathscr{M}$ is a smooth map assigning to each $x \in U$ a positively oriented orthonormal frame $F(x)$ of $T_x\mathscr{M}$.

Let $\nabla$ be the Riemannian connection on $\mathscr{M}$ (cf. [23]). For a given OMF $F$ defined on $U \subset \mathscr{M}$, the Christoffel symbols associated to $F$ are defined by

$$\nabla_{F^i} F^j = \sum_k \Gamma^k_{ij} F^k,$$

where $1 \leqslant i, j, k \leqslant 2$. The connection form $\omega$ is the mapping defined on $U$ such that, for every $x \in U$, $\omega_x$ is the linear application from $T_x\mathscr{M}$ to the set of $2 \times 2$ skew-symmetric matrices given as follows. For $i, j, k = 1, 2$, the $(i, j)$th coefficient of $\omega_x(F^k)$ is equal to $\Gamma^k_{ij}$.

Let $\gamma : J \to \mathscr{M}$ be an absolutely continuous curve in $\mathscr{M}$ with $J$ compact interval of $\mathbb{R}$. Set $X(t) := \dot{\gamma}(t)$ a.e. in $J$. Let $Y : J \to T\mathscr{M}$ be an absolutely continuous assignment such that, for every $t \in J$, $Y(t) \in T_{\gamma(t)}\mathscr{M}$. We say that $Y$ is parallel with respect to (or along) $\gamma$ if $\nabla_X Y = 0$ for a.e. $t \in J$. In the domain of an OMF $F$, that relation can be written as

$$\dot{Y}^k = -\sum_{ij} \Gamma^k_{ij} X^i Y^j$$

or, equivalently,

$$\dot{Y} = -\omega(X)Y.$$

In a similar manner, an OMF $(Y^1, Y^2)$ is parallel with respect to (or along) $\gamma$ if $\nabla_X Y^1 = 0$. Recall that $\gamma$ is a geodesic if $\dot{\gamma}(t)$ is parallel along $\gamma$, that is $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. If $x \in \mathscr{M}$ and $v \in T_x\mathscr{M}$ there exists a unique geodesic $\gamma$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$ and we write $\gamma(t) = \mathrm{e}^{tv}x$.

We identify SO(2) with the circle $S^1$ by the application that associates to $\theta \in [0, 2\pi)$, $R^{\theta}$, the rotation of (oriented) angle $\theta$. For every $x \in \mathscr{M}$ and $\theta \in \mathrm{SO}(2)$, let $R^{\theta}_x : T_x\mathscr{M} \to T_x\mathscr{M}$ be the rotation of angle $\theta$ in $T_x\mathscr{M}$, i.e., for every $v \in T_x\mathscr{M}$, $R^{\theta}_x(v)$ is the unique vector of $T_x\mathscr{M}$ having the same length as $v$ and such that the oriented angle between the angle between $v$ and $R^{\theta}_x(v)$ is equal to $\theta$. For $t \in \mathbb{R}$ and an angle $\theta$, $\mathrm{e}^{tf_{\theta}}$ denotes the geodesic flow at time $t$ of angle $\theta$ on $\mathscr{M}$, i.e., for $(x, v) \in U\mathscr{M}$, $\mathrm{e}^{tf_{\theta}}(x, v)$ is the point reached at time $t$ by the geodesic starting at $x$ at time 0 in the direction $R^{\theta}_x(v)$. If $\theta = 0$, we refer to $\mathrm{e}^{tf_0}$ as the geodesic flow.

Set $\mathscr{P} := U\mathscr{M}$. Define the right action $A$ of SO(2) on $\mathscr{P}$ by

$$\mathscr{P} \times \mathrm{SO}(2) \mapsto \mathscr{P},$$

$$((x, v), \theta) \mapsto A_{\theta}(x, v) := (x, R^{\theta}_x(v)).$$

The action $A$ induces an equivalence relation $\sim$ on $\mathscr{P}$ as follows: $(x_1, v_1) \sim (x_2, v_2)$ if there exists $\theta \in \mathrm{SO}(2)$ such that $A_{\theta}(x_1, v_1) = (x_2, v_2)$.

That action is clearly free (see Definition 6). Therefore, $\mathscr{M}$ is the quotient space of $\mathscr{P}$ by $\sim$ and the canonical projection $\pi : \mathscr{P} \to \mathscr{M}$ is differentiable. For every $x \in \mathscr{M}$, the set $\pi^{-1}(x)$ is called the fiber over $x$ and it is a closed submanifold diffeomorphic to SO(2). Moreover, for every $x \in \mathscr{M}$, there exists a neighborhood $U$ of $x$ such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathrm{SO}(2)$. To see that, simply choose an OMF $F$ defined in a neighborhood of $x \in \mathscr{M}$ and, for every $y \in U$, parameterize $\pi^{-1}(y)$ by the oriented angle of any $v \in T_y\mathscr{M}$ with $F^1(y)$. If $(x, v) \in \mathscr{P}$ and $a \in \mathrm{SO}(2)$, we may write $A_a(x, v)$ simply as $(x, va)$. It is also clear that, for every $x \in \mathscr{M}$, the set of orthonormal frames of $T_x\mathscr{M}$ can be identified with $\pi^{-1}(x)$ and thus is diffeomorphic with SO(2).

We just showed that $\mathscr{P}$ is a principal fiber bundle over $\mathscr{M}$ with Lie group SO(2) and group action $A$.

We need one more definition for later use.

**Definition 1.** Given $a \in \mathrm{SO}(2)$, the geodesic flow $\mathrm{e}^{ta}$ induces a map $\exp(ta)$ on $\mathscr{P}$ defined as follows. For $(x, v) \in \mathscr{P}$, let $t \to b(t) \in \mathrm{SO}(2)$ be a vector field parallel along $\gamma_a := \mathrm{e}^{ta}x$ with $b(0) = v$. Then,

$$\exp(ta)(x, v) = (\mathrm{e}^{ta}x, b(t)).$$

## 3. Rolling body problem

In this section, we briefly recall how the rolling without slipping nor spinning of a manifold $\mathscr{M}_1$ onto another one $\mathscr{M}_2$ was defined in [15,1,7,19]. We first start with the definition of the state space and then proceed by characterizing the rolling dynamics.

### 3.1. State space

Let $\mathscr{D}$ be the diagonal of $\mathrm{SO}(2) \times \mathrm{SO}(2)$, i.e. the elements of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ that can be written $(a, a)$. There is an action DA of $\mathscr{D}$ on $\mathscr{P}_1 \times \mathscr{P}_2$ defined as follows. Consider $((x_1, v_1), (x_2, v_2)) \in \mathscr{P}_1 \times \mathscr{P}_2$ and $(a, a) \in \mathscr{D}$. Set

$$\mathrm{DA}_{(a,a)}((x_1, v_1), (x_2, v_2)) = ((x_1, v_1 a), (x_2, v_2 a)).$$

Two points $((x_1, b_1), (x_2, b_2))$ and $((x_1, c_1), (x_2, c_2))$ of $\mathscr{P}_1 \times \mathscr{P}_2$ are equivalent along with $\sim_{\mathrm{DA}}$, the equivalence relation induced by DA if and only if $b_2 b_1^{-1} = c_2 c_1^{-1}$.

Notice that $\mathscr{P}_1 \times \mathscr{P}_2$ is a six dimensional fiber bundle over $\mathscr{M}_1 \times \mathscr{M}_2$ with group $\mathrm{SO}(2) \times \mathrm{SO}(2)$. We define $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ as the quotient space of $\mathscr{P}_1 \times \mathscr{P}_2$ by $\sim_{\mathrm{DA}}$. This is a five-dimensional principal fiber bundle over $\mathscr{M}_1 \times \mathscr{M}_2$ with Lie group $\mathrm{SO}(2)$.

We use $\pi_{\mathscr{M}_1}$ and $\pi_{\mathscr{M}_2}$ to denote the canonical projections on $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. We can endow $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ of a Riemannian metric $\langle \cdot, \cdot \rangle$ defined as follows. At a point $x = (x_1, x_2, a)$ ($a \in \mathrm{SO}(2)$) for $v = (v_1, v_2, as)$ ($s$ a skew-symmetric $2 \times 2$ matrix) in $T_x \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$,

$$\langle v, v \rangle \overset{\mathrm{def}}{=} \tfrac{1}{2}(\langle v_1, v_1 \rangle_1 + \langle v_2, v_2 \rangle_2 - \mathrm{tr}(s^2)).$$

### 3.2. Rolling dynamics

Next, we describe the motion of one body rolling on top of another one so that the contact point of the first follows a prescribed absolutely continuous curve on the second.

Let $F_1$ and $F_2$ be two OMFs defined on the chart domains of $\alpha_1, \alpha_2$. Let $b_i(t) = F_i(\gamma_i(t))R_i(t)$ be parallel along $\gamma_i^{\alpha_i}$ and $R := R_2(t)R_1(t)^{-1} \in \mathrm{SO}(2)$ which, by definition, measures the relative position of $F_2$ with respect to $F_1$ along $(\gamma_1^{\alpha_1}, \gamma_2^{\alpha_2})$. The variation of $R_i$ along $\gamma_i^{\alpha_i}$, for $i = 1, 2$, is given by $\dot{R}_i = -\omega_i(\dot{\gamma}_i^{\alpha_i})R_i$.

Let $F_1$ and $F_2$ be two OMFs defined on the chart domains of $\alpha_1, \alpha_2$. For $i = 1, 2$, let $b_i(t)$ be any OMF parallel along $\gamma_i^{\alpha_i}$. Then $b_i(t) = F_i(\gamma_i(t))R_i(t)$, $R_i \in \mathrm{SO}(2)$, where the variation of $R_i$ along $\gamma_i^{\alpha_i}$, for $i=1, 2$, is given by $\dot{R}_i = -\omega_i(\dot{\gamma}_i^{\alpha_i})R_i$.

Given an a.c. curve $\gamma_1 : [0, T] \to \mathscr{M}_1$, the rolling of $\mathscr{M}_2$ on $\mathscr{M}_1$ without slipping nor spinning along $\gamma_1$ is characterized by a curve $\Gamma = (\gamma_1, \gamma_2, R) : [0, T] \to \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ defined as follows. Up to initial conditions, no slipping amounts to

$$M^{\alpha_2}\dot{\gamma}_2^{\alpha_2}(t) = RM^{\alpha_1}\dot{\gamma}_1^{\alpha_1}(t) \tag{1}$$

and no spinning to

$$\dot{R}R^{-1} = R\omega_1(\dot{\gamma}_1^{\alpha_1})R^{-1} - \omega_2(\dot{\gamma}_2^{\alpha_2}). \tag{2}$$

Since the $\omega_i$'s are $2 \times 2$ skew-symmetric, Eq. (2) reduces to

$$\dot{R}R^{-1} = \omega_1(\dot{\gamma}_1^{\alpha_1}) - \omega_2(\dot{\gamma}_2^{\alpha_2}). \tag{3}$$

If we fix a point $x = (x_1, x_2, R_0) \in \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$, a curve $\gamma_1$ on $\mathscr{M}_1$ starting at $x_1$ defines entirely the curve $\Gamma$ by Eqs. (1) and (3).

Therefore we can give the following definition:

**Definition 2.** $\mathscr{M}_2$ rolls on $\mathscr{M}_1$ without slipping or spinning if, for every $x=(x_1, x_2, R_0) \in \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ and a.c. curve $\gamma_1 : [0, T] \to \mathscr{M}_1$ starting at $x_1$, there exists an a.c. curve $\Gamma : [0, T] \to \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ with $\Gamma(t) = (\gamma_1(t), \gamma_2(t), R(t))$, $\Gamma(0) = x$ and for every $t \in [0, T]$, on appropriate coordinate systems, Eqs. (1) and (3) are satisfied. We call the curve $\Gamma(t)$ an admissible trajectory.

The above definition of the rolling of $\mathscr{M}_2$ on $\mathscr{M}_1$ without slipping nor spinning is given in coordinates (i.e. depends on the choice of charts and OMF). A coordinate-free definition of rolling as a standard control system $\mathscr{S}_R$ is given in [7] and is briefly reported in the Appendix for the reader's convenience.

**Remark 1.** In general, it is not possible to get a global basis for the distribution $\Delta$ (see Appendix A) and therefore to define globally the dynamics of the control system $\mathscr{S}_R$. One notable exception occurs when one of the manifolds is a plane [7]. Therefore, addressing the motion planning efficiently becomes a delicate issue since most of the standard techniques are based on a global vector field expression of the dynamics of a control system.

**Remark 2.** When the manifolds $\mathscr{M}_1$ and $\mathscr{M}_2$ are oriented surfaces of $\mathbb{R}^3$ with metrics induced by the Euclidean metric of $\mathbb{R}^3$, there are two possible ways to define the rolling problem, depending on the respective (global) choice of normal vectors for $\mathscr{M}_1$ and $\mathscr{M}_2$. Indeed, the orientation of the tangents planes of an oriented surface $\mathscr{M}$ is determined by the choice of a Gauss map, i.e. a continuous normal vector $n : \mathscr{M} \to S^2$. There are two such normal vectors, $n$ and $-n$. If $\mathscr{M}$ is (strictly) convex, these two normal vectors are called inward and outward.

For the rolling problem, recall that we identify two oriented OMFs $b_1$ and $b_2$. In the case of surfaces, this identification is equivalent to identify the two OMFs $B_1$ and $B_2$ of $\mathbb{R}^3$, each of them made up of $b_i$ and $n_i$, the normal vector. Since $b_1$ is identified with $b_2$, then $n_1 = \varepsilon_R n_2$, where $\varepsilon_R = \pm 1$ and only depends on the choice of orientations of $\mathscr{M}_1$ and $\mathscr{M}_2$.

An easy but fundamental property of the rolling is given by the following (see [7]):

**Proposition 1.** *Let* $\Gamma(t) = (\gamma_1(t), \gamma_2(t), R(t))$, $t \in [0, T]$, *be an admissible trajectory. If* $\gamma_1$ *is a finitely broken geodesic on* $\mathscr{M}_1$, *then* $\gamma_2$ *is a finitely broken geodesic on* $\mathscr{M}_2$ *and R is constant.*

## 4. Quantization of the rolling body problem

### 4.1. Definitions

In this paragraph, we describe our quantization procedure. The quantization of the control system $\mathscr{S}_R$ we propose proceeds in two steps as defined next.

1. Fix a finite number of base points $p_1, \ldots, p_n$ on $\mathscr{M}_1$ and notice that $p_i \times T_1\mathscr{M}_2$ can be imbedded as a three-dimensional submanifold of $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$.
2. Define control quantas, see [14], by choosing, for each $i = 1, \ldots, n$, a finite set of a.c. closed loops $\gamma_l^i$ on $\mathscr{M}_1$, $l = 1, \ldots, m_i$, based at $p_i$.

**Definition 3.** Let $\mathscr{L}_p$ be the set of a.c. closed loops on $\mathscr{M}_1$, parameterized by arclength, based at $p$ and $\mathscr{L}_p^g$ its subset made of concatenations of a finite number of geodesic segments.

For every $\gamma \in \mathscr{L}_p$, we let $A_\gamma : \mathscr{P}_2 \to \mathscr{P}_2$ be defined as follows. For $(x_2, R) \in \mathscr{P}_2$, $A_\gamma(x_2, R)$ is the terminal point of $\Gamma$, the admissible curve of $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ associated to the initial point $(p, x_2, R)$ and curve $\gamma$, cf. Definition 2.

**Proposition 2.** $A_\gamma$ *is a symplectomorphism* (*i.e. a diffeomorphism preserving the volume form*) *of* $\mathscr{P}_2$.

**Proof.** It is a diffeomorphism because $A_\gamma$ is the flow map of a smooth time varying vector field. By density of $\mathscr{L}_p^g$ in $\mathscr{L}_p$, it is enough to show that $A_\gamma$ preserves the volume form for $\gamma \in \mathscr{L}_p^g$. By concatenation we reduce to a geodesic flow which is a symplectomorphism. □

The set $\mathscr{L}_p$ has a natural group structure given by concatenation and by defining $\gamma^{-1}$ the path $\gamma$ run backwards in time. This notation obviously extends to any a.c. path of finite length on $\mathscr{M}_2$. Note that $\mathscr{L}_p^g$ is a subgroup of $\mathscr{L}_p$.

We are now ready to define a quantization of our rolling body problem.

**Definition 4.** Given the two bodies $\mathscr{M}_1$, $\mathscr{M}_2$, a quantization of the rolling body problem of $\mathscr{M}_1$ onto $\mathscr{M}_2$ is given by assigning

(Q1) a finite set of points $p_i \in \mathscr{M}_1$, $i = 1, \ldots, n$,
(Q2) for $i = 1, \ldots, n$, a finite set of a.c. closed curves $\gamma_l^i$, $l = 1, \ldots, m_i$, based at $p_i$ and parameterized by arclength,

(Q3) for $1 \leqslant i < j \leqslant n$, a transit path $\tau_{ij}$ that is an a.c. curve parameterized by arclength connecting $p_i$ to $p_j$.

The assumption of arclength is not necessary for the definitions and several properties listed below. However, it simplifies several computations.

**Remark 3.** As for the case of the rolling of a polyhedron on a plane (see [3]) it is not difficult to see that the groups $\mathscr{L}^g_{p_i}$, $i = 1, \ldots, n$, are in fact conjugate, i.e., for every $i, j = 1, \ldots, n$ there exists a bijection between $\mathscr{L}^g_{p_i}$ and $\mathscr{L}^g_{p_j}$. It is defined as the assignment $\gamma \mapsto \gamma_{ij}$, where $\gamma_{ij}$ is the concatenation of $\tau_{ij}^{-1}$, $\gamma$ and $\tau_{ij}$.

Our basic problem is to analyze the effect of quantization of the rolling body system. This will be done through the study of an algebraic object $\mathscr{Q}$ that we call the quantization group, defined next.

**Definition 5.** Fix $p_1$ in $\mathscr{M}_1$. For every $i = 2, \ldots, n$ and $l = 1, \ldots, m_i$, $\gamma^i_l$ can be associated to an element of $\mathscr{L}^g_{p_1}$, thanks to Remark 3. Then the quantization group $\mathscr{Q}$ associated to the quantization of the rolling body problem of $\mathscr{M}_1$ onto $\mathscr{M}_2$ is the subgroup of $\mathscr{L}^g_{p_1}$ generated by the $\gamma^1_l$'s.

As a consequence of Remark 3, it is clear that $\mathscr{Q}$ does not depend on the choice of the reference point $p_1$.

We can now reformulate the motion planning problem of the rolling to the following statement: given $(x, v) \in \mathscr{P}_2$, study the orbit $O^q_{(x,v)}$ by $\mathscr{Q}$ (for a definition of orbit, see Definition 6 below). Since the latter is countable, so is $O^q_{(x,v)}$ and the best one can hope (with a controllability perspective in mind) is $O^q_{(x,v)}$ to be dense in $\mathscr{P}_2$. This is called approachability in [4]. In any case, the approach we propose is based on the investigation of $\mathscr{Q}$.

**Remark 4.** A quantization, according to Definition 4, automatically gives a way of encoding the information for control algorithms. In fact one can set a finite alphabet of symbols corresponding to closed and transit paths: $\Sigma = \{\gamma^i_l : i = 1, \ldots, n,\ l = 1, \ldots, m_i\} \cup \{\tau_{ij} : 1 \leqslant i < j \leqslant n\}$. Regarding $\Sigma$ as an alphabet, we consider the set of words generated by concatenations of symbols in $\Sigma$ that obeys specific rules. More precisely:

- $\gamma^{i_1}_l$ concatenates with $\gamma^{i_2}_m$ if and only if $i_1 = i_2$;
- $\gamma^{i_1}_l$ concatenates with $\tau_{i_2 j}$ if and only if $i_1 = i_2$;
- $\tau_{ki_1}$ concatenates with $\tau_{i_2 j}$ if and only if $i_1 = i_2$.

For encoded controls, it appears that the possibility of using feedbacks permits one to treat larger classes of systems with simplified algorithms, see [5,9,11]. However, for the rolling bodies problem, the use of feedback is not so efficient. Indeed, the system has only a local coordinate expression, hence only local feedbacks can be easily defined. On the other hand, quantization is a technique for global reachability problem. Also, a smooth global feedback necessarily presents a set of zeroes (more than one) for topological reasons.

### 4.2. Study of $\mathscr{Q}$ through group actions

In the case where $\mathscr{M}_2$ admits a group action that extends to a group action of $\mathscr{P}_2$, it is possible to deduce some information on the group $\mathscr{Q}$ and eventually to get a precise description of that group.

We first need basic definitions for the group actions we consider (cf. [18]).

**Definition 6.** If $G$ is a group and $X$ a set, we say that $G$ acts on $X$ on the left (resp. on the right) if the following holds: to every $g \in G$, is associated a bijection $a_g$ of $X$ so that the identity element of $G$, $\mathrm{id}_G$, is the identity bijection of $X$ and, for every $g_1, g_2 \in G$ and $x \in X$, $a_{g_1}(a_{g_2}(x)) = a_{g_1 g_2}(x)$ (resp. $a_{g_1}(a_{g_2}(x)) = a_{g_2 g_1}(x)$). For $x \in X$, the set $O^G_x = \{a_g(x) | \ g \in G\}$ is called the orbit of $x$ by $G$.

If $G$ is a Lie group and $X$ is a differentiable manifold, then $a_g$ is a diffeomorphism of $X$ and the mapping $a : X \times G \to X$ that associates $a_g(x)$ to $(g, x)$ is differentiable.

If $H$ is a subgroup of $G$, then the group action of $G$ on $X$ restricts in an obvious manner to a group action of $H$ on $X$. For $x \in X$, $O^H_x$ is the orbit of $x$ by $H$.

For $x \in X$, let $\mathscr{H}_x$ be the isotropy group associated to $x$, i.e. the subgroup of $G$ so that $g \in H_x$ if $a_g(x) = x$. The action $a$ is said to be free if, for every $x$, $H_x$ reduces to $\mathrm{id}_G$, the identity of $G$.

The action $a$ is said to be transitive if, for every $x, x' \in X$ there exists $g \in G$ such that $a_g(x) = x'$.

In this case, $X$ is said to be a homogeneous $G$-space. Clearly, every orbit $O_x^G$ is equal to $X$.

Going back to the rolling problem, we assume that $\mathscr{M}_2$ admits a group action that extends to a group action of $\mathscr{P}_2$. More precisely, the following holds.

**Definition 7.** Consider a (left) action $a : \mathscr{M}_2 \times G \to \mathscr{M}_2$ of a group $G$ on $\mathscr{M}_2$ such that, for every $g \in G$, $(a_g)_\star(x) : T_x\mathscr{M}_2 \to T_{a_g(x)}\mathscr{M}_2$ is an isometry preserving orientation between the two vector spaces $T_x\mathscr{M}_2$ and $T_{a_g(x)}\mathscr{M}_2$. Then the action $a$ can be extended to an action $\tilde{a} : \mathscr{P}_2 \times G \to \mathscr{P}_2$ with $\tilde{a}_g(x,v) = (a_g(x), (a_g)_\star(x)(v))$. Note that, in coordinates, $(a_g)_\star(x)$ can be represented by an element of SO(2).

Owing to the fact that SO(2) is commutative, the next key property holds true.

**Proposition 3.** *If $\gamma \in \mathscr{L}_p$, then for every $g \in G$ and $(x,v) \in \mathscr{P}_2$,*

$$\tilde{a}_g(A_\gamma(x,v)) = A_\gamma(\tilde{a}_g(x,v)). \tag{4}$$

**Proof.** It is enough to show the proposition for $\gamma \in \mathscr{L}_p^g$. By composition and using the notations of Definition 1, we consider (4) with $A_\gamma$ replaced by $\exp(tb)$ for $t \in \mathbb{R}$ and $b \in$ SO(2), for $t \in \mathbb{R}$ and $a$ in SO(2) corresponding to a rotation of angle $\theta$ of $v$. If $t = 0$, the conclusion trivially holds; thus it is enough to establish the equality obtained by differentiating (4).

At time 0 the differentiation gives

$$\begin{aligned}(\tilde{a}_g)_\star(x,v)\cdot(R_x^\theta(v), -\omega(R_x^\theta(v))(v)) \\ = (R^\theta_{a_g(x)}((a_g)_\star(x)(v)), \\ -\omega(R^\theta_{a_g(x)}((a_g)_\star(x)(v)))((a_g)_\star(x)(v))).\end{aligned}$$

By definition of $\tilde{a}$ we have, for every $(v_1, v_2) \in T_{(x,v)}\mathscr{P}_2$,

$$(\tilde{a}_g)_\star(x,v)\cdot(v_1,v_2) = ((a_g)_\star(x)(v_1), (a_g)_\star(x)(v_2)).$$

Thus the left-hand side of the above equation can be written as

$$((a_g)_\star(x)(R_x^\theta(v)), (a_g)_\star(x)(-\omega(R_x^\theta(v))(v))).$$

Hence, using the linearity of $\omega$, it amounts to ask

$$(a_g)_\star(x)\cdot R_x^\theta = R^\theta_{a_g(x)}\cdot(a_g)_\star(x)$$

for every $x \in \mathscr{M}$ and $\theta \in$ SO(2). This is guaranteed since they are elements of SO(2) in coordinates. Moreover, the differentiation at any time can be computed similarly. We thus conclude. □

This implies that every orbit $O_{(x,v)}$ is isomorphic to $G$.

Assume now that the action $\tilde{a}$ is transitive. Then Proposition 3 exactly says that for every $\gamma \in \mathscr{L}_p$, $A_\gamma$ is $G$-equivariant (cf. [18]). In other words, $A_\gamma$ is an automorphism of the homogeneous $G$-space $\mathscr{P}_2$ (i.e. a $G$-equivariant isomorphism of $\mathscr{P}_2$). There exists a complete description of the group of automorphisms of a homogeneous $G$-space (cf. [18]) which reads in the present situation as follows:

**Proposition 4.** *Assume that $\tilde{a}$ acts transitively on $\mathscr{P}_2$. Then $\mathscr{P}_2$ is a homogeneous $G$-space and, for every $\gamma \in \mathscr{L}_p$, $A_\gamma$ is an element of* Aut$(\mathscr{P}_2)$, *the group of automorphisms of $\mathscr{P}_2$ given next.*

*Let $(x_0, v_0) \in \mathscr{P}_2$ and $H$ its isotropy group. Let $N(H)$ be the normalizer of $H$ that is*

$$N(H) = \{g \in G \mid gHg^{-1} = H\}. \tag{5}$$

*Then* Aut$(\mathscr{P}_2)$ *is isomorphic to $N(H)/H$. If the action $\tilde{a}$ is free i.e. $H = \{\mathrm{id}_G\}$, then* Aut$(\mathscr{P}_2)$ *is isomorphic to $G$.*

For subsequent applications of Proposition 4 let us explicit the isomorphism Aut$(\mathscr{P}_2) \sim G$ in the case of a free action.

Fix $(x_0, v_0) \in \mathscr{P}_2$. Let $\psi : G \to \mathscr{P}_2$ defined by $\psi(g) = a_g(x_0, v_0)$. This map is well-defined since $\tilde{a}$ is transitive and it is a bijection since $\tilde{a}$ is free. The isomorphism between Aut$(\mathscr{P}_2)$ and $G$ is simply the correspondence that associates to $\phi \in$ Aut$(\mathscr{P}_2)$ the element of $G$, $g_\phi := \psi^{-1}(\phi(x_0, v_0))$. Using (4), there is even a clearer way to read that correspondence. If $\phi \in$ Aut$(\mathscr{P}_2)$, then we have for every $g \in G$,

$$\phi(\psi(g)) = \psi(g \cdot g_\phi), \tag{6}$$

simply saying that through the map $\psi$, an automorphism of $\mathscr{P}_2$ acts on $G$ as a right multiplication operator.

Recall that the control object we had to study for the rolling problem was $O^q_{(x,v)}$. If $\mathscr{M}_2$ admits a group action that extends to a group action of $\mathscr{P}_2$ which is free and transitive, then, from what precedes, we have to understand how large is the closure of the quantization group $Q$ in $G$.

## 5. Motion planning by quantized rolling

In this section we show for specific choices of $\mathscr{M}_2$ ($\mathscr{M}_1$ arbitrary) how to treat the motion planning problem via quantization.

### 5.1. Rolling on a plane

We assume in this section that $\mathscr{M}_2$ is the plane. Then

$$\mathscr{P}_2 \simeq \mathrm{SE}(2),$$

as Lie groups in the sense that we can identify $\mathscr{P}_2$ with SE(2) endowed of its algebraic structure of semidirect product $\mathbb{R}^2 \times_R S^1$. Given $(t, \theta)$, $(t', \theta') \in \mathrm{SE}(2)$ we define

$$(t, \theta) \cdot (t', \theta') = (t + R(\theta)t', \theta + \theta'),$$

where $R(\theta) : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation of angle $\theta$.

The Lie group SE(2) acts on the plane $\mathbb{R}^2$ as the group of Euclidean motions of the plane that preserve the orientation. Clearly this action extends to $\mathscr{P}_2$ and coincides with the group action of SE(2) on itself defined by the left multiplication. This action is free and transitive. All the conditions of the previous section are satisfied. Let us read them.

For $\gamma \in \mathscr{L}_p$, $A_\gamma$ commutes with the left action of SE(2) over itself. Thus for every element $s \in \mathrm{SE}(2)$ we can write

$$A_\gamma s(\mathrm{id}) = s A_\gamma(\mathrm{id}),$$

where $\mathrm{id} = \mathrm{id}_{\mathrm{SE}(2)}$. Hence we get the following.

**Proposition 5.** *$A_\gamma$ acts as a right multiplication by the element $A_\gamma(\mathrm{id})$. Therefore $Q$ (identified with the group of actions $A_\gamma$ for $\gamma \in Q$) is a subgroup of $\mathrm{SE}(2)_r \subset \mathrm{Diff}(\mathrm{SE}(2))$ the set of right multiplication by elements of* SE(2).

Given $\gamma \in Q$, we can determine the element $(t_\gamma, \theta_\gamma) \in \mathrm{SE}(2)$ which represents $A_\gamma(\mathrm{id})$ in the case where $\mathscr{M}_1$ is a surface of $\mathbb{R}^3$ and the arc $\dot\gamma$ (taking values in the unit sphere $S^2$) is piecewise absolutely continuous. In that case, let $\kappa_g$ be the geodesic curvature of $\gamma$. It is a measurable real-valued function. We allow $\dot\gamma$ to have a finite number of discontinuities at time $0 < t_1 < \cdots < t_m \leqslant T$. For each of them, set $\theta_i$ as the oriented angle between $\dot\gamma(t_i-)$ and $\dot\gamma(t_i-)$ (if $t_m = T$, then $t_m+$ is the limit as the time tends to $0+$).

**Proposition 6.** *Let $\gamma \in Q$, $\gamma : [0, T] \to \mathscr{M}_1$, with geodesic curvature $\kappa_g$. Set $\theta : [0, T] \to S^1$, $\theta(t) = \varepsilon_R \sum_{t_i \leqslant t} \theta_i + \int_0^t \kappa_g(s)\, \mathrm{d}s$, where $\varepsilon_R$ was defined in Remark* 2.

*Then, if $A_\gamma(\mathrm{id})$ is represented by $(t_\gamma, \theta_\gamma) \in \mathrm{SE}(2)$, we have*

$$t_\gamma = \int_0^t \mathrm{e}^{i\theta(s)}\, \mathrm{d}s, \quad \theta_\gamma = \theta(T), \tag{7}$$

*where $\mathrm{e}^{\mathrm{i}\theta(s)}$ represents the unit vector of the plane* $(\cos(\theta(s)), \sin(\theta(s)))^{\mathrm{T}}$.

A proof of the above proposition is provided in Appendix A. Moreover, $\theta_\gamma$ can be related to the total curvature enclosed by $\gamma$, thanks to the Gauss–Bonnet theorem.

As proposed in [4], the structure of the reachable sets of a quantized system can be investigated by looking at subgroups of the group of actions. We are thus led to consider the subgroups of SE(2).

The same approach was used in [3] where the set of reachable points for a polyhedron rolling on a plane is investigated. In this case we obtain exactly the same results. The possible subgroups of SE(2) are one of the following:

1. $H \times_R G$, where $H$ is a dense subgroup of $\mathbb{R}^2$ generated by three elements $t_i$ such that $t_3 = \alpha_1 t_1 + \alpha_2 t_2$ with $\alpha_1$ or $\alpha_2$ irrational numbers and $G$ is a dense subgroup of $S^1$ generated by an element $\theta$ such that $\theta/\pi$ is an irrational number.
2. $H \times_R G$, where $H$ is a dense subset of $\mathbb{R}^2$ and $G$ is a finite subgroup of $S^1$ generated by an element $\theta = 2\pi/q$, with $q$ rational. In this case $G$ contains $q$ elements.
3. $H \times_R G$, where $H$ is a discrete subgroup of $\mathbb{R}^2$ generated by two elements and $G$ is a finite

subgroup of $S^1$ generated by one element $\theta=2\pi/q$ with $q \in \{1, 2, 3, 4, 6\}$.

### 5.2. Rolling on the sphere

We assume in this section that $\mathscr{M}_2$ is the unit sphere $S^2$. In that case, we have

$$\mathscr{P}_2 \simeq \mathrm{SO}(3),$$

as Lie groups in the sense that we can identify $\mathscr{P}_2$ with SO(3) such that, if $S^2$ is centered at $(0, 0, 0)^{\mathrm{T}}$, then $(e_3, e_1) \in \mathscr{P}_2$ is identified with $\mathrm{id}_3$, the identity of SO(3). Here $(e_i)_{i=1,2,3}$ is the canonical basis of the Euclidean space $\mathbb{R}^3$.

In an entirely similar manner as before we get

**Proposition 7.** *$A_\gamma$ acts as a right multiplication by the element $A_\gamma(\mathrm{id})$. $\Gamma$ (the group of actions $A_\gamma$) is a subset of $\mathrm{SO}(3)_r \subset \mathrm{Diff}(\mathrm{SO}(3))$ the set of right multiplication by elements of SO(3).*

Moreover, we can characterize the rotation $R_\gamma \in \mathrm{SO}(3)$ representing $A_\gamma$ when $\mathscr{M}_1$ is a surface of $\mathbb{R}^3$ and the arc $\dot\gamma$ is piecewise absolutely continuous. We allow $\dot\gamma$ to have a finite number of discontinuities at time $0<t_1<\cdots<t_m \leqslant T$. For each of them, set $\theta_i$ as the oriented angle between $\dot\gamma(t_i-)$ and $\dot\gamma(t_i-)$ and $R_i \in \mathrm{SO}(3)$, the rotation given by

$$R_i = \begin{pmatrix} \cos(\varepsilon_R\theta_i) & \sin(\varepsilon_R\theta_i) & 0 \\ -\sin(\varepsilon_R\theta_i) & \cos(\varepsilon_R\theta_i) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\varepsilon_R$ was defined in Definition 2.

**Proposition 8.** *Let $\gamma \in Q$, $\gamma : [0, T] \to \mathscr{M}_1$, with geodesic curvature $\kappa_g$. Let s be the* SO(3)*-valued function defined a.e. in* $[0, T]$ *by*

$$s(t) = \begin{pmatrix} 0 & \kappa_g(t) & -1 \\ -\kappa_g(t) & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

*Then $A_\gamma(\mathrm{id}) = R(T^+)$, where R is the solution of the Cauchy problem defined by the initial condition $R(0)=\mathrm{id}_3$ and the impulse differential equation*

$$\dot R R^{-1} = s, \quad R(t_i+) = R(t_i)R_i.$$

A proof of the above proposition is provided in Appendix A.

We are thus led to consider the subgroups of SO(3). The simplest subgroups are those which are dense or finite in some copy of $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ that fixes an axis:

1. A dense subgroup of SO(2) generated by a rotation around a fixed axis of angle $\theta$ with $\theta/\pi \notin \mathbb{Q}$.
2. A finite subgroup of SO(2) generated by a rotation around a fixed axis of angle $\theta=2\pi/q$ with $q \in \mathbb{Q}$.

The other possible subgroups of SO(3) are one of the following (see [10]):

3. A dense subgroup of SO(3).
4. A finite subgroup of SO(3), called point group which belongs to the following list:

   4.1. Dihedral group $D_h$ (for $h \in \mathbb{N}\backslash\{0\}$) consisting of $2h$ elements: $h$ of them are rotations of angle $\theta = k2\pi/h,\ k = 0, \ldots, h-1$ about the same axis and carry an $h$-sided polygon $P$ into itself. The remaining $h$ are rotations of angle $\pi$ about the $h$ symmetry axes of $P$. $D_h$ is generated by 2 elements.

   4.2. Tetrahedral group $T$ of order 12 consisting of rotations carrying a regular tetrahedron onto itself. Nine of these rotations are rotations of angle $k2\pi/3$ about the axis through a vertex and the opposite face centroid. The remaining three rotations are rotation of angle $\pi$ about the axis through opposite midpoints of edges. There are two generators that are two rotations about axis through two different vertices.

   4.3. Octahedral Group $O$ of order 24 carrying a cube onto itself. The elements are given by nine rotations of angle $k2\pi/3$ about the four interior diagonals, the nine rotations of angle $k\pi/2$ about the axis through opposite face centroids and the six rotations of angle $\pi$ about the axis through opposite edges midpoints. The group is generated by one rotation of angle $2\pi/3$ about one interior diagonal and one rotation of angle $\pi/2$ about one axis through opposite faces centroids.

   4.4. Icosahedral Group $I$ of order 60 carrying an icosahedron onto itself.

**Example.** Consider a generic convex body $\mathscr{M}_1$ rolling on a sphere $\mathscr{M}_2$. Assume that the two bodies are in contact through a point $p_1$ of $\mathscr{M}_1$ and the north pole of $\mathscr{M}_2$. We choose a geodesic direction $v$ and consider the point $p_2$ on $\mathscr{M}_1$ along the corresponding geodesic and at distance $\ell$ corresponding to the geodesic distance on $\mathscr{M}_2$ among two vertices of an inscribed tetrahedron. Then we choose two closed paths at $p_1$ and $p_2$ that produce pure rotation of $2\pi/3$ on $\mathscr{M}_2$. This is achieved by a path that produces a pure rotation and encloses a region of $\mathscr{M}_1$ with $2\pi/3$ total curvature. Such a quantization generates a group action on $\mathscr{M}_2$ that corresponds to the tetrahedral subgroup of SO(3).

*5.3. Rolling on a body of revolution*

We consider $\mathscr{M}_2$ a body of revolution given by rotating, along the positive *x*-axis, the graph of a concave function:

$$\phi : [0, 1] \to \mathbb{R}, \quad \phi(0) = \phi(1) = 0. \tag{8}$$

We can describe the body by $(x, \theta)$, where $x \in [0, 1]$ and $\theta \in S^1$. Notice that there are singularities of this parameterization at $(0, \theta)$ and $(1, \theta)$.

The manifold $\mathscr{P}_2$ has no more a natural Lie group structure and the action of $A_\gamma$'s are more difficult to describe. However there are natural symmetries of the body, namely

$$S^\theta(x', \theta') = (x, \theta' + \theta).$$

If we are interested in motion planning regardless of the coordinate $\theta$ of the final position, we can take the quotient of $\mathscr{P}_2$ over the action of $(S^\theta, S^\theta_*)$. This defines an equivalence relation $\sim$ and

$$\frac{\mathscr{P}_2}{\sim} \simeq \mathscr{M}_2.$$

Let us describe this identification better. Consider the singular parameterization $[0, 1] \times S^1$ and call the point $(1, \theta)$ north pole and the point $(0, \theta)$ south pole. We let the north (resp. south) pole correspond to $((1, \theta), \psi)$ (resp. $((0, \theta), \psi)$) for any $\theta, \psi \in S^1$. Finally each $((x, \theta), \psi)$ is identified with $(x, S^\theta_*\psi)$, see Fig. 1.

Consider the action over $\mathscr{P}_2/\sim$ of a geodesic path (not closed) $\gamma$ over $\mathscr{M}_1$. Let us describe in detail the case of a sphere, for simplicity, being the general case entirely similar. There exist two fixed points $(\frac{1}{2}, \bar{\theta})$ and $(\frac{1}{2}, \bar{\theta} + \pi)$. Thus the action of each $\gamma$ is given by

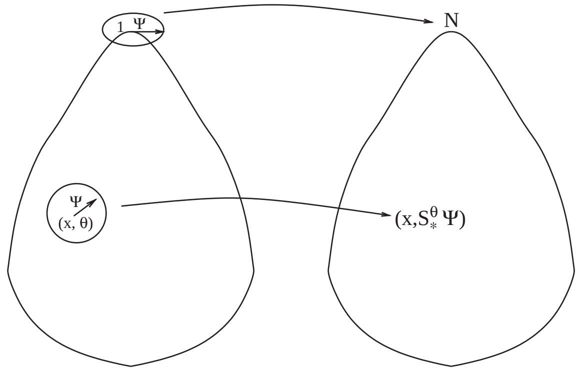


Fig. 1. Body of revolution.

**Proposition 9.** *Let $\mathscr{M}_2$ be the sphere, that is $\phi(x) = r\sqrt{1 - (x - \frac{1}{2})^2}$. Then for every geodesic $\gamma$ on $\mathscr{M}_1$, $A_\gamma$ acts on $\mathscr{P}_2/\sim$ as a rotation along the axis through $(\frac{1}{2}, \bar{\theta})$ and $(\frac{1}{2}, \bar{\theta} + \pi)$.*

To consider the action of a closed curve piecewise geodesic $\gamma$ on $\mathscr{M}_1$, it is enough to understand the action of a rotation in the fiber on $\mathscr{P}_2$. We obviously have that if $R$ is such a rotation then

$$R(x, \theta) = (x, \theta + \psi)$$

for some fixed $\psi \in S^1$. Thus the action corresponds to a rotation with axis passing through the poles. Finally we obtain

**Proposition 10.** *The action of $\Gamma$ over $\mathscr{P}_2/\sim$ is the same as the $(\mathbb{R}^2, +)$ action on $S^2$ where $(v_1, v_2)$ acts by rotating of angle $v_1$ around the axis through the poles and rotating of angle $v_2$ around the axis through $(\frac{1}{2}, \bar{\theta})$ and $(\frac{1}{2}, \bar{\theta} + \pi)$.*

The planning problem is now easy since each quantization group $Q$ corresponds to a subgroup of the additive group $(\mathbb{R}^2, +)$. It is well known that we have three possibilities:

1. $Q$ is discrete and is generated by two independent vectors;
2. $Q$ is dense on the whole $\mathbb{R}^2$;
3. $Q$ is the direct sum of a discrete and a dense subgroup of $(\mathbb{R}, +)$.

## 6. Topological and metric homogeneous quantization

In this section, we consider the quantization of the rolling body problem with uniformity properties from the topological and metric point of view.

**Definition 8.** Consider a quantization of the rolling of a body $\mathscr{M}_1$ on a body $\mathscr{M}_2$.

The quantization is said to be topological if there exists a geodesic tassellation $\mathscr{T}$ of $\mathscr{M}_1$ such that points $p_i$ are vertices of $\mathscr{T}$, the closed curves and transit paths correspond to unions of edges of $\mathscr{T}$.

A topological quantization is homogeneous if the corresponding tassellation is homogeneous, i.e. each vertex has the same number of incident edges and each face has the same number of edges.

A topological quantization is metric homogeneous if it is a topological homogeneous quantization and the corresponding tassellation $\mathscr{T}$ satisfies the following. Each edge has the same length and, for every vertex and each pair of adjacent incident edges, the corresponding tangent vectors form the same angle.

**Remark 5.** Consider the rolling body problem (for a Riemannian manifold $\mathscr{M}_1$) where the control consists of assigning at each point a finite number of possible initial directions (of geodesics) and lengths. Giving a metric homogeneous quantization means precisely to assign a quantization for such a control problem with uniformity of lengths and directions, generating a reachable set on $\mathscr{M}_1$ that is precisely the set of vertices of a tassellations.

The main problem is the possibility of constructing homogeneous quantizations, if they even exist. However, from a practical point of view, it allows an easy implementation of control algorithms as sequences from a finite alphabet, i.e. an encoding of control actions, see Remark 4. This also bears the presence of finite bandwidth communication channels [5].

Consider now a general convex surface $\mathscr{M}_1$ and a homogeneous tassellation $\mathscr{T}$ of $\mathscr{M}_1$. One can define the stereographic projection $\pi_p(\mathscr{T})$ of $\mathscr{T}$ from a point $p$ in the interior of any face of $\mathscr{T}$ onto a plane $\Pi$ tangent $\mathscr{M}_1$ at a point $\sigma \neq p$ in the interior of any face. Then $\pi_p(\mathscr{T})$ is a planner graph $G(\mathscr{T})$ and any closed path on $\mathscr{T}$ corresponds to a closed path on $G(\mathscr{T})$. Therefore, recalling the results of [3]:

**Theorem 1.** *The fundamental group of $G(\mathscr{T})$ at a node $\pi_p(p_i)$ is a free group generated by a finite number of elements of the generator set. The number of generators is equal to the number of faces of the polyhedron minus one.*

From this theorem, since the group $\mathscr{Q} \subset \mathscr{L}^g_{p_i}$ is homomorphic to the fundamental group at the node $\pi_p(p_i)$ of $G(\mathscr{T})$, also the former group has a finite number of generators. This allows one to compute the reachable set of the rolling of $\mathscr{M}_1$ onto $\mathscr{M}_2$ by composing the actions of the single generators, reducing the analysis of orbits to a standard case. This approach is exploited in [3] and we use it in the next section for the rolling of the sphere onto a plane.

### *6.1. Homogeneous tassellation of the sphere*

We start discussing the first nontrivial case for $\mathscr{M}_1$, that is the sphere. A homogeneous tassellation is indicated by a couple $(a, b)$, where $a$ is the number of edges for each face and $b$ is the number of incident edges at each vertex. A well-known result gives [12]:

**Theorem 2.** *A homogeneous tassellation of the sphere is of one of the following types*: (3,3), (3,4), (4,3), (3,5) *and* (5,3).

Using Euler–Poincare characteristic and Euler relation, from $(a, b)$ one determines the number of faces, edges and vertices. Consider, on the sphere, the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$. Given an inscribed polyhedron, we can generate a tassellation by projection of edges and vertices from the center of the sphere. We immediately get the following:

**Theorem 3.** *A metric homogeneous topological quantization of the sphere is of one of the following types*:

- (3,3) *corresponding to an inscribed tetrahedron,*
- (3,4) *corresponding to an inscribed octahedron,*
- (4,3) *corresponding to an inscribed cube,*
- (3,5) *corresponding to an inscribed icosahedron,*

- (5,3) *corresponding to an inscribed dodecahedron*.

For each graph on the sphere we can define the dual graph by mapping edges to edges, faces to vertices and vice versa, keeping the incidence relations. This induces a duality relation on the set of graphs that in turn gives a duality relation on the set of inscribed polyhedra. In this way, one gets the following relations:

- tetrahedron–tetrahedron,
- octahedron–cube,
- icosahedron–dodecahedron.

Considering duality maps, one can easily check:

**Proposition 11.** *Consider the rolling of the sphere $\mathscr{M}_1$ on the plane $\mathscr{M}_2$. The rolling obtained by a metric homogeneous topological quantization of the sphere produces the same reachable sets of the rolling of the dual polyhedron*.

Then we are reduced to the case considered in [3] and we get

**Theorem 4.** *Consider the rolling of the sphere $\mathscr{M}_1$ on the plane $\mathscr{M}_2$ given by a metric homogeneous topological quantization of $\mathscr{M}_1$. Then we have one of the following cases*:

- *If the tassellation is of type* (3,3) (*rolling a tetrahedron*), *then the reachable set is discrete in* $\mathbb{R}^2 \times S^1$ *corresponding to a triangle tassellation of the plane*.
- *If the tassellation is of type* (3,4) (*rolling a cube*), *then the reachable set is discrete in* $\mathbb{R}^2 \times S^1$ *corresponding to a square tassellation of the plane*.
- *If the tassellation is of type* (4,3) (*rolling a octahedron*), *then the reachable set is discrete in* $\mathbb{R}^2 \times S^1$ *corresponding to a triangle tassellation of the plane*.
- *If the tassellation is of type* (3,5) (*rolling a dodecahedron*), *then the reachable set is discrete in* $S^1$ ($\pi/3\,\mathbb{Z}$ *subgroup*) *and dense in* $\mathbb{R}^2$.
- *If the tassellation is of type* (5,3) (*rolling an icosahedron*), *then the reachable set is dense in* $\mathbb{R}^2 \times S^1$.

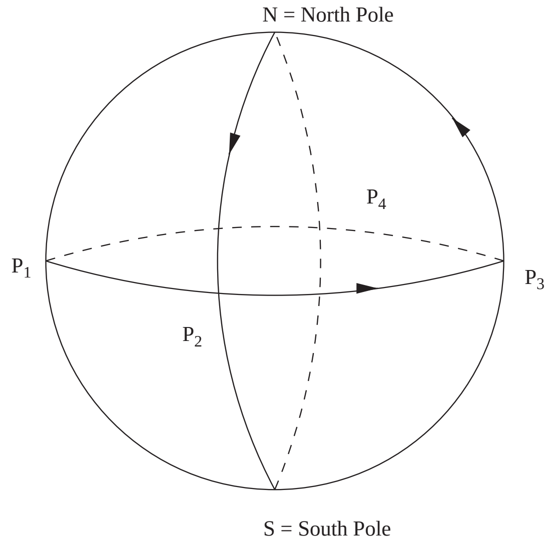


Fig. 2. Example of homogeneous tassellation.

**Example.** Let $\mathscr{M}_1$ be a sphere and $\mathscr{M}_2$ a plane. We fix the quantization corresponding to the tassellation of topological type (3,4). This is simply obtained by taking the two poles, four equidistant points on the equator and, as paths, the geodesics connecting each pole with each point on the equator and every pair of adjacent points on the equator, see Fig. 2. This corresponds to the paths naturally defined by inscribing an octahedron in the sphere.

As observed above, this quantization produces an action on the plane $\mathscr{M}_2$ identical to that given by the rolling of a cube on the plane. This action is induced by a subgroup of $\mathbb{R}^2 \times S^1$ generated by one pure rotation of angle $\pi/2$ and one translation.

Note that we can individuate the paths on $\mathscr{M}_1$ acting as generators of such a subgroup. The pure rotation is given by the closed path that starts at one pole, reaches a point on the equator, goes to an adjacent point on the equator and finally back to the pole (as indicated in Fig. 2). The total curvature contained in this path is precisely $\pi/2$, while the translation is generated by any path running along a maximal circle.

### 6.2. Other homogeneous quantizations

Unfortunately, the analysis of homogeneous quantization on general convex surfaces (with a Riemannian metric) is complicated. In particular, the existence of homogeneous tassellations is often not granted. As an example, consider a body of revolution $\mathscr{M}_1$ obtained rotating along the positive $x$-axis of the graph of a concave function $\phi$, as in Eq. (8). Assume that $\mathscr{M}_1$ has the Riemannian metric inherited by $\mathbb{R}^3$ and consider a tassellation $\mathscr{T}$ of type (3,4) corresponding to an inscribed octahedron. If we fix two vertices $p_1$ and $p_2$ in the two poles and the other four on the circle $C$ obtained by rotating the point $(x, \phi(x))$, $0<x<1$, then the following conditions are necessary for metric homogeneity.

- The circle $C$ should be a geodesic, thus $\phi'(x)=0$.
- The vertices $p_3, \dots, p_6$ are equidistant on $C$.
- Finally to have equal lengths of edges

$$\frac{\pi}{2}\phi(x) = \int_0^x \sqrt{1+(\phi'(x))^2}\,\mathrm{d}x = \int_x^1 \sqrt{1+(\phi'(x))^2}\,\mathrm{d}x.$$

Then not all bodies of revolution admit such a kind of quantization.

## Appendix A

### A.1. Coordinate-free definition of the rolling body problem

Let us show how to formulate the rolling of $\mathscr{M}_1$ on $\mathscr{M}_2$ without slipping or spinning as a standard control system, see [13].

We may rewrite (1) and (3) as follows: if the state $x$ is represented by $(x_1, x_2, R)$ then, on appropriate charts, there exists a measurable function $u$ with values in $\mathbb{R}^2$ called the control (and depending on the particular chart we are using) such that

$$\dot{x}_1 = u_1 X_1^1 + u_2 X_2^1, \tag{9}$$

$$\dot{x}_2 = u_1 (X^2 R)_1 + u_2 (X^2 R)_2, \tag{10}$$

$$\dot{R}R^{-1} = \sum_{i=1}^{2} u_i [\omega_1(X_i^1) - \omega_2((X^2R)_i)], \tag{11}$$

where $X_i^1$ and $(X^2R)_i$ are the $i$th columns of $X^1$ and $X^2R$. Note also that $u=(u_1, u_2)$, appearing in the Eqs. (9)–(11), now depends both on the particular chart and on the OMF we are using. Conversely, given $x \in \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$, $u : [0, T] \to \mathbb{R}^2$ integrable and a covering of $\mathscr{M}_1$ by neighborhoods with appropriate OMFs $X^1$, let $x_1 : [0, T] \to \mathscr{M}_1$ be the a.c. curve defined by $x_1(0)=\pi_{\mathscr{M}_1}(x)$ and $\dot{x}_1(t)=u_1(t)X_1^1(x_1(t))+u_2(t)X_2^1(x_1(t))$ for almost every $t \in [0, T]$. We can then associate to $u$ an admissible curve $\Gamma$ with values in $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ starting at $x$. We denote by $\mathscr{A}d$ the set of admissible controls, i.e. functions $u : [0, T] \to \mathbb{R}^2$ which are integrable ($[0, T]$ depends on $u$ in general). We can rewrite the above equations as follows:

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \tag{12}$$

where we have for the state $x$ in the domain of the ad hoc chart and $1 \leqslant i \leqslant 2$, $F_i(x) = (X_i^1, (X^2R)_i, T_iR)^{\mathrm{T}}$, with $T_i = \omega_1(X_i^1) - \omega_2((X^2R)_i)$.

The vector fields $F_i$ generate locally a 2-dimensional $C^\infty$ distribution $\Delta$ on $\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$, for which the $F_i$'s are a local $C^\infty$ basis. To see that, first define a distribution $\widetilde{\Delta}$ on $\mathscr{O}(\mathscr{M}_1, \mathscr{M}_2)$ as follows. If $x = (p_1, p_2, R_1, R_2) \in \mathscr{O}(\mathscr{M}_1, \mathscr{M}_2)$, then $\widetilde{\Delta}(x)$ is the set of tangent vectors $v = (v_1, v_2, s_1R_1, s_2R_2) \in T_x\mathscr{O}(\mathscr{M}_1, \mathscr{M}_2)$ such that $R_1^{-1}v_1 = R_2^{-1}v_2$, $s_1 = -\omega_1(v_1)$ and $s_2 = -\omega_2(v_2)$. Next notice that $\widetilde{\Delta}(x)$ is invariant by the diagonal action DA and finally pass to the quotient in order to obtain $\Delta$. The above definition of $\Delta$ is clearly independent of the choice of OMF, see [23]. The distribution $\Delta$ is simply the assignment $x \mapsto \Delta(x)$ where $x \in \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ and $\Delta(x)$ is the subspace of $T_x\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ of vectors $(v_1, v_2, sR)$ where $v_2 = Rv_1$and $s = \omega_1(v_1) - \omega_2(v_2)$. Moreover, the admissible trajectories are the a.c. curves $\gamma : [0, T] \to \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ such that $\dot{\gamma}(t) \in \Delta(\gamma(t))$ for almost every $t \in [0, T]$.

Therefore, the rolling of a manifold $\mathscr{M}_2$ onto another manifold $\mathscr{M}_1$ is defined by the control system $\mathscr{S}_R = (\mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2), \mathbb{R}^2, \Delta, \mathscr{A}d)$, which is driftless and affine in the control.

### A.2. Proofs of Propositions 6 and 8

In this section, we provide an argument for Propositions 6 and 8. For that purpose, we assume that $\mathscr{M}_1$ and $\mathscr{M}_2$ are two surfaces of $\mathbb{R}^3$ with appropriate charts. Let $\varepsilon_R$ be the relative orientation of $\mathscr{M}_1$ with respect to $\mathscr{M}_2$.

Let $\gamma_1$ be an a.c. curve of $\mathscr{M}_1$ parameterized by arclength and such that $\dot\gamma_1$ is continuous.

We first reformulate the rolling of $\mathscr{M}_2$ along $\gamma_1$ in terms of Darboux frames as defined next.

**Definition A.1.** Let $\gamma : [0, T] \to \mathbb{R}^3$ be an arc of an oriented surface $\mathscr{M}$ of $\mathbb{R}^3$, parameterized by arclength and so that $\dot\gamma$ is absolutely continuous. Set $v := \dot\gamma$ and $u := n \times v$, where $n$ is the normal tangent vector. The Darboux frame $D : [0, T] \to SO(3)$ is the rotation matrix

$$D(t) := (v(t), u(t), n(t))^{\mathrm{T}}.$$

For a.e. $t \in [0, T]$, $\dot D D^{-1}$ is a skew-symmetric matrix given by

$$\dot D(t) D^{-1}(t) = \begin{pmatrix} 0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0 \end{pmatrix}, \tag{13}$$

where $\kappa_n$ and $\tau_g$ are called respectively the normal curvature and the geodesic torsion. These two functions can be expressed in terms of the second fundamental form $\mathscr{I}\mathscr{I}$ of $\mathscr{M}$ (we dropped the index $\alpha$)

$$\kappa_n(t) = \mathscr{I}\mathscr{I}(v(t), v(t)),$$
$$\tau_g(t) = \mathscr{I}\mathscr{I}(v(t), u(t)). \tag{14}$$

Note that $\kappa_n$ and $\tau_g$ are continuous functions.

Consider $\Gamma = (\gamma_1, \gamma_2, R) : [0, T] \to \mathscr{RO}(\mathscr{M}_1, \mathscr{M}_2)$ characterizing the rolling of $\mathscr{M}_2$ along $\gamma_1$. We next express the rolling problem dynamics in terms of the Darboux frames $D^1$ of $\gamma_1$ and $D^2$ of $\gamma_2$.

**Lemma A.1.** *With the notations given above, the rolling of $\mathscr{M}_2$ along $\gamma_1$ is characterized by $\Gamma = (\gamma_1, \gamma_2, R) : [0, T] \to \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3)$ with $\kappa_g^1 = \kappa_g^2$.*

**Proof.** The conclusion follows directly from Eq. (2) and [23, p. 390]. It is noteworthy that the statement of Lemma A.1 can actually serve to establish the dynamics of the rolling problem. Using [23, p. 387], one can even show that this definition does not depend on the choice of OMF. □

The proof of Proposition 6 is now immediate. Note that, for the plane, $\mathscr{I}\mathscr{I} \equiv 0$ so $\kappa_n = \tau_g \equiv 0$. Clearly $\theta_R = \int_0^T \kappa_g(t)\,\mathrm{d}t$ and, since $t_R = \int_0^T \dot\gamma_2 = \int_0^T v^2$, the proof is finished.

For Proposition 8, first note that a natural choice of $n$ is $\gamma$ itself. So, by using the third line in the matrix of Eq. (13), we get $v = \dot n = -\kappa_n v - \tau_g u$. This implies that $\kappa_n \equiv -1$ and $\tau_g \equiv 0$. We conclude as before.

The general case in both Propositions 6 and 8 (finite number of discontinuity for $\dot\gamma$) can be now proven by simply noticing that a discontinuity of $\dot\gamma$ at $t$ such that $\dot\gamma(t^-)$ and $\dot\gamma(t^+)$ exist can be modelled as a motion in the corresponding fiber, i.e. by keeping the normal vector constant.

## References

[1] A. Agrachev, Yu. Sachkov, An intrinsic approach to the control of rolling bodies, Proceedings of the IEEE International Conference on Decision and Control, vol. 1, 1999, pp. 431–435.

[2] J.E. Bertram, The effect of quantization in sampled feedback systems, Trans. AIEE Appl. Ind. 77 (1958) 177–181.

[3] A. Bicchi, Y. Chitour, A. Marigo, Reachability and steering of rolling polyhedra: a case study in discrete nonholonomy, IEEE Trans. Automat. Control 49 (5) (2004) 710–726.

[4] A. Bicchi, A. Marigo, B. Piccoli, On the reachability of quantized control systems, IEEE Trans. Automat. Control 47 (4) (2002) 546–563.

[5] A. Bicchi, A. Marigo, B. Piccoli, Encoding steering control with symbols, in: Proceedings of the IEEE International Conference on Decision and Control, 2003.

[6] A. Bicchi, D. Prattichizzo, S.S. Sastry, Planning motions of rolling surfaces, in: Proceedings of the IEEE International Conference on Decision and Control, 1995.

[7] A. Chelouah, Y. Chitour, On the motion planning of rolling surfaces, Forum Math. 15 (5) (2003) 727–758.

[8] Y. Chitour, B. Piccoli, Controllability for discrete systems with a finite control set, Math. Control Signals Systems 14 (2) (2001) 173–193.

[9] M. Egerstedt, R.W. Brockett, Feedback can reduce the specification complexity of motor programs, IEEE Trans. Automat. Control 48 (2) (2003) 213–223.

[10] A. Fassler, E. Stiefel, Group Theoretical Methods and their Applications, Birkhauser, Boston, 1992.

[11] E. Frazzoli, M.A. Dahleh, E. Feron, Real-time motion planning for agile autonomous vehicles, AIAA J. Guidance Control Dyn. 25 (1) (2002) 116–129.

[12] B. Grünbaum, Convex Polytopes, Wiley, New York, 1967.

[13] V. Jurdjevic, Geometric Control Theory, Cambridge University Press, Cambridge, 1997.

[14] A. Marigo, A. Bicchi, Steering driftless nonholonomic systems by control quanta, in: Proceedings of the IEEE International Conference on Decision and Control, 1998.

[15] A. Marigo, A. Bicchi, Rolling bodies with regular surface: controllability theory and applications, IEEE Trans. Automat. Control 45 (9) (2000) 1586–1599.
[16] A. Marigo, Y. Chitour, A. Bicchi, Manipulation of polyhedral parts by rolling, Proceedings of the IEEE International Conference on Robotics and Automation, vol. 4, 1997, pp. 2992–2997.
[17] A. Marigo, B. Piccoli, A. Bicchi, A group theoretic characterization of quantized control systems, in: Proceedings of the IEEE International Conference on Decision and Control, 2002.
[18] W.S. Massey, Algebraic Topology: An Introduction, Springer, New York, 1970.
[19] R.M. Murray, Z. Li, S.S. Sastry, A Mathematical Introduction to Robotic Manipulation, CRC Press, Boca Raton, 1994.
[20] T. Sakai, Riemannian Geometry, AMS Transactions on Mathematical Monograph no. 149, 1996.
[21] J.B. Slaughter, Quantization errors in digital control systems, IEEE Trans. Autom. Control 9 (1964) 70–74.
[22] R. Sorrentino, A. Bicchi, Dexterous manipulation through rolling, in: Proceedings of the IEEE International Conference on Robotics and Automation, 1995, pp. 452–457.
[23] M. Spivak, A Comprehensive Introduction to Differential Geometry, vol. III, Publish or Perish, Berkeley, CA, 1975.