Forum Math. **15** (2003), 727–758

**Forum Mathematicum**


# On the motion planning of rolling surfaces

A. Chelouah and Y. Chitour

(Communicated by Michel Fliess)

**Abstract.** In this paper, we address the issue of motion planning for the control system $\mathscr{S}_R$ that results from the rolling without slipping nor spinning of a two dimensional Riemannian manifold $M_1$ onto another one $M_2$. We present two procedures to tackle the motion planning problem when $M_1$ is a plane and $M_2$ a convex surface. The first approach rests on the Liouvillian character of $\mathscr{S}_R$. More precisely, if just one of the manifolds has a symmetry of revolution, then $\mathscr{S}_R$ is shown to be a Liouvillian system. If, in addition, that manifold is convex and the other one is a plane, then a maximal linearizing output is explicitely computed. The second approach consists of the use of a continuation method. Even though $\mathscr{S}_R$ admits nontrivial abnormal extremals, we are still able to successfully apply the continuation method if $M_2$ admits a stable periodic geodesic.

2000 Mathematics Subject Classification: 12H05, 53A40, 73B99, 93C10, 93C95.

## 1 Introduction

In this paper, we study the rolling without slipping nor spinning of a two dimensional Riemannian manifold $M_1$ onto another one $M_2$. This is a classical problem in mechanics with several applications in robotics and it is usually assumed that $M_1$ and $M_2$ are embedded surfaces in $\mathbb{R}^3$, cf. [21], [24] and references therein. Let us mention an important particular case: when $M_1$ is an Euclidean plane and $M_2$ is the unit sphere, the rolling problem is called the plate-ball problem. Recently, active research focused around two central issues of control theory, the controllability question and the motion planning problem (MPP for short) (cf. [21] for references). Recall that a control system $\mathscr{S}$ is said to be completely controllable (CC) if any pair of points of its state space can be joined by an admissible trajectory of $\mathscr{S}$. On the other hand, the MPP is the problem of finding a procedure that, for every pair $(p, q)$ of the state space of a control system $\mathscr{S}$, effectively produces a control $u_{p,q}$ giving rise to an admissible trajectory steering $p$ to $q$.

Marigo and Bicchi, in [21], provide sufficient geometric conditions on the surfaces $M_1$ and $M_2$ in order to ensure complete controllability. Jurdjevic, however, adopts in [15], a more intrinsic approach of the rolling problem in order to study the time-optimal control aspect of the plate-ball problem. Rather than embedding the plane $M_1$ and the unit sphere $M_2$ in $\mathbb{R}^3$, he formulates the plate-ball problem as a left-invariant control system $\mathcal{S}_R$ over the Lie group $SO(3) \times \mathbb{R}^2$. The most spectacular result of Jurdjevic's paper [15] is probably the fact that time-optimal trajectories of the center of the unit sphere are solutions to the Euler elastica problem. In [5], Bryant and Hsu consider the general rolling problem as an example of a rank-two distribution on a five-dimensional manifold obtained as the quotient of a six-dimensional fiber bundle by an SO(2)-action. Then, in [2], Agrachev and Sachkov, in the spirit of Jurdjevic, proved that for general two-dimensional Riemannian manifolds $M_1$ and $M_2$, the control system $\mathcal{S}_R$ is CC if and only if $M_1$ and $M_2$ are not isometric. In [17], Kiss, Lévine and Lantos address the motion planning problem of rigid bodies and provide a classification according to the dimension, the number of fingers manipulating the bodies, the model type (dynamic or kinematic) and their structural properties (flat or Liouvillian). In the present paper, we start with a complete and precise definition of $\mathcal{S}_R$ as a 4-tuple $(\mathcal{RO}(M_1, M_2), \mathbb{R}^2, \Delta, \mathcal{Ad})$ where the state space $\mathcal{RO}(M_1, M_2)$ is a five dimensional manifold, $\mathbb{R}^2$ is the control space, the distribution $\Delta$ is a $C^\infty$ assignment $p \mapsto \Delta(p)$ of rank two and $\mathcal{Ad}$ is the set of admissible controls. In particular, we obtain the state space $\mathcal{RO}(M_1, M_2)$ as a circle bundle over $M_1 \times M_2$. Finally, we describe the structures of the possible reachable sets, recovering the controllability results of Agrachev and Sachkov.

The main core of the paper is devoted to the motion planning problem for $\mathcal{S}_R$ when it is completely controllable and when $M_1$ is a plane and $M_2$ is convex. The MPP for the rolling problem is considered as an important test case because it represents the next stage of difficulty after the class of chained systems (again see [21] for more details and complete references). Until now, the most significant result is the ingenious algorithm proposed by Li and Canny ([19]) when $M_2$ is a ball. It seems however difficult to generalize that algorithm to more general manifolds $M_2$. In this paper, we propose two different approaches to address the MPP for convex surfaces $M$ rolling on a plane. The first one is based on the Liouvillian character of $\mathcal{S}_R$. Liouvillian systems were first introduced in [7], using the differential algebra setting, as a natural extension of flat systems. We give here a new formulation, well suited for the problem under consideration, which uses the language of diffieties and the infinite prolongation theory ([9, 11, 12, 13, 30]). As a matter of fact, we show that the control system $\mathcal{S}_R$ belongs to the class of Liouvillian systems. Recall that one of the main properties of flat systems related to the MPP is the possibility to obtain, from the flat output and a finite number of its time derivatives, the system trajectories without any integration. Liouvillian systems share a similar property. Of course, since they are not flat, Liouvillian systems do not possess a flat output but a variable called *partial (or maximal) linearizing output* that plays a similar role. From this variable and a finite number of its time derivatives, the system trajectories can be obtained by means of a finite number of elementary integrations, called quadratures. When $M_1$ admits a symmetry of revolution and $M_2$ is a plane, we are able to compute a maximal linearizing output, which reduces the MPP to a purely algebraic problem.

Our second approach to the MPP is based on the well-known continuation method (also called homotopy method or continuous Newton's algorithm-[3]-) which goes back to Poincaré. The continuation method (CM) is often used for solving nonlinear equations of the form $F(x) = y$, where $x$ is the unknown and $F : X \to Y$ is surjective. The CM proceeds by starting from a value $x_0$ of $x$ and its corresponding image $y_0 = F(x_0)$, then by joining $y_0$ to the given $y$ by a continuous path $\pi$ and by trying to lift $\pi$ to a path $\Pi$ so that $F \circ \Pi = \pi$. To construct such a path $\Pi$ which is defined only implicitly, we may differentiate $F(\Pi(s)) = \pi(s)$ to get $DF(\Pi(s))\dot{\Pi}(s) = \dot{\pi}(s)$. The latter is satisfied if we can solve $\dot{\Pi}(s) = P(\Pi(s))\dot{\pi}(s)$ where $P(x)$ is a right inverse of $DF(x)$. Therefore, solving $F(x) = y$ amounts to first show that $P(\Pi(s))$ exists (for instance if $DF(\Pi(s))$ is surjective) and second to prove that the ODE in $X$, $\dot{\Pi}(s) = P(\Pi(s))\dot{\pi}(s)$, (also called the Wazewski equation-[29]-) admits a global solution. The singularities of $F$, *i.e.*, $x \in X$ for which $DF(x)$ is not surjective, are therefore expected to cause difficulties. In the context of the MPP, the CM was introduced in [28] and developed in [8]. The map $F$ is now an end-point map from the space of admissible inputs to the state space. Its singularities are exactly the abnormal extremals of the sub-Riemannian metric induced by the dynamics of the system, which are usually a major obstacle for the CM to apply efficiently to the MPP (cf. [8]). In the case of $\mathscr{S}_R$, the distribution $\Delta$ admits non trivial abnormal extremals and their projections on the state space are the horizontal geodesics of $\mathscr{RO}(M_1, M_2)$. However, if $M_1$ is a plane and $M_2$ is compact and satisfies a mild extra condition (existence of a stable periodic geodesic), we show that the CM provides complete answers to the MPP.

The balance of this paper is organized as follows. In section 2, the control system $\mathscr{S}_R$ is introduced and studied. Section 3 contains the Liouvillian approach of the MPP for the rolling problem and finally in the fourth section, we tackle the MPP using the continuation method.

## 2 Notations and first properties of the control system

### 2.1 Definition of the state space

All manifolds considered hereafter are two-dimensional, connected, oriented $C^\infty$ Riemannian manifolds. We also assume the manifolds to be complete in the sense of the Hopf-Rinow theorem (cf. [25]). We call convex surface such a manifold $M$ if in addition it is simply connected and of positive curvature $K$. A classical result states that $M$ can be embedded as a convex surface in $\mathbb{R}^3$ (cf. [18]). If $P$ is a matrix, we use $P^T$, and $\mathrm{tr}(P)$ respectively to denote the transpose of $P$, and the trace of $P$ respectively. For $\psi \in S^1$, we use $R_\psi$ to denote the rotation of angle $\psi$ and $(\varepsilon_i)_{i=1,\dots,5}$ to denote the canonical basis of $\mathbb{R}^5$.

Let $M$ be a manifold and $\{U_\alpha, \varphi^\alpha\}_{\alpha\in\mathscr{A}}$ an atlas on $M$. For $\alpha, \beta \in \mathscr{A}$ such that $U_\alpha \cap U_\beta$ is not empty, we denote by $J_{\beta\alpha}$ the jacobian matrix of $\varphi^\beta \circ (\varphi^\alpha)^{-1}$ the coordinate transformation on $\varphi^\alpha(U_\alpha \cap U_\beta)$. For $\alpha \in \mathscr{A}$, the Riemannian metric $g := \langle \cdot, \cdot \rangle$ is represented by the symmetric definite positive matrix $G^\alpha$. The geodesic coordinates on $M$ are charts $(v, w)$ defined such that $G^\alpha$ is diagonal with $g_{11} = 1$ and $g_{22} = B^2(v, w)$. The function $B$ is defined in an open neighborhood of $(0, 0)$ (the domain of the chart) and satisfies $B(0, w) = 1$, $B_v(0, w) = 0$ and $B_{vv} + KB = 0$, where $K$ denotes the Gaussian curvature of $M$ at $(v, w)$ and $B_v$ $(B_{vv})$ is the (double) partial derivative of $B$ with respect to $v$. The curve $(0, w)$ defined for $w$ in a neighborhood of 0 is a geodesic and is called the basis of the coordinate chart. If the manifold has a symmetry of revolution, we can suppose that $B$ only depends on $v$.

Let $I^\alpha = \sqrt{G^\alpha}$ be the symmetric definite positive matrix such that $(I^\alpha)^2 = G^\alpha$. If $f$ is a frame, *i.e.*, an ordered basis for $T_pM$, then $f^\beta = J_{\beta\alpha} f^\alpha$ and it is orthonormal if in addition $I^\alpha f^\alpha$ is an orthogonal matrix. Since $M$ is oriented, we may assume all the $\det J_{\beta\alpha}$ are positive.

Let $\mathrm{O}^+(M)$ be the set of all positively oriented orthonormal frames $f$ for all tangent spaces $T_pM$. There is an effective right action of $\mathrm{SO}(2)$ on $\mathrm{O}^+(M)$ given by $f \cdot N = I^\alpha f^\alpha N$ where $f \in \mathrm{O}^+(M)$ and $N \in \mathrm{SO}(2)$.

Let $\mathcal{O}^+(M)$ be the principal bundle over $M$ defined by (cf. [26] p. 7)

$$(2.1) \qquad \mathcal{O}^+(M) = \{\mathrm{O}^+(M), M, \mathrm{SO}(2), \mathrm{SO}(2), \{U_\alpha, \varphi^\alpha\}\}.$$

Given two manifolds $M_1$ and $M_2$, let $\mathcal{O}(M_1, M_2)$ be the 6-dimensional coordinate bundle obtained as the fiber product of $\mathcal{O}^+(M_1)$ by $\mathcal{O}^+(M_2)$. The fiber is $\mathrm{SO}(2) \times \mathrm{SO}(2)$ and the Lie group that acts on it is $\mathrm{SO}(2) \times \mathrm{SO}(2)$.

The group $\mathrm{SO}(2)$ acts on $\mathcal{O}(M_1, M_2)$ as the diagonal of the action of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ on $\mathcal{O}(M_1, M_2)$. Let us denote $dg$ this action, which is acting without fixed point. We take the quotient of $\mathcal{O}(M_1, M_2)$ by $dg$ and obtain a manifold of dimension 5. We use $\mathcal{RO}(M_1, M_2)$ to denote $\mathcal{O}(M_1, M_2)/dg$.

Since $\mathrm{SO}(2)$ is commutative, the manifold $\mathcal{RO}(M_1, M_2)$ can be made to a 5-dim. fiber bundle with $\mathrm{SO}(2)$ as fiber and group bundle. We use $\pi_{M_1}$ and $\pi_{M_2}$ resp. to denote the canonical projections on $M_1$ and $M_2$ resp.. The Riemannian metric $\langle \cdot, \cdot \rangle$ at a point $x = (x_1, x_2, R)$ ($R \in \mathrm{SO}(2)$) is defined as follows: for $v = (v_1, v_2, Rs)$ ($s$ a skew-symmetric $2 \times 2$ matrix) in $T_x\mathcal{RO}(M_1, M_2)$, we have

$$\langle v, v \rangle := \frac{1}{2}(\langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle - \mathrm{tr}(s^2)).$$

**Remark 1.** The construction of the state space given here already appears in [5]. In [2], Agrachev and Sachkov do not specify any orientation in their definition of the circle bundle $\mathcal{RO}(M_1, M_2)$. We do so for reasons that become clear later when we consider the case of one of the manifold being a plane. □

## 2.2 Statement of the control problem

Let $M_1$ and $M_2$ be two manifolds. Given an absolutely continuous (a.c. for short) curve $c_1 : [a, b] \to M_1$, we will define the rolling of $M_2$ on $M_1$ without slipping nor spinning along $c_1$ by defining a curve $C = (c_1, c_2, R) : [a, b] \to \mathcal{RO}(M_1, M_2)$ next. First, consider $c : [a', b'] \to M_1$ and $d : [a', b'] \to M_2$, two a.c. curves entirely defined on some charts of $M_1$ and $M_2$. Let $Y^1$ ($Y^2$ resp.) be the positively oriented Orthonormal Moving Frame (OMF for short) parallel along $c$ ($d$ resp.). We have

$Y^1 = X^1 \cdot A_1$ and $Y^2 = X^2 \cdot A_2$ for $A_1, A_2 \in \mathrm{SO}(2)$. Then $A_1$ ($A_2$ resp.) measures the relative position of $X^1$ ($X^2$ resp.) with respect to $Y^1$ ($Y^2$ resp.) along $c$ ($d$ resp.) and $A_2 A_1^{-1} \in \mathrm{SO}(2)$ measures the relative position of $X^2$ with respect to $X^1$ along $(c, d)$. The variation of $A_i$ along $c_i$, for $i = 1, 2$, is given by $\dot{A}_i = -\omega_i(\dot{c}_i^{\alpha_i}) A_i$, where $\omega_i(\dot{c}_i^{\alpha_i})$, $i = 1, 2$, is the evaluation of the (Cartan) connection $\omega_i$ associated to $X^i$ along the curve $c_i$. Then, up to initial conditions, the curves $c_2$ and $R$ are defined by

$$(2.2) \qquad I^{\alpha_2} \dot{c}_2^{\alpha_2}(t) = R I^{\alpha_1} \dot{c}_1^{\alpha_1}(t).$$

and

$$(2.3) \qquad \dot{R} R^{-1} = R \omega_1(\dot{c}_1^{\alpha_1}) R^{-1} - \omega_2(\dot{c}_2^{\alpha_2}).$$

Since the $\omega_i$'s are $2 \times 2$ skew-symmetric, equation (2.3) reduces to

$$(2.4) \qquad \dot{R} R^{-1} = \omega_1(\dot{c}_1^{\alpha_1}) - \omega_2(\dot{c}_2^{\alpha_2}).$$

Therefore, if we fix a point $x = (x_1, x_2, R_0) \in \mathscr{RO}(M_1, M_2)$, a curve $c_1$ on $M_1$ starting at $x_1$ defines entirely the curve $C$ by the equations (2.2) and (2.3). We say that $M_2$ rolls on $M_1$ without slipping nor spinning if, for every $x = (x_1, x_2, R_0) \in \mathscr{RO}(M_1, M_2)$ and a.c. curve $c_1 : [a, b] \to M_1$ starting at $x_1$, there exists an a.c. curve $C : [a, b] \to \mathscr{RO}(M_1, M_2)$ with $C(t) = (c_1(t), c_2(t), R(t))$, $C(a) = x$ and for every $t \in [a, b]$ and each appropriate coordinate system, the equations (2.2) and (2.4) are satisfied. We say that $C = (c_1, c_2, R) : [a, b] \to \mathscr{RO}(M_1, M_2)$ is an admissible trajectory starting at $x$ if $M_2$ rolls on $M_1$ without slipping nor spinning along $c_1$.

In addition, we can rewrite the equations (2.2) and (2.4) in local coordinates as follows: if $X^1$ and $X^2$ are two appropriate oriented OMF and if the state $x$ is represented by the triple $(c_1, c_2, R)$ then for almost all $t$ such that we remain in the domain of an appropriate chart, there exists a measurable function $u$ with values in $\mathbb{R}^2$ called the control (and depending on the particular chart we are using) such that

$$(2.5) \qquad \dot{c}_1 = u_1 X_1^1 + u_2 X_2^1,$$

$$(2.6) \qquad \dot{c}_2 = u_1 (X^2 R)_1 + u_2 (X^2 R)_2,$$

$$(2.7) \qquad \dot{R} R^{-1} = \sum_{i=1}^{2} u_i [\omega_1(X_i^1) - \omega_2(X^2 R)_i],$$

where $X_i^1$ and $(X^2 R)_i$ are the $i$-th columns of $X^1$ and $X^2 R$. Note also that $u = (u_1, u_2)$, appearing in the equations (2.5)–(2.7), now depends both on the particular chart and OMF we are using. Conversely, one can see that, given $x \in \mathscr{RO}(M_1, M_2)$, $u : [a, b] \to \mathbb{R}^2$ integrable and a covering of $M_1$ by neighborhoods such that on each of one we have defined an OMF $X^1$, we can consider an a.c. curve $c_1 : [a, b] \to M_1$ with $c_1(a) = \pi_{M_1}(x)$ and $\dot{c}_1(t) = u_1(t) X_1^1(c_1(t)) + u_2(t) X_2^1(c_1(t))$ for almost every

$t \in [a, b]$. We can then associate to $u$ an admissible trajectory $C$ starting at $x$. Let $\mathscr{A}d$ be the set of admissible controls, *i.e.*, the functions $u : [a, b] \to \mathbb{R}^2$ which are integrable ($[a, b]$ depends on $u$ in general). We can rewrite the above equations as follows:

$$(2.8) \qquad \dot{x} = u_1 F_1(x) + u_2 F_2(x),$$

where we have for the state $x$ in the domain of the ad hoc chart and $1 \le i \le 2$, $F_i(x) = (X_i^1, (X^2 R)_i, T_i R)^T$, with $T_i = \omega_1(X_i^1) - \omega_2((X^2 R)_i)$.

It is not difficult to see that the $F_i$'s generate locally a 2-dimensional $C^\infty$ distribution $\Delta$ on $\mathscr{RO}(M_1, M_2)$, for which the $F_i$'s are a local $C^\infty$ basis. First, define a distribution $\tilde{\Delta}$ on $\mathscr{O}(M_1, M_2)$ as follows: If $x = (p_1, p_2, A_1, A_2) \in \mathscr{O}(M_1, M_2)$, then $\tilde{\Delta}(x)$ is the set of tangent vectors $v = (v_1, v_2, s_1 A_1, s_2 A_2) \in T_x \mathscr{O}(M_1, M_2)$ such that $A_1^{-1} v_1 = A_2^{-1} v_2$, $s_1 = -\omega_1(v_1)$ and $s_2 = -\omega_2(v_2)$. Next notice that $\tilde{\Delta}(x)$ is invariant by the diagonal action of SO(2) on $\mathscr{O}(M_1, M_2)$ and finally pass to the quotient in order to obtain $\Delta$. The above definition of $\Delta$ is independent of the choice of OMF. The distribution $\Delta$ is simply the assignment $x \mapsto \Delta(x)$ where $x \in \mathscr{RO}(M_1, M_2)$ and $\Delta(x)$ is the subspace of $T_x \mathscr{RO}(M_1, M_2)$ of vectors $(v_1, v_2, sR)$ where $v_2 = R v_1$ and $s = \omega_1(v_1) - \omega_2(v_2)$. Moreover, the admissible trajectories are the a.c. curves $\gamma : [a, b] \to \mathscr{RO}(M_1, M_2)$ such that $\dot{\gamma}(t) \in \Delta(\gamma(t))$ for almost every $t \in [a, b]$.

**Remark 2.** Both equations (2.2) and (2.3) have easy physical meanings and generalizations in the case where the rolling occurs with slipping or spinning. Equation (2.2) simply says that the curves $c_1$ and $c_2$ have the same arc length or that the contact point has relative speed equal to 0, *i.e.*, $M_2$ does not slip on $M_1$. This equation becomes

$$I^{\alpha_2}(\dot{c}_2^{\alpha_2}(t) + v_r(t)) = R I^{\alpha_1} \dot{c}_1^{\alpha_1}(t),$$

when $v_r$ stands for the relative speed of $M_2$ with respect to $M_1$. As for equation (2.3), it is a consequence of the no spinning condition. Its generalization is

$$\dot{R} R^{-1} + \omega_2(\dot{c}_2^{\alpha_2}) - \omega_1(\dot{c}_1^{\alpha_1}) = s_r(t),$$

where the skew-symmetric matrix $s_r$ measures the relative spin of $M_2$ with respect to $M_1$ at the point of contact. □

The previous formulation of the rolling of a manifold $M_2$ onto another manifold $M_1$ can be summarized by considering the control system $\mathscr{S}_R = (\mathscr{RO}(M_1, M_2), \mathbb{R}^2, \Delta, \mathscr{A}d)$. It is driftless and affine in the control and the state space has dimension five. In local coordinates the state $x$ is represented by the 3-tuple $(c_1, c_2, R)$ and if $X^1$ and $X^2$ are OMF on the domain of the chart, equation (2.8) represents the dynamics of the control system.

For $x \in \mathscr{RO}(M_1, M_2)$, $RS(x)$ denotes the reachable set from $x$ by admissible trajectories of $\Delta$, *i.e.*, the (local) integral curves of the vector fields $F_i$'s. Then if we use

$G_\Delta$ to denote the pseudo-group of local diffeomorphisms generated by the local flows of the $F_i$, we have that $G_\Delta(x)$, the orbit of $\Delta$ through $x$ is equal to $RS(x)$, thanks to the symmetric structure of $\mathscr{S}_R$ ($p \in RS(q)$ is equivalent to $q \in RS(p)$). As an important consequence for our problem, we have by the Orbit Theorem ([16]) that for every $x \in \mathscr{RO}(M_1, M_2)$, $RS(x)$ is a connected (immersed) submanifold of $\mathscr{RO}(M_1, M_2)$.

## 2.3 Lie algebraic structure of the control system

We first prove a proposition which is a fundamental property of the rolling problem.

**Proposition 1.** *Let* $u : [t_1, t_2] \to \mathbb{R}^2$ *an admissible control that gives rise to the admissible trajectory* $C = (c_1, c_2, R) : [t_1, t_2] \to \mathscr{RO}(M_1, M_2)$ *according to the equations* (2.5)–(2.7). *Then the following statements are equivalent:*

(a) *the curve* $c_1 : [t_1, t_2] \to M_1$ *is a geodesic;*

(b) *the curve* $c_2 : [t_1, t_2] \to M_2$ *is a geodesic;*

(c) *the curve* $C : [t_1, t_2] \to \mathscr{RO}(M_1, M_2)$ *is an horizontal geodesic.* □

*Proof of Proposition 1.* Since (c) implies (a) and (b) trivially, we have to show that (a) is equivalent to (b) and (a) implies (c). It is also clear that it is enough to prove that (a) implies (b) and (a) implies (c). Without loss of generality, we assume that $c_1 : [0, d] \to M_1$ is parameterized by arc length. Let $C(0) = (c_1(0), c_2(0), R_0)$. We choose an OMF $X^1$ in such a way that it is adapted to a fixed orthonormal frame of $T_{c_1(0)}M_1$ and $X_1^1(c_1(t)) = \dot{c}_1(t)$ for $t \in [0, d]$. On $M_2$, we also choose an OMF $X^2$ adapted to some fixed orthonormal frame of $T_{c_2(0)}M_2$. For $s = 1, 2$, if $\gamma$ is a minimizing geodesic of $M_s$ parameterized by arc length, then $\omega_s(\dot{\gamma}) = 0$. It implies that, for $s, i = 1, 2$, $\nabla^s_{X_i^s} X_i^s = 0$, where $\nabla^s$ is the Levi-Civita connection on $M_s$. By using $X^1$, $X^2$, the equations (2.5)–(2.7) become

$$\dot{c}_1 = X_1^1, \tag{2.9}$$

$$\dot{c}_2 = (X^2 R)_1, \tag{2.10}$$

$$R^{-1}\dot{R} = -\omega_2((X^2 R)_1). \tag{2.11}$$

The two last equations represent a first order differential system in $(c_2, R)$ with initial condition $((X^2 R_0)_1, R_0)$. Since $((X^2 R_0)_1, R_0)$ is a solution of the previous system and by uniqueness of the solution, $(c_2, R)$ is then constant and equal to $((X^2 R_0)_1, R_0)$. Therefore $c_2$ is a geodesic of $M_2$ and $C : [0, d] \to \mathscr{RO}(M_1, M_2)$ is an horizontal geodesic. □

**Remark 3.** More generally, let us fix a point $(p_1, p_2) \in M_1 \times M_2$ and two OMF $X^1$ and $X^2$ adapted to some fixed orthonormal frames of $T_{p_1}M_1$ and $T_{p_2}M_2$. Then rolling without slipping nor spinning along a geodesic $c_1 : [0, d] \to M_1$ starting at $p_1$ produces a curve $(c_1, c_2, R)$ in $\mathscr{RO}(M_1, M_2)$ with $c_2$ a geodesic of $M_2$ starting at $p_2$, $R$

being constantly equal to $R_0$ and there exist $\alpha_1, \dots, \alpha_n$ with $\sum_{i=1}^{2} \alpha_i^2 = 1$ such that for small $t > 0$

$$\dot{c}_1 = \sum_{i=1}^{2} \alpha_i X_i^1, \quad \omega_1(\dot{c}_1) = 0 \quad \text{and} \quad \dot{c}_2 = \sum_{i=1}^{2} \alpha_i (X^2 R_0)_i, \quad \omega_2(\dot{c}_2) = 0.$$

In addition, it is clear that we can replace the word "geodesic" by "once-broken geodesic". In particular, along such a curve the relative orientation remains constant. □

We next compute some Lie bracket of the $F_i$'s defined in (2.8) at $(p_1, p_2, R_0) \in \mathscr{RO}(M_1, M_2)$. Rewrite equation (2.8) using the fact that $\mathscr{RO}(M_1, M_2)$ is a circle bundle and taking geodesic coordinates for $M_1$ and $M_2$ at $p_1$ and $p_2$ respectively. Then take $R$ as $JR_\psi$, with $J := \mathrm{diag}(1, -1)$, and consider coordinates $X = (v_1, w_1, v_2, w_2, \psi)$ in some neighborhood of $(0, \psi_0)$ in $\mathbb{R}^4 \times S^1$. The control system $\mathscr{S}_R$ can be written as

$$\text{(2.12)} \quad \dot{X} = u_1 F_1(X) + u_2 F_2(X),$$

with

$$\text{(2.13)} \quad F_1(X) = \begin{pmatrix} 1 & 0 & \cos(\psi) & -\dfrac{\sin(\psi)}{C} & -\dfrac{C_{v_2}}{C}\sin(\psi) \end{pmatrix}^T,$$

$$F_2(X) = \begin{pmatrix} 0 & \dfrac{1}{B} & -\sin(\psi) & -\dfrac{\cos(\psi)}{C} & \dfrac{B_{v_1}}{B} - \dfrac{C_{v_2}}{C}\cos(\psi) \end{pmatrix}^T,$$

where $B$ and $C$ are used to define geodesic coordinates on $M_1$ and $M_2$ respectively. Let $K_1$ and $K_2$ be the curvatures on $M_1$ and $M_2$ respectively. We get after computations that

$$\text{(2.14)} \quad [F_1, BF_2] = B(K_2 - K_1)\varepsilon_5, \quad [F_1, \varepsilon_5] = \frac{\varepsilon_2 + B_{v_1}\varepsilon_5 - BF_2}{B},$$

$$[BF_2, \varepsilon_5] = B(F_1 - \varepsilon_1).$$

Then we finally obtain

$$\text{(2.15)} \quad \det(F_1, BF_2, [F_1, BF_2], [F_1, [F_1, BF_2]], [BF_2, [F_1, BF_2]]) = B^3 (K_1 - K_2)^3.$$

## 2.4 Case where $M_1 = \mathbb{R}^2$

The situation under consideration is of particular interest for us since we will try later to solve the MPP in that context. Moreover, the control system $\mathscr{S}_R$ presents worth-noticing features when one of the manifolds is the Euclidean plane. First remark that

$\mathscr{RO}(\mathbb{R}^2, M_2)$ is simply equal to $\mathbb{R}^2 \times T_1 M_2$, where $T_1 M_2$ is the unit tangent bundle of $M_2$. To see that first notice that $O^+(M)$ is equal to $T_1 M$ for a two dimensional manifold $M$ and $T_1 \mathbb{R}^2 = \mathbb{R}^2 \times S^1$. Therefore $\mathscr{O}(\mathbb{R}^2, M_2)$ is equal to $\mathbb{R}^2 \times S^1 \times T_1 M_2$. It is then easy to see that taking the quotient by the action of $dg$ simply cancels the $S^1$-factor. On the other hand, the distribution $\Delta$ admits global $C^\infty$ orthonormal basis over the state space $\mathbb{R}^2 \times T_1 M_2$. Indeed, if $(\varepsilon_i)_{i=1,2}$ is an orthonormal basis of $\mathbb{R}^2$, we can write, for $z = (x, y) \in \mathbb{R}^2 \times T_1 M_2$, the basis $(F_1(z), F_2(z))$ of $\Delta(z)$ defined in (2.12) as

$$(2.16) \quad F_1(z) = (\varepsilon_1 \quad f(y))^T, \quad F_2(z) = (\varepsilon_2 \quad h(y))^T,$$

where $f$ is the infinitesimal generator of the geodesic flow on $T_1 M_2$ and $h$ is a vector field on $T_1 M_2$ whose integral curves are also geodesics: for a system of coordinates $(T_1 U, \bar{y}, v)$, we have

$$h(\bar{y}, v) = \left( -R_{\bar{y}}(v) \left( \sum_{i,j} \Gamma_{ij}^k v^i R_{\bar{y}}(v)^j \right)_{k=1,2} \right)^T,$$

where $R_{\bar{y}}$ stands for the rotation of angle $\frac{\pi}{2}$ in the $S^1$-fiber above $\bar{y} \in M_2$. Let $g$ be the vector field of $T_1 M_2$ which generates the rotation of angle $\frac{\pi}{2}$ in the $S^1$-fiber. We have the following Lie bracket relations between $f, g$ and $h$ (cf. [8]),

$$(2.17) \quad [f, g] = h, \quad [g, h] = f, \quad [h, f] = K_2 g.$$

For every $z \in \mathbb{R}^2 \times T_1 M_2$, $B_0(z) = (F_1(z), F_2(z), \sqrt{2}F_3(z), \sqrt{2}F_4(z), \sqrt{2}F_5(z))$ defines an orthonormal basis of $T_z(\mathbb{R}^2 \times T_1 M_2)$. We can define other basis of $T_z(\mathbb{R}^2 \times T_1 M_2)$ as follows: for $\alpha \in S^1$, let

$$(2.18) \quad B_\alpha(z) = (F_1^\alpha(z), F_2^\alpha(z), \sqrt{2}F_3(z), \sqrt{2}F_4^\alpha(z), \sqrt{2}F_5^\alpha(z)),$$

where

$$(F_1^\alpha(z), F_2^\alpha(z))^T = R_\alpha(F_1(z), F_2(z))^T$$

$$(F_4^\alpha(z), F_5^\alpha(z))^T = R_\alpha(F_4(z), F_5(z))^T.$$

Notice that $(F_1^\alpha(z), F_2^\alpha(z))$ is also a basis of $\Delta(z)$. If $M_2$ is the unit sphere of radius one, $\mathbb{R}^2 \times T_1 M_2$ is simply equal to $\mathbb{R}^2 \times SO(3)$ and (2.16) provides the dynamics of the plate-ball problem as given in [15].

## 2.5 On the controllability of rolling surfaces

As a consequence of equation (2.15), we recover a result proved by Agrachev and Sachkov in [2].

**Theorem 1.** *For $x \in \mathscr{RO}(M_1, M_2)$, the reachable set $RS(x)$ is an immersed manifold of $\mathscr{RO}(M_1, M_2)$ of dimension two or five. If in addition, the two manifolds are simply connected, then the control system $\mathscr{S}_R$ is completely controllable if and only if $M_1$ and $M_2$ are not isometric (by an isometry of positive determinant).* □

*Proof of Theorem 1.* Thanks to the symmetry of $\mathscr{S}_R$ and to the Orbit theorem (cf. [16]), the proof of the previous theorem reduces to study the following alternative

(i) for every $x \in \mathscr{RO}(M_1, M_2)$, there exists $x' \in RS(x)$ such that $K_2(\pi_{M_2}(x')) \neq K_1(\pi_{M_1}(x'))$;

(ii) there exists $x \in \mathscr{RO}(M_1, M_2)$ such that for every $x' \in RS(x)$ we have $K_2(\pi_{M_2}(x')) = K_1(\pi_{M_1}(x'))$.

If case (i) holds then thanks to equation (2.15), the Lie Algebraic Rank Condition (LARC) is satisfied at $x'$, *i.e.*, if $\Omega$ is the Lie algebra of vector fields generated by the $F_i$'s then $\dim \Omega(y) = 5$. By the Orbit theorem, we conclude that the dimension of $RS(x)$ is equal to five and, since $\mathscr{S}_R$ is symmetric (in the sense that the reachability relation for $\mathscr{S}_R$ is a symmetric relation), for every $x \in \mathscr{RO}(M_1, M_2)$, $RS(x)$ contains an open neighborhood of $x$ and so is an open subset. By the same argument, $\mathscr{RO}(M_1, M_2) \backslash RS(x)$ is open, and then $RS(x)$ is closed. As $\mathscr{RO}(M_1, M_2)$ is connected, $RS(x) = \mathscr{RO}(M_1, M_2)$ for every $x \in \mathscr{RO}(M_1, M_2)$.

If case (ii) holds, then $[F_1, BF_2] = 0$ on $RS(x)$ (where $B$ and $C$ are defined locally). Set $x = (x_1, x_2, R_0)$. We prove next by a direct computation that $M_1$ and $M_2$ are locally isometric. It is enough to do it in a neighborhood of $x_1$. First notice that every point $x' \in RS(x)$ in a neighborhood of $x$ can be reached by $\gamma$, the concatenation of $\gamma_1$ an integral curve of $F_1$ and $\gamma_2$ an integral curve of $BF_2$. Since $\gamma_1$ and $\gamma_2$ are geodesics then $\gamma$ is a once broken geodesic. Thanks to Proposition 1, we obtain that $R$ remains constant along $\gamma$ and equal to $R_0$, *i.e.*, we can take with no loss of generality $\psi = 0$ for every chart. We get that, in coordinates, $F_1$ and $BF_2$ are given by

$$(2.19) \quad F_1(z) = (1 \quad 0 \quad 1 \quad 0 \quad 0)^T, \quad BF_2(z) = \left(0 \quad 1 \quad 0 \quad -\frac{B}{C} \quad B_{v_1} - \frac{C_{v_2}}{C} B\right)^T.$$

For $(v, w)$ in a neighborhood of 0, we consider the inputs $(v, 0)$ and $(0, w)$ defined for $t \in [0, 1]$. Let $u^1$ and $u^2$ be resp. the concatenation of $(0, w)$ followed by $(v, 0)$ and $(v, 0)$ followed by $(0, w)$ resp.. Both inputs steer 0 to $(v, w, v, -w, 0)$. We use $w_2(t)$ to denote the fourth coordinate of the trajectory associated to $u^2$ for $t \in [1, 2]$. We easily show that

$$(2.20) \quad w_2(t) = -w \int_1^t \frac{B(v, (s-1)w)}{C(v, w_2(s))} ds.$$

Set $f(t) = \frac{\partial w_2}{\partial v}(t)$. Differentiating (2.20) and using the fact that $\dot{\psi} = 0$ yields to the following inequality

$$|f(t)| \leq C_0 \int_1^t |f(s)|\, ds.$$

This immediately implies that $f = 0$. By computing its value when $v = 0$, we get that $w_2(t) = -(t-1)w$ for all $v, w$. Using again (2.20), we have $B(v, w) = C(v, -w)$ and therefore

$$BF_2(z) = (0 \quad 1 \quad 0 \quad -1 \quad 0)^T.$$

We conclude that $M_1$ and $M_2$ are locally isometric with a symmetry of positive determinant. Moreover, along any admissible trajectory $R$ remains constant. Assume that $R = R_0$ (modulo parallel transport along trajectories of $\mathcal{S}_R$). Recall that $R_0$ can be seen as a isometry from $T_{x_1}M_1$ to $T_{x_2}M_2$. For every minimal geodesic $\gamma_1 : [0, l] \to M_1$ starting at $x_1$, consider $\gamma_2 : [0, l] \to M_2$, the geodesic of $M_2$ starting at $x_2$ with tangent vector $R_0(\dot{\gamma}_1(0))$. Consider now the map $T : M_1 \to M_2$ defined by $T(\gamma_1(t)) = \gamma_2(t)$. Since, $(\gamma_1, \gamma_2, R_0)$ belongs to $RS(x)$, the conditions of Ambrose's theorem are verified (see Theorem 5.1 in [25]) and since $M_2$ is simply connected, we deduce that $M_1$ and $M_2$ are isometric. □

**Remark 4.** In the case where the two manifolds are convex surfaces embedded in $\mathbb{R}^3$, Marigo and Bicchi give a beautiful geometric description of case (ii): each manifold is the image of the other one by the reflection with respect to the (common) tangent plane to $M_1$ and $M_2$ at the contact point and that geometric property holds during the rolling of $M_2$ on $M_1$ (cf. [21]). Note that the previous reflection is a symmetry with determinant $-1$, which is different from the symmetry given in the proof above. This comes from the fact that the Marigo and Bicchi modelization embeds the rolling problem in $\mathbb{R}^3$. □

## 3 The rolling body problem is Liouvillian

The class of Liouvillian systems (cf. [7, 6]) was recently introduced as a natural extension of differential flat systems (cf. [10]). As well as for flat systems, open loop trajectories can be obtained by a simple parameterization of a particular variable called "the partial or maximal linearizing output" modulo quadratures, *i.e.*, elementary integrations. In the next subsections, we give an alternative definition to [7] of the class of Liouvillian systems and we prove that the control system (2.12) is Liouvillian when one of the two bodies admits a symmetry of revolution, *e.g.*, $C(x, y) = C(x)$.

### 3.1 Liouvillian systems

Liouvillian systems were initially defined in the differential algebra setting. We give here a new formulation using the language of diffieties and infinite dimensional geometries. This definition is well suited to prove the Liouvillian character of the control system $\mathcal{S}_R$. For the sake of convenience, we first recall some facts concerning the theory of diffieties and the Lie-Bäcklund approach to equivalence and flatness (see [9, 11, 12, 13, 30]).

Let $I$ be a countable set of cardinality $\ell$, which may be finite or not. Let $\mathbb{R}^I$ be the linear space of all real functions $x = (x^i)$ on $I$. The space $\mathbb{R}^I$ has the natural topology of the Euclidean space if $I$ is finite and the Fréchet topology otherwise. The elements $x^i$, $i \in I$, are called coordinates. For an open set $U \subset \mathbb{R}^I$ we denote by $C^\infty(U)$ the space of all real functions on $U$ that depend on finitely many coordinates and are smooth as functions of a finite number of variables. A chart on a set $M$ is a 3-tuple $(U, \varphi, \mathbb{R}^I)$, where $U$ is a subset of $M$, $\varphi$ is a bijection of $U$ onto an open subset $\varphi(U)$. The notions of smooth charts and smooth atlases can be defined as in the finite dimensional case. The set $M$, equipped with an equivalence class of smooth atlases, is called a $C^\infty$ $\mathbb{R}^I$-manifold. The number $\ell$ does not depend on a chart $(U, \varphi, \mathbb{R}^I)$ and is called the dimension of the smooth manifold $M$.

A diffiety is a pair $\mathscr{M} = (M, CTM)$ where $M$ is a $C^\infty$ $\mathbb{R}^I$-manifold and $CTM$ a finite dimensional involutive distribution on $M$. The distribution $CTM$ is called *Cartan distribution* and its dimension the *Cartan dimension* of $\mathscr{M}$. Local smooth sections of $CTM$ are called *Cartan fields*. We are only concerned here with the case of ordinary diffieties, *i.e.*, the dimension of $CTM$ is equal to 1. For the sake of convenience, we use without distinction the notations $(M, CTM)$ and $(M, \partial)$ to denote the ordinary diffiety $\mathscr{M}$, where $\partial$ is basis vector field of $CTM$. Let $\mathscr{M} = (M, CTM)$ be a diffiety with $\dim CTM = 1$. Let $(U, \varphi, \mathbb{R}^I)$ be a chart on $M$ and $\partial$ be a basis vector field of $CTM$ on $U$, then the 4-tuple $(U, \varphi, \mathbb{R}^I, \partial)$ is called a chart on $\mathscr{M}$. A smooth mapping $\phi : M \to N$ is called a *Lie-Bäcklund morphism* of a diffiety $\mathscr{M} = (M, CTM)$ into a diffiety $\mathscr{N} = (N, CTN)$, written $\phi : \mathscr{M} \to \mathscr{N}$, if it is compatible with the Cartan distributions $CTM$ and $CTN$, *i.e.*, $\phi_*(CTM) \subset CTN$, where $\phi_* : TM \to TN$ is the tangent mapping and $TM$ (resp. $TN$) the tangent bundle of $M$ (resp. $N$).

Consider now the ordinary diffiety $\mathscr{F}_m = (F_m, CTF_m)$, where $F_m = \mathbb{R} \times \mathbb{R}^\infty_m$[1], and let $(U, \varphi, \mathbb{R} \times \mathbb{R}^\infty_m, \partial_{F_m})$ be a chart on $\mathscr{F}_m$ with local coordinates $\{t, w_i^{(v)} \mid i = 1, \dots, m; v \geq 0\}$ and basis Cartan field

$$\partial_{F_m} = \frac{\partial}{\partial t} + \sum_{i=1}^{m} \sum_{v \geq 0} w_i^{(v+1)} \frac{\partial}{\partial w_i^{(v)}}.$$

The diffiety $\mathscr{F}_m$, as above defined, is usually called *trivial diffiety* and plays a central role in the Lie-Bäcklund approach of flatness.

A diffiety $\mathscr{M}$ is said to be *(locally) of finite type* if there exists a (local) Lie-Bäcklund submersion $\pi : \mathscr{M} \to \mathscr{F}_m$ such that the fibers are finite dimensional. The integer $m$ is called the *(local) differential dimension* of $\mathscr{M}$ (cf. [11]).

**Definition 1** ([11, 12]). A *system* is a (local) Lie-Bäcklund fiber bundle $\sigma_M = (\mathscr{M}, \mathbb{R}, \lambda)$, where

- $\mathscr{M}$ is a diffiety of finite type where a Cartan field $\partial_M$ has been chosen once for all;

[1] $\mathbb{R}^\infty_m = \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the product of a countably infinite number of copies of $\mathbb{R}^m$.

- $\mathbb{R}$ is endowed with a canonical structure of a diffiety, with global coordinate $t$ and Cartan field $\partial/\partial t$;
- $\lambda : \mathscr{M} \to \mathbb{R}$ is a Lie-Bäcklund submersion such that $\lambda_*(\partial_M) = \partial/\partial t$, where $\lambda_*$ is the tangent mapping of $\lambda$. □

The system $(\mathscr{F}_m, \mathbb{R}, \mathrm{pr})$, where pr is the natural projection mapping $\mathrm{pr} : \{t, w_i^{(\nu)}\} \to t$ and $\mathscr{F}_m$ a trivial diffiety, is called a *trivial system*. Two systems $(\mathscr{M}, \mathbb{R}, \lambda)$ and $(\mathscr{N}, \mathbb{R}, \delta)$ are said to be *(differentially) equivalent* iff

- $\phi_*(\partial_M) = \partial_N$, where $\phi : \mathscr{M} \to \mathscr{N}$ is a Lie-Bäcklund isomorphism and $\phi_*$ the tangent mapping of $\phi$;
- $\lambda = \phi^*\delta$, where $\phi^*$ is the dual mapping of $\phi$.

A system $(\mathscr{M}, \mathbb{R}, \lambda)$ is said to be *(locally) differentially flat*, or simply *flat* if it is (locally) equivalent to a trivial system. If $\{t, y_i^{(\nu)} \mid i = 1, \dots, m; \nu \geq 0\}$ are local coordinates of $\mathscr{F}_m$ then $y = (y_1, \dots, y_m)$ is called a *flat* or *linearizing output*.

A diffiety $\mathscr{S} = (S, CTS)$ is called a *subdiffiety* of a diffiety $\mathscr{M} = (M, CTM)$ if $S$ is a submanifold of $M$ and $CTS = TS \cap CT_SM$, *i.e.*, the natural embedding $\imath : \mathscr{S} \to \mathscr{M}$ is a Lie-Bäcklund immersion[2]. The fiber bundle $T_SM$ denotes here the restriction of the vector bundle $TM$ on $S$, *i.e.*,

$$T_SM = \bigcup_{p \in S} T_pM.$$

The tangent mapping $\imath_* : TS \to TM$ is injective and the image $\imath_*(TS) \subset T_SM$. If $\mathscr{M}$ is of finite type, then clearly $\mathscr{S}$ is of finite type as well.

**Definition 2** (Subsystem). A system $\sigma_S = (\mathscr{S}, \mathbb{R}, \delta)$ is said to be a *subsystem* of $\sigma_M = (\mathscr{M}, \mathbb{R}, \lambda)$, written $\sigma_S \subset \sigma_M$, iff

- $\mathscr{S}$ is a subdiffiety of $\mathscr{M}$;
- The restriction $\imath^*\lambda = \delta$, where $\imath^*$ is the dual mapping of the natural embedding $\imath : \mathscr{S} \to \mathscr{M}$. □

Consider the system $\sigma_M = (\mathscr{M}, \mathbb{R}, \lambda)$ with differential dimension $m$. Since $\mathscr{M}$ is of finite type, there exists a Lie-Bäcklund submersion $\pi : \mathscr{M} \to \mathscr{F}_m$ such that its fibers are finite dimensional, say $n$. Assume now that $\sigma_M$ is not flat and $\sigma_S = (\mathscr{S}, \mathbb{R}, \delta)$ is a flat subsystem of $\sigma_M$ with a flat output given by $y = (y_1, \dots, y_m)$. Define the canonical bundle morphism $\rho : TM \to TM/TS$ that takes a vector $\zeta \in T_pM$, $p \in M$, to its equivalence class $\zeta + T_pS$ and let $\tau : TM/TS \to M$ be the fiber bundle whose fibers $\tau^{-1}(p)$, $p \in M$, are finite dimensional. If $\{t, \eta_1, \dots, \eta_s, u_i^{(\nu)} \mid i = 1, \dots, m; \nu \geq 0\}$ are local coordinates of $\mathscr{S}$ then the Cartan distribution of $\mathscr{S}$ is spanned by

[2] Since we consider only diffieties of Cartan dimension 1, $CTS = CT_SM$ here.

$$\partial_S = \frac{\partial}{\partial t} + \sum_{j=1}^{s} F_j^1 \frac{\partial}{\partial \eta_j} + \sum_{i=1}^{m} \sum_{v \geq 0} u_i^{(v+1)} \frac{\partial}{\partial u_i^{(v)}},$$

where $F_j^1$ are $C^\infty$ functions on $S$. A local smooth section $\zeta$ of $TM/TS$ is given by

$$\zeta = \sum_{j=1}^{\Delta=n-s} F_j^2 \frac{\partial}{\partial \xi_j},$$

where $F_j^2$ are $C^\infty$ functions on $M$, with $\{t, \xi_1, \dots, \xi_\Delta, \eta_1, \dots, \eta_s, u_i^{(v)} \mid i = 1, \dots, m; v \geq 0\}$ local coordinates of $\mathscr{M}$.

**Definition 3** (Defect). Let $\sigma_M$ and $\sigma_S$ two systems such that

- $\sigma_S \subset \sigma_M$;
- $\sigma_S$ is flat with a flat output $y$.

Then $\sigma_S$ is called a *partial flat subsystem* of $\sigma_M$ and the flat output $y$ of $\sigma_S$, a *partial linearizing output* of $\sigma_M$. If, in addition, that flat output $y$ is such that $\Delta = \dim \tau^{-1}(p)$, with $p \in M$ and $\tau : TM/TS \to M$ the aforementioned fiber bundle, is minimal, then $\Delta$ is called the *defect*, $\sigma_S$ a *maximal flat subsystem* and $y$ a *maximal linearizing output* of $\sigma_M$. □

Consider now the classical dynamics

(3.1) $\quad \dot{x} = F(x, u), \quad (x, u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m,$

where $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$ and $F = (F_1, \dots, F_n)$ is a $m$-tuple of $C^\infty$ functions on $X \times U$. To (3.1) we can associate a diffiety $\mathscr{M} = (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_m^\infty, \partial)$ with local coordinates $\{t, x_1, \dots, x_n, u_i^{(v)} \mid i = 1, \dots, m; v \geq 0\}$ and Cartan field

$$\partial = \frac{\partial}{\partial t} + \sum_{j=1}^{n} F_j \frac{\partial}{\partial x_j} + \sum_{i=1}^{m} \sum_{v \geq 0} u_i^{(v+1)} \frac{\partial}{\partial u_i^{(v)}}.$$

A subsystem of (3.1) is given by a diffiety $\mathscr{S} = (S, \partial_S)$, with local coordinates $\{t, \eta_1, \dots, \eta_s, u_i^{(v)} \mid i = 1, \dots, m; v \geq 0\}$ and a basis Cartan field

$$\partial_S = \frac{\partial}{\partial t} + \sum_{j=1}^{s} F_j^1(\eta, u) \frac{\partial}{\partial \eta_j} + \sum_{i=1}^{m} \sum_{v \geq 0} u_i^{(v+1)} \frac{\partial}{\partial u_i^{(v)}},$$

where $\eta = (\eta_1, \dots, \eta_s) \in X^1 \subset \mathbb{R}^s$ and $F_j^1$ are $C^\infty$ functions on $X^1 \times U$. A local section $\zeta$ of $TM/TS$ is given by

$$\zeta = \sum_{j=1}^{\Delta=n-s} F_j^2(\eta, \xi, u) \frac{\partial}{\partial \xi_j},$$

where $\xi = (\xi_1, \dots, \xi_\Delta) \in X^2 \subset \mathbb{R}^\Delta$ and $F_j^2$ are $C^\infty$ functions on $X^1 \times X^2 \times U = X \times U$. The vector $\xi$ represents only the complement of $\eta$ (by renumbering the $x_i$'s if needed) to form the vector $x$, *i.e.*, $x = (\eta, \xi)$.

$$\dot{x} = \begin{pmatrix} \dot{\eta} \\ \dots \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F^1(\eta, u) \\ \dots\dots\dots \\ F^2(\eta, \xi, u) \end{pmatrix}.$$

**Definition 4** (Liouvillian Systems). Let $\sigma_M = (\mathscr{M}, \mathbb{R}, \lambda)$ a system of differential dimension $m$ and $\sigma_S = (\mathscr{S}, \mathbb{R}, \delta)$ a flat subsystem of $\sigma_M$. Then $\sigma_M$ is said to be *Liouvillian* iff there exists a nested chain of subsystems $\sigma_S = \sigma_{S_0} \subset \sigma_{S_1} \subset \cdots \subset \sigma_{S_\Delta} = \sigma_M$, with $\sigma_{S_j} = (\mathscr{S}_j, \mathbb{R}, \delta_j)$ and $\mathscr{S}_j = (S_j, \partial_j)$, such that, for $j = 1, \dots, \Delta$, either

(i) $\partial_j = \alpha_j \partial / \partial \xi_j + \partial_{j-1}$, $\alpha_j \in C^\infty(S_{j-1})$, or

(ii) $\partial_j = \alpha_j \xi_j \partial / \partial \xi_j + \partial_{j-1}$, $\alpha_j \in C^\infty(S_{j-1})$.

If $\sigma_S$ is maximal (resp. partial), *i.e.*, $\Delta$ is the defect of $\sigma_M$, then $\sigma_M$ is called *maximal Liouvillian system* (resp. *partial Liouvillian system*). □

According to the definition, a local section $\zeta_j$ of $TS_j / TS_{j-1}$ is given either by

(i) $\zeta_j = \alpha_j(\eta, \xi_1, \dots, \xi_{j-1}) \partial / \partial \xi_j$ (hence $\dot{\xi}_j = \alpha_j(\eta, \xi_1, \dots, \xi_{j-1})$ and $\xi_j = \int \alpha_j$) or

(ii) $\zeta_j = \alpha_j(\eta, \xi_1, \dots, \xi_{j-1}) \xi_j \partial / \partial \xi_j$ (hence $\dot{\xi}_j = \alpha_j(\eta, \xi_1, \dots, \xi_{j-1}) \xi_j$ and $\xi_j = e^{\int \alpha_j}$).

**Remark 5.** Notice that an arbitrary linearizing output $y$ for $\sigma_S$ does not necessarily give rise to a Liouvillian system. Therefore, the Liouvillian character of a system depends on the choice of $y$. □

## 3.2 The rolling problem and its motion planning

Let us consider the kinematic equations of motion of the contact point between two bodies rolling on top of each other described in geodesic coordinates by (2.12)

$$\begin{aligned}
\dot{v}_1 &= u_1, \\
\dot{w}_1 &= \frac{1}{B} u_2, \\
\dot{v}_2 &= \cos(\psi) u_1 - \sin(\psi) u_2, \\
\dot{w}_2 &= -\frac{1}{C} \sin(\psi) u_1 - \frac{1}{C} \cos(\psi) u_2, \\
\dot{\psi} &= -\frac{C_{v_2}}{C} \sin(\psi) u_1 + \left( \frac{B_{v_1}}{B} - \frac{C_{v_2}}{C} \cos(\psi) \right) u_2.
\end{aligned} \tag{3.2}$$

The above system is equivalent to the reduced system

(3.3a) $\dot{v}_2 = \cos(\psi)\dot{v}_1 - B\sin(\psi)\dot{w}_1,$

(3.3b) $\dot{w}_2 = -\dfrac{1}{C}\sin(\psi)\dot{v}_1 - \dfrac{B}{C}\cos(\psi)\dot{w}_1,$

(3.3c) $\dot{\psi} = -\dfrac{C_{v_2}}{C}\sin(\psi)\dot{v}_1 + \left(B_{v_1} - \dfrac{C_{v_2}}{C}B\cos(\psi)\right)\dot{w}_1,$

since $u_1 = \dot{v}_1$ and $u_2 = B\dot{w}_1$.

Different situations will be considered throughout this section:

**(h1)** One of the two bodies has a symmetry of revolution, $C(v_2, w_2) = C(v_2)$.

**(h2)** One of the two bodies is a plane, $B \equiv 1$, and the other has a symmetry of revolution, $C(v_2, w_2) = C(v_2)$.

**(h3)** One of the two bodies is a plane, $B \equiv 1$, and the other is a ball, $C = \cos(v_2)$. This case is often referred as the plate-ball problem.

**Proposition 2.** *System* (3.2) *is not differentially flat.* □

*Proof of Proposition 2.* The proof rests on the Goursat theorem (cf. [4, 22, 23]). As a matter of fact, a two-inputs driftless controllable system is flat if and only if it can be put under the Goursat normal form. The Pfaffian system associated to (3.3) is generated by the one-forms

$$\alpha^1 := dv_2 - \cos(\psi)\,dv_1 + B\sin(\psi)\,dw_1,$$

$$\alpha^2 := d\psi + \frac{C_{v_2}}{C}\sin(\psi)\,dv_1 - \left(B_{v_1} - \frac{C_{v_2}}{C}B\cos(\psi)\right)dw_1, \tag{3.4}$$

$$\alpha^3 := dw_2 + \frac{1}{C}\sin(\psi)\,dv_1 + \frac{B}{C}\cos(\psi)\,dw_1.$$

Denote by $I^{(0)}$ the $C^\infty(\mathscr{RO}(M_1, M_2))$-module generated by $\{\alpha^1, \alpha^2, \alpha^3\}$. From $I^{(0)}$, the derived systems are constructed inductively as follows

$$I^{(k+1)} = \{\beta \in I^{(k)} \mid d\beta \equiv 0 \bmod I^{(k)}\}, \quad k \in \mathbb{N} \tag{3.5}$$

where $\wedge$ is the wedge product and $d\beta$ the exterior differential of $\beta$. For instance, $I^{(1)} = \{\beta \in I^{(0)} \mid d\beta \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = 0\}$. The derived systems form a chain of Pfaffian systems called the derived flag

$$I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(k)} \supset \cdots$$

A straightforward calculation shows that

$$d\alpha^1 = \sin(\psi)\,d\psi \wedge dv_1 + \sin(\psi)B_{v_1}\,dv_1 \wedge dw_1 + B\cos(\psi)\,d\psi \wedge dw_1,$$

$$d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = 0,$$

$$d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = \left(-B_{v_1v_1} + \left(\frac{C_{v_2}}{C}\right)_{v_2} B - \left(\frac{C_{v_2}}{C}\right)^2 B\right) dv_1 \wedge dw_1 \wedge dv_2 \wedge d\psi \wedge dw_2,$$

$$d\alpha^3 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = 0,$$

$$d\alpha^1 \wedge \alpha^1 \wedge \alpha^3 = d\alpha^1 \wedge dv_2 \wedge dw_2 + B\,d\psi \wedge dv_1 \wedge dw_1 \wedge dw_2,$$

$$d\alpha^3 \wedge \alpha^1 \wedge \alpha^3 = \frac{1}{C}\cos(\psi)\,d\psi \wedge dv_1 \wedge dv_2 \wedge dw_2 + \frac{1}{C}B_{v_1}\cos(\psi)\,dv_1 \wedge dw_1 \wedge dv_2 \wedge dw_2 - \frac{B}{C}\sin(\psi)\,d\psi \wedge dw_1 \wedge dv_2 \wedge dw_2 + \frac{B}{C^2}d\psi \wedge dv_1 \wedge dv_2 \wedge dw_1 + \left(\frac{1}{C}\right)_{v_2} B\,dv_2 \wedge dv_1 \wedge dw_2 \wedge dw_2.$$

Using now relations $B_{v_1v_1} + K_1B = 0$ and $C_{v_2v_2} + K_2C = 0$, where $K_1$, $K_2$ are the Gaussian curvatures (see section 2.1), $d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3$ writes

$$d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = (K_1 - K_2)B\,dv_1 \wedge dw_1 \wedge dv_2 \wedge d\psi \wedge dw_2.$$

Assume that system (3.2) is controllable (see theorem 1), then $K_1 \neq K_2$ and $d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \neq 0$. So, we get

$$I^{(0)} = \{\alpha^1, \alpha^2, \alpha^3\},$$

$$I^{(1)} = \{\alpha^1, \alpha^3\},$$

$$I^{(2)} = \{0\}.$$

The rank condition on the derived flag is not satisfied and it follows that (3.4) is not flat. □

**Lemma 1.** *Under the assumption* (h1)*, system* (3.2) *has a defect of 1.* □

*Proof of Lemma 1.* We only need here to exhibit a Pfaffian system of dimension 2, with class $I^{(0)} = 4$ (see [4]), satisfying the conditions of the Goursat theorem (cf. [14]). Consider the subsystem defined by (3.3a) and (3.3c), *i.e.*,

$$\begin{aligned} \dot{v}_2 &= \cos(\psi)\dot{v}_1 - B\sin(\psi)\dot{w}_1, \\ \dot{\psi} &= -\frac{C_{v_2}}{C}\sin(\psi)\dot{v}_1 + \left(B_{v_1} - \frac{C_{v_2}}{C}B\cos(\psi)\right)\dot{w}_1. \end{aligned} \tag{3.6}$$

Then the associated Pfaffian system is generated by the two one-forms $\{\alpha^1, \alpha^2\}$ given by (3.4). Denote by $I^{(0)}$ the $C^\infty(\mathscr{RO}(M_1, M_2))$-module generated by $\{\alpha^1, \alpha^2\}$. A straightforward calculation shows that

$$\begin{aligned} d\alpha^1 &= \sin(\psi)\,d\psi \wedge dv_1 + \sin(\psi)B_{v_1}\,dv_1 \wedge dw_1 + B\cos(\psi)\,d\psi \wedge dw_1, \\ d\alpha^1 \wedge \alpha^1 &= d\alpha^1 \wedge dv_2 + B\,d\psi \wedge dv_1 \wedge dw_1, \\ d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 &= 0, \\ d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 &= (K_1 - K_2)B\,dv_1 \wedge dw_1 \wedge dv_2 \wedge d\psi, \quad K_1 \neq K_2. \end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned} I^{(0)} &= \{\alpha^1, \alpha^2\}, \\ I^{(1)} &= \{\alpha^1\}, \\ I^{(2)} &= \{0\}, \end{aligned}$$

and the conditions of the Goursat theorem are generically fulfilled. □

**Remark 6.** The assumption that one of the two bodies admits a symmetry of revolution, *i.e.*, $C(v_2, w_2) = C(v_2)$, is necessary to ensure that (3.6) is independent of the variable $w_2$, and so it is a subsystem of (3.3). □

**Proposition 3.** *Under the assumption* (h1)*, system* (3.2) *is Liouvillian.* □

*Proof of Proposition 3.* From Lemma 1, we showed that the reduced subsystem (3.6) is flat. Hence, there is a maximal linearizing output, say $z = (z_1, z_2)$, such that every variable in (3.6), *i.e.*, $(v_1, w_1, v_2, \psi)$, can be expressed as a function of $z$ and a finite number of its time derivatives, so it is for $u_1$ and $u_2$ since $u_1 = \dot{v}_1$ and $u_2 = B\dot{w}_1$. From equation (3.3b), $\dot{w}_2$ can also be expressed as a function of $z$ and its time derivatives and it readily follows that system (3.2) is Liouvillian (see definition 4). □

Despite the last proposition, there is no constructive way of finding a maximal linearizing output. However, under the assumption (h2), a maximal linearizing output can be explicitly computed. Let us first rewrite (3.2) with $B \equiv 1$ and $C(v_2, w_2) = C(v_2)$

$$
\begin{aligned}
\dot{v}_1 &= u_1, \\
\dot{w}_1 &= u_2, \\
\dot{v}_2 &= \cos(\psi)u_1 - \sin(\psi)u_2, \\
\dot{w}_2 &= -\frac{1}{C}(\sin(\psi)u_1 + \cos(\psi)u_2), \\
\dot{\psi} &= -\frac{C_{v_2}}{C}(\sin(\psi)u_1 + \cos(\psi)u_2) = C_{v_2}\dot{w}_2.
\end{aligned}
\tag{3.8}
$$

**Proposition 4.** *A maximal linearizing output for system* (3.8) *is given by*

$$
\begin{aligned}
X &= v_1 - v_2\cos(\psi), \\
Y &= w_1 + v_2\sin(\psi).
\end{aligned}
\tag{3.9}
$$

□

*Proof of Proposition 4.* A straightforward computation shows that

$$
\begin{aligned}
\dot{X} &= \sin(\psi)(v_2C_{v_2} - C)\dot{w}_2, \\
\dot{Y} &= \cos(\psi)(v_2C_{v_2} - C)\dot{w}_2,
\end{aligned}
$$

so

$$
\tan(\psi) = \frac{\dot{X}}{\dot{Y}}, \tag{3.10}
$$

and it follows that

$$
\psi = \alpha(\dot{X}, \dot{Y}). \tag{3.11}
$$

From equation (3.10), we get

$$
\dot{\psi} = \frac{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}}{\dot{X}^2 + \dot{Y}^2},
$$

so

$$\frac{C_{v_2}}{v_2 C_{v_2} - C} = \frac{\ddot{X}\cos(\psi) - \ddot{Y}\sin(\psi)}{\dot{X}^2 + \dot{Y}^2}. \tag{3.12}$$

We obtain an equation where the solution[3] $v_2$ is of the form

$$v_2 = \mathfrak{b}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}). \tag{3.13}$$

Equations (3.11) and (3.13) show that $\psi$ and $v_2$ can be written as a function of the maximal linearizing output (3.9) and a finite number of its time derivatives. From $(X, Y, v_2, \psi)$, we finally deduce

$$\begin{aligned} v_1 &= X + v_2\cos(\psi) = \mathfrak{c}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}), \\ w_1 &= Y - v_2\sin(\psi) = \mathfrak{d}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}), \\ u_1 &= \dot{v}_1 = \mathfrak{e}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}, X^{(3)}, Y^{(3)}), \\ u_2 &= \dot{w}_1 = \mathfrak{f}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}, X^{(3)}, Y^{(3)}), \\ w_2 &= \int \frac{1}{C_{v_2}}\dot{\psi}, \end{aligned} \tag{3.14}$$

and the proposition follows. □

**Remark 7.** Under the assumption (h3), equation (3.12) can be written

$$v_2 + \cot(v_2) = \frac{\dot{X}^2 + \dot{Y}^2}{\ddot{X}\cos(\psi) - \ddot{Y}\sin(\psi)}.$$ □

[3] It is easy to prove that equation (3.12) admits a solution in the neighborhood of $v_2 = 0$. Let $g(v_2)$ be defined by

$$g(v_2) := \frac{C_{v_2}}{v_2 C_{v_2} - C},$$

then

$$\left.\frac{dg(v_2)}{dv_2}\right|_{v_2=0} = K_2 > 0 \quad \text{(convexity assumption)}$$

where $K_2$ is the Gaussian curvature. Then, from the implicit function theorem, equation (3.12) admits, in the neighborhood of $v_2 = 0$, a solution of the form

$$v_2 = \mathfrak{b}(X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}).$$

As for flat systems, it is now clear that a simple parameterization of the maximal linearizing output leads to open-loop trajectories for the state and input variables.

## 4 The continuation method

For the rest of the section, we assume that $M_1 = \mathbb{R}^2$ and set $M := \mathscr{RO}(\mathbb{R}^2, M_2) = \mathbb{R}^2 \times T_1 M_2$. We have $\mathscr{S}_R = (M, \mathbb{R}^2, \Delta, H)$, where $M_2$ is a convex compact surface subject to the condition (C) given below in (4.14) and $H = L^2([0,1], \mathbb{R}^2)$. Then $\mathscr{S}_R$ is completely controllable. Let $K$ be the curvature function of $M_2$. Set $K_{\min} = \inf_{M_2} K > 0$ and $K_{\max} = \sup_{M_2} K$. We use $\|u(t)\|$ and $\|u\|_H$ respectively to denote $(\sum_{i=1}^2 u_i^2(t))^{1/2}$ and $(\int_0^1 \|u(t)\|^2 \, dt)^{1/2}$. If $I = [t, t']$ is a subinterval of $[0,1]$, we use $\|u\|_I$ or $\|u\|_{[t,t']}$ to denote $(\int_t^{t'} \|u(t)\|^2 \, dt)^{1/2}$. In particular, if $u, v \in H$, then $(u, v)_H = \int_0^1 u^T(t) v(t) \, dt$.

### 4.1 The continuation method

We apply the continuation method (CM for short) to the motion planning problem for $\mathscr{S}_R$. From the brief description of the CM given in the introduction, we specify the map $F$ to be the end-point $\phi_p : H \to M$ associated to some fixed $p \in M$. (For more details and complete justifications regarding the CM cf. [8].) For $u \in H$ and $p \in M$, let $\gamma_{p,u}$ be the trajectory of $\mathscr{S}_R$ starting at $p$ for $t = 0$ and corresponding to $u$. Then for $v \in H$, $\phi_p(v)$ is given by

$$\phi_p(v) := \gamma_{p,v}(1).$$

Recall that $\phi_p(v)$ is defined for every $v \in H$. The MPP can be reformulated as follows: for every $p, q \in M$, exhibit a control $u_{p,q} \in H$ such that

$$\phi_p(u_{p,q}) = q. \tag{4.1}$$

In other words, for fixed $p$, we must find a map $i_p : M \to H$ such that $\phi_p \circ i_p =$ identity, *i.e.*, we are looking for a right inverse of $\phi_p$. It can be shown that such a right inverse exists in a neighborhood of any point $u \in H$ such that $D\phi_p(u)$ is surjective. Therefore, it is reasonable to expect difficulties with the singular points of $\phi_p$, *i.e.*, the controls $v \in H$ where $\operatorname{rank} D\phi_p(v) < 5$. Let then $S_p$ be the set of singular points of $\phi_p$ and $\phi_p(S_p)$ the set of singular values.

The application of the CM to the MPP is thus decomposed in two steps. In the first one, we have to characterize (when possible) $S_p$ and $\phi_p(S_p)$. The second step consists of lifting paths $\pi : [0,1] \to M$ avoiding $\phi_p(S_p)$ to paths $\Pi : [0,1] \to H$ such that for every $s \in [0,1]$

$$\phi_p(\Pi(s)) = \pi(s). \tag{4.2}$$

Differentiating (4.2) yields to

(4.3) $$D\phi_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s).$$

If $D\phi_p(\Pi(s))$ has full rank, then (4.3) can be solved for $\Pi(s)$ by taking $\Pi$ such that

(4.4) $$\frac{d\Pi}{ds}(s) = P(\Pi(s)) \cdot \frac{d\pi}{ds}(s),$$

where $P(v)$ is a right inverse of $D\phi_p(v)$ when $v \in H/S_p$. (For instance, we can choose $P(v)$ to be the Moore-Penrose pseudo-inverse of $D\phi_p(v)$.)

We are then led to study the Wazewski equation (4.4) called the Path Lifting Equation (PLE) as an ODE in $H$. To successfully apply the CM to the MPP, we have to resolve two issues:

(a) Non degeneracy: the path $\pi$ has to be chosen so that, for every $s \in [0, 1]$, $\pi(s) \notin \phi_p(S_p)$ and then $D\phi_p(\Pi(s))$ has always full rank;

(b) Non explosion: to solve (4.1), the PLE defined in (4.4) must have a global solution on $[0, 1]$.

Local existence and uniqueness of the solution of the PLE hold as soon as $\phi_p$ is of class $C^2$. One can show that the singular points of $\phi_p$ are exactly the controls $u$ that give rise to the abnormal extremals of the sub-Riemannian metric defined by $\Delta$ (cf. [20] for the ad hoc definitions). Resolving (b) amounts to prove some estimates on line integrals along trajectories.

To evaluate $D\phi_p(u)$, for $u \in H$, we first need to define a field of covectors along $\gamma_{p,u}$. For $z \in T^*_{\phi_p(u)}M$, let $\lambda_{z,u} : [0, 1] \to T^*M$ be the field of covectors along $\gamma_{p,u}$ such that it satisfies (in coordinates) the adjoint equation along $\gamma_{p,u}$ with terminal condition $z$, *i.e.*, $\lambda_{z,u}$ is a.c., $\lambda_{z,u}(1) = z$ and for a.e. $t \in [0, 1]$

(4.5) $$\dot{\lambda}_{z,u}(t) = -\lambda_{z,u}(t) \cdot \left( \sum_{i=1}^{2} u_i(t) DF_i(\lambda_{z,u}(t)) \right).$$

If $X$ is a $C^\infty$ vector field over $M$, the switching function $\varphi_{X,z,u}(t)$ associated to $X$ is the evaluation of $\lambda \cdot X(x)$, the Hamiltonian function of $X$ along $(\gamma_{p,u}, \lambda_{z,u})$, *i.e.*, for $t \in [0, 1]$,

$$\varphi_{X,z,u}(t) := \lambda_{z,u}(t) \cdot X(\gamma_{p,u}(t)).$$

Then $D\phi_p(u)$ can be computed thanks to the following formula: for $z \in T^*_{\phi_p(u)}M$ and $u, v \in H$,

(4.6) $$z \cdot D\phi_p(u)(v) = (v, \varphi_{z,u})_H,$$

where the switching function vector $\varphi_{z,u}(t)$ is given by $\varphi_{z,u}(t) := (\varphi_{F_1,z,u}(t), \varphi_{F_2,z,u}(t))^T$ (cf. (2.16) for a definition of the $F_i$'s). Recall that $S_p$ is the set of controls $u$ for which $D\phi_p(u)$ looses rank, *i.e.*, there exists $z \in T^*_{\phi_p(u)}M$, $\|z\| = 1$, such that $\varphi_{z,u} \equiv 0$ on $[0,1]$.

To attack issue (b), we need to relate, for a regular value $u$ of $\phi_p$, $P(u)$ to $\varphi_{z,u}$. This is done through the next equation

$$\|P(u)\| = \left(\inf_{\|z\|=1} z^T D\phi_p(u) D\phi_p(u)^T z\right)^{-1/2} = \left(\inf_{\|z\|=1} \|\varphi_{z,u}\|_H^2\right)^{-1/2}.$$

If one has a linear growth of $\|P(u)\|$ with respect to $\|u\|$, for an appropriate choice of $u$, then one resolves issue (b) by applying Gronwall lemma to the PLE (4.4). To achieve such estimates, the knowledge of the dynamics of $\varphi_{z,u}$ is necessary: if $X$ is a $C^\infty$ vector field over $M$, we have for a.e. $t \in [0,1]$

$$\dot{\varphi}_{X,z,u}(t) = \sum_{i=1}^{2} u_i(t) \varphi_{[F_i,X],z,u}(t). \tag{4.7}$$

To simplify the subsequent notations, we use $\varphi_{i,z,u}$, for $i = 1, \ldots, 5$, to denote the switching functions associated respectively to the vector fields $F_1$, $F_2$, $\varepsilon_5$, $F_1 - \varepsilon_1$, and $F_2 - \varepsilon_2$. Using (2.17), the time derivatives of the $\varphi_i$'s for $i = 1, \ldots, 5$ are given by

$$\dot{\varphi}_1 = -u_2 K \varphi_3, \tag{4.8}$$

$$\dot{\varphi}_2 = u_1 K \varphi_3, \tag{4.9}$$

$$\dot{\varphi}_3 = -u_2 \varphi_4 + u_1 \varphi_5, \tag{4.10}$$

$$\dot{\varphi}_4 = -u_2 K \varphi_3, \tag{4.11}$$

$$\dot{\varphi}_5 = u_1 K \varphi_3. \tag{4.12}$$

The non degeneracy issue (cf. issue (a)) is resolved by the next proposition:

**Proposition 5.** *Let $p \in M$. Then, $S_p = \{(v\cos\theta, v\sin\theta) \mid v \in H, \theta \in S^1\}$ and $\phi_p(S_p)$ is equal to the union of all horizontal geodesics starting at p.* □

*Proof of Proposition 5.* Consider a nonzero singular input $u = (u_1, u_2)$. Using equations (4.8), (4.9), (4.11) and (4.12), there exists $z \in T^*_{\phi_p(u)} T_1 M_2$, $\|z\| = 1$, such that

$$u_2 K \varphi_3 = u_1 K \varphi_3 = \dot{\varphi}_4 = \dot{\varphi}_5 = 0.$$

Since $\dot{\varphi}_3 \varphi_3 = 0$, we get that $\varphi_3$ is constant. Since $K > 0$ and $u \neq 0$, we deduce that $\varphi_3 \equiv 0$. Moreover $\varphi_i \equiv \varphi_i(1)$ for $i = 4, 5$. Since $z \neq 0$, then $\varphi_4(1)$ or $\varphi_5(1)$ is not equal to zero. By (4.10), we get that for a.e. $t \in [0,1]$

$$u_1(t)\sin\theta - u_2(t)\cos\theta = 0,$$

where $\cos\theta = \frac{\varphi_4(1)}{\sqrt{\varphi_4(1)^2+\varphi_5(1)^2}}$ and $\sin\theta = \frac{\varphi_5(1)}{\sqrt{\varphi_4(1)^2+\varphi_5(1)^2}}$.

We can also rewrite equation (4.10) as

$$-\dot{x}_2\cos\theta + \dot{x}_1\sin\theta = 0,$$

which implies that the projection of $\gamma_{p,u}$ on $\mathbb{R}^2$ is a line. By Proposition 1, we conclude that $\gamma_{p,u}$ is an horizontal geodesic. □

The next proposition summarizes the fact that if there is a certain linear growth of the norm of the Moore-Penrose pseudo-inverse $P(u)$ with respect to $\|u\|$, then the CM applies successfully:

**Proposition 6.** *Let $K$ be a closed subset of $M$ such that*

(i) *$K$ is disjoint from $\overline{\phi_p(S_p)}$, where $\overline{\phi_p(S_p)}$ is the closure of $\phi_p(S_p)$;*

(ii) *there exists $c_K > 0$ such that for every $u \in H$ with $\phi_p(u) \in K$ and $z \in T^*_{\phi_p(u)}M$, $\|z\| = 1$, we have*

$$(4.13)\qquad \|u\|_H\|\varphi_{z,u}\|_H \geq c_K.$$

*Then for every path $\pi : [0,1] \to K$ of class $C^1$ and every control $\bar{u} \in H$ such that $\phi_p(\bar{u}) = \pi(0)$ the solution of the PLE defined in* (4.4) *with initial condition $\bar{u}$ exists globally on the interval* $[0,1]$. □

In order to resolve the motion planning problem, an appropriate application of Proposition 6 is required: we must choose the point $p$ to define $\phi_p$, determine a "large" closed set $K$ subject to (i) and (ii) and finally, lift enough paths $\pi : [0,1] \to K$ to conclude.

To obtain $K$ as "large" as possible, we need a "small" singular set $\phi_p(S_p)$. The condition (C) mentioned in the introduction and defined next serves for that purpose. It says that $M_2$ admits a periodic geodesic $\gamma$ which is stable for the geodesic flow. Let $d_2$ be the distance function associated to the Riemannian metric of $M_2$. The curve $\gamma : \mathbb{R}^+ \to T_1M_2$ is a geodesic of $T_1M_2$ and there exists $L \geq \frac{2\pi}{\sqrt{K_{\max}}}$ such that $\gamma(t+L) = \gamma(t)$ for all $t \geq 0$ (cf. [18]). Then we use $G$ to denote the closed subset of $T_1M_2$, $\gamma([0,L])$. For $\rho > 0$, let $N_\rho(G)$ be the open set of points $y \in T_1M_2$ such $d_2(y,G) < \rho$. Let $\phi(y,t)$ be the geodesic flow of $T_1M_2$. Condition (C) is now given by

$$(4.14)\quad (C)\qquad \begin{aligned} &\exists\rho_0 > 0,\quad \forall\rho < \rho_0,\quad \exists\eta > 0,\quad \forall y_0 \text{ such that } y_0 \in N_\rho(G),\\ &\forall t \geq 0,\quad \phi(y_0,t) \in N_\eta(G). \end{aligned}$$

We assume that $\rho_0$ and $\frac{\rho_0}{K_{\min}}$ are small enough in order for $N_{\rho_0}(G)$ to be included in the domain of a chart of geodesic coordinates with basis $\gamma$. In particular we choose $\psi$ to be equal to zero along $\gamma$.

**Remark 8.** Condition (C) holds for any convex compact surface having a symmetry of revolution. Indeed, let $r : M_2 \to \mathbb{R}^+$ be the distance function to the axis of revolution. The level set of $r$ corresponding to its maximum value is a closed geodesic which satisfies condition (C), thanks to Clairault's relation (cf. [18]). Moreover, the above condition is generic within the convex compact surfaces verifying $\frac{K_{\min}}{K_{\max}} > \frac{1}{4}$ and is suspected to be generic within all the convex compact surfaces, cf. [18] for more results. □

**Remark 9.** The periodic stable geodesic $\gamma$ defined above can be replaced by any "geodesically stable" closed set.

## 4.2 Planning strategy

We now describe how to apply the CM to solve the motion planning problem for $\mathscr{S}_R$. We assume that $M$ satisfies condition (C) defined previously. The control system $\mathscr{S}_R$ can now be written (cf. section 2.4) as follows:

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \qquad (4.15) \\ (P_2\mathscr{S}_R)\, \dot{y} &= u_1 f(y) + u_2 h(y), \end{aligned}$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in T_1 M_2$.

For $\rho \in (0, \rho_0)$, define $K_\rho$ to be the complement in $M$ of $Sg \times N_\rho(G)$, where $Sg$ is the open line segment of $\mathbb{R}^2$ between the points $(-1, 0)$ and $(1, 0)$. Since $\gamma$ is periodic, $N_\rho(G)$ is diffeomorphic to the product of a small two-dimensional ball and a closed path in $T_1 M_2$. Therefore $C_\rho$ is closed and arcwise-connected. For $(x, y) \in M$ such that $y \in G$, there exists a unique line $\mathscr{L}_{x,y}$ in $\mathbb{R}^2$ such that $\mathscr{L}_{x,y} \times \gamma$ is the horizontal geodesic going through $(x, y)$, cf. Proposition 1.

Let $(x^0, y^0)$ and $(x^1, y^1)$ two points of $M$. Since the contact distribution $(f, h)$ satisfies the Strong Bracket Generating Condition (SBGC), the CM solves the MPP for $(P_2\mathscr{S}_R)$ (cf. [8] and [27]). Therefore, we may assume that $y^0$ belongs to $G$. By taking an appropriate orthonormal basis of $\mathbb{R}^2$, we may assume that $x^0 = 0$ and $\mathscr{L} = \mathscr{L}_{0,y^0}$ is the first coordinate axis. We choose the point $p$ used in Proposition 6 to be $(0, y^0)$ and $\phi$ denotes $\phi_p$. Moreover, we can assume that $x^1 = (2, 0)$ and $y_1 \notin N_\rho(G)$. If it is not the case (*i.e.*, if $y_1 \in N_\rho(G)$) we displace $y_1$ using, several times if necessary, the input $u^\varepsilon : [0, 1] \to \mathbb{R}^2$, $0 < \varepsilon \ll 1$, defined as follows

$$u^{\varepsilon}(t) = \begin{cases} (0,\varepsilon) & \text{if } 0 \le t \le \frac{1}{4}, \\ (\varepsilon,0) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ (0,-\varepsilon) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \\ (-\varepsilon,0) & \text{if } \frac{3}{4} \le t \le 1. \end{cases}$$

Finally we conclude thanks to Proposition 6 and the following lemma, whose proof is deferred in the appendix:

**Lemma 2.** *With the previous notations, there exists* $\bar{\rho} \in (0, \rho_0)$ *such that for every* $\rho \in (0, \bar{\rho})$, $K_\rho$ *satisfies the hypothesis* (i) *and* (ii) *of Proposition 6.*

Then for every path $\pi : [0,1] \to K_\rho$ of class $C^1$ and every control $\bar{u} \in H$ such that $\phi(\bar{u}) = \pi(0)$ the solution of the PLE defined in (4.4) with initial condition $\bar{u}$ exists globally on $[0,1]$.

## Appendix: Proof of Lemma 2

We are given $\rho \in \left(0, \frac{\rho_0}{4}\right)$, an input $u \in H$ such that $\gamma_{(0,y_0),u}$ steers $(0, y^0)$ to $\phi(u) = ((a,0), y)$ with $|a| \ge 1$ and $y \notin N_{4\rho}(G)$ and $z \in T^*_{\phi(u)} T_1 M_2$ such that $\|z\| = 1$. The switching functions $\varphi_i$, $i = 1, \dots, 5$, satisfy equations (4.8)–(4.12). If $k$ is a function defined on a time interval $I$ bounded by $t$ and $t'$, we use $[k]_t^{t'}$ (or $[k]_I$ if $t \le t'$) to denote $k(t') - k(t)$. The main point in the proof is of course to find $\rho > 0$ small enough such that (4.13) holds for $K_\rho$. For any $K_\rho$, part (i) of Proposition 6 is verified. Then, to get part (ii) of Proposition 6, one tries to find some function $\Psi : T^*M \to \mathbb{R}$ whose evaluation along $(\gamma_{(0,y_0),u}, \lambda_{z,u})$ (again denoted by $\Psi$) verifies:

(a) there exists $[t_0, t_1] \in [0,1]$ such that $|[\Psi]_{t_0}^{t_1}| \ge C_0(\rho)$;

(b) for a.e. $t \in [t_0, t_1]$, $\dot{\Psi}(t) = u_1(t) H_1(t) + u_2(t) H_2(t)$ where the $H_i$'s are bounded functions by some $C_1(\rho)$ (the $C_i(\rho)$'s are constants depending on $\rho$ and $M_2$);

(c) we can majorize the $|H_i|$'s by the $\varphi_1$, $\varphi_2$ and some their time derivatives.

Then, by integrating by parts enough time, one tries to end up with an inequality of the type

$$C(\rho) \le |[\Psi]_{t_0}^{t_1}| \le C'(\rho) \sum_{i=1}^{2} \int_{t_0}^{t_1} |u_i(t)| \cdot \|\varphi(t)\| \, dt.$$

Applying Cauchy-Schwarz to the right-hand side of the previous inequality leads to (4.13). The proof of Lemma 2 requires the consideration of several cases and, for each case, an adapted function $\Psi$ and an adapted time interval $[t_0, t_1]$ have to be determined.

Integrating (4.8)–(4.12), we obtain

**Lemma 3.** *For every* $t, t' \in [0,1]$, *we have*

$$(4.16)\quad [\varphi_5]_t^{t'} = [\varphi_2]_t^{t'}, \quad [\varphi_4]_t^{t'} = [\varphi_1]_t^{t'}$$

*and*

$$(4.17)\quad [\varphi_3]_t^{t'} = -(\varphi_4(1) - \varphi_1(1))[x_2]_t^{t'} + (\varphi_5(1) - \varphi_2(1))[x_1]_t^{t'} - \int_t^{t'} u_2\varphi_1 + \int_t^{t'} u_1\varphi_2. \qquad \square$$

*Proof of Lemma 3.* Equation (4.16) is trivial to obtain. By using it, (4.17) is obtained from (4.10) as

$$\dot\varphi_3(t) = -u_2(t)\varphi_4(t) + u_1(t)\varphi_5(t) = -\dot x_2(t)(\varphi_1(t) + \varphi_4(1) - \varphi_1(1)) + \dot x_1(t)(\varphi_2(t) + \varphi_5(1) - \varphi_2(1)). \qquad \square$$

Define $\Phi^0 : M \to \mathbb{R}$ by $\Phi^0(x,y,z) = d_2(y,G)^2$ and $\Phi^1 : M \to \mathbb{R}$ by $\Phi^1(x,y,z) = \Phi^0(x,y,z) + \|z\|^2$. We simply use $\Phi_z^0$ and $\Phi_z^1$ from $[0,1]$ to $\mathbb{R}$ to denote the evaluation of $\Phi^0$ and $\Phi^1$ along $(\gamma_{(0,y_0),u}, \lambda_{z,u})$. For $i = 1,2$, the time derivative of $\Phi_z^i$ can be written $u_1(t)G_1^i(t) + u_2(t)G_2^i(t)$ for a.e. $t \in [0,1]$.

Let $t_e \in (0,1)$ the smallest time such that $d(\gamma_{(0,y_0),u}(t), G) = 2\rho$. Then $t_e > 0$ and $\gamma_{(0,y_0)(t)} \in N_{2\rho}(G)$ for $t \le t_e$. We face the following alternatives

**case 1):** there exists $\bar t \in [t_e, 1]$ such that for some $i = 1,2$, $|\varphi_i(\bar t)| \ge C_0\rho^2$;

**case 2):** for every $t \in [t_e, 1]$ and $i = 1,2$, $|\varphi_i(t)| < C_0\rho^2$.

- **case 2-1):** there exists $\bar t \in [t_e, 1]$ such that $|\varphi_3(\bar t)| \ge C_1\rho^2$;
- **case 2-2):** for every $t \in [t_e, 1]$ $|\varphi_3(t)| < C_1\rho^2$,

where $C_0, C_1$ are constants independent on $\rho$ and determined later. We first treat case 1) and we assume it holds for $i = 1$. Consider an interval $[t_*, t^*]$ of $[t_e, 1]$ where $\varphi_1(t) \ge C_0\rho^2/2$ and containing some $\bar t'$ such that $|\varphi_1(\bar t')| \ge C_0\rho^2$. Define now $t_0$, $t_1$ and $\Psi$ as follows: if $\varphi_1(t) > C_0\rho^2/2$ for $t \in [t_e, 1]$, take $t_0 = t_e$, $t_1 = 1$ and $\Psi = \Phi^0$; otherwise take $t_0$ such that $\varphi_1(t_0) = C_0\rho^2/2$, $t_1 = \bar t'$, $C_0\rho^2/2 \le \varphi_1(t) \le C_0\rho^2$ for $t$ between $t_0$ and $\bar t'$ and finally $\Psi = \varphi_1$. In both cases, $|[\Psi]_{t_0}^{t_1}| \ge 3C_0\rho^2/4$ and for a.e. $t \in [t_0, t_1]$, $\dot\Psi(t) = u_1(t)H_1(t) + u_2(t)H_2(t)$ where the $H_i$'s are bounded functions by some $C_0 > 0$ only depending on $M_2$ over $[t_0, t_1]$. Therefore for a.e. $t \in [t_0, t_1]$ we have

$$\dot\Psi(t) = u_1(t)\varphi_1(t)G_1(t) + u_2(t)\varphi_1(t)G_2(t),$$

with $G_i(t) = H_i(t)/\varphi_1(t)$, $i = 1,2$. The $G_i$'s are bounded over $[t_0, t_1]$ by some $C(\rho) > 0$ and after integrating between $t_0$ and $t_1$ and applying Cauchy-Schwarz inequality, we get

$$\frac{3}{4}C_0\rho^2 \leq |[\Psi]_{t_0}^{t_1}| = \left|\int_{t_0}^{t_1} u_1(t)\varphi_1(t)G_1(t) + u_2(t)\varphi_1(t)G_2(t)\right|$$

$$\leq C(\rho)\int_{t_0}^{t_1} |u_1(t)+u_2(t)|\,|\varphi_1(t)| \leq C'(\rho)\|u\|_{[t_0,t_1]}\|\varphi_1\|_{[t_0,t_1]},$$

which leads to (4.13).

We now consider case 2-1). Proceeding exactly as in case 1) with $\varphi_1$ replaced by $\varphi_3$, we get that there exists $\Psi : [t_0, t_1]$ such that $|[\Psi]_{t_0}^{t_1}| \geq \frac{3}{4}C_1\rho^2$ and functions $G_i$'s bounded over $[t_0, t_1]$ by some $C(\rho) > 0$ such that for a.e. $t \in [t_0, t_1]$ we have

$$\dot{\Psi}(t) = u_1(t)\varphi_3(t)G_1(t) + u_2(t)\varphi_3(t)G_2(t).$$

By using (4.8) and (4.9), we rewrite the previous equation as

$$\dot{\Psi}(t) = \dot{\varphi}_1(t)G_1'(t) + \dot{\varphi}_2(t)G_2'(t),$$

where the $G_i'$'s satisfy the same hypothesis as the $G_i$'s. Integrating by part we get

$$[\Psi]_{t_0}^{t_1} = [\varphi_1 G_1' + \varphi_2 G_2']_{t_0}^{t_1} + \sum_{i,j=1}^{2}\int_{t_0}^{t_1} u_i(t)\varphi_j(t)G_{ij}''(t),$$

where the $G_{ij}''$'s satisfy the same hypothesis as the $G_i$'s. Indeed this follows from the fact that $|\varphi_4|$ and $|\varphi_5|$ remain bounded by 2 thanks to (4.16). Once $C_1$ is chosen, we determine $C_0$ in such a way that

$$|[\Psi - (\varphi_1 G_1' + \varphi_2 G_2')]_{t_0}^{t_1}| \geq \frac{1}{2}C_1\rho^2.$$

We immediately get that for some $i$ and $j \in \{1, 2\}$

$$C'(\rho) \leq \|u_i\|_{[t_0,t_1]}\|\varphi_j\|_{[t_0,t_1]},$$

and then (4.13) follows.

We finally consider the last case 2-2). Define $z' = \lambda_{z,u}(t_e)$. We have $\|z - z'\| \leq C_1\rho^2$ because of (4.16). We next assume that for all $t \in [0, t_e]$ and $i = 1, 2$, $|\varphi_i(t)| < C_0\rho^2$ and $|\varphi_3(t)| < C_1\rho^2$. Otherwise, we are back to either case 1) or case 2-1) with $[0, t_e]$ instead of $[t_e, 1]$. For simplicity, we assume that $t_e = 1$ and then $\gamma_{(0,y_0)(t)} \in N_{2\rho}(G)$ for $t \in [0, 1)$ and $d(\phi(u), G) = 2\rho$. Then, the estimate (4.13) will be a consequence of the following lemma

**Lemma 4.** *Let an input $u \in H$ such that $\gamma_{(0,y_0),u}$ steers $(0, y^0)$ to $\phi(u) = ((a, 0), y)$ with $|a| \geq 1$ and $z \in T^*_{\phi(u)}T_1M_2$ such that $\|z\| = 1$. Moreover we assume that*

(1) *for every* $t \in [0,1]$, $\gamma_{(0,y_0)}(t) \in N_{2\rho}(G)$, $|\varphi_i(t)| < C_0\rho^2$ *for* $i = 1,2$ *and* $|\varphi_3(t)| < C_1\rho^2$;

(2) *for every* $t < t' \in [0,1]$, *we suppose that*

$$(4.18) \quad \|u\|_{[t,t']}\|\varphi\|_{[t,t']} < C_2\rho^2,$$

*where* $C_2$ *is a positive constant depending on* $C_0$ *and* $C_1$. *Then for every* $t \in [0,1)$, $\gamma_{(0,y_0)}(t) \in N_{3\sqrt{C_1}\rho}(G)$.

*Proof of Lemma 4.* Using (4.17) applied to $t = 0$ and $t' = 1$, we get that $|\varphi_5(1)| < 2C_1\rho^2$ and from (4.16) we obtain that for every $t \in [0,1]$, $|\varphi_5(t)| < 3C_1\rho^2$. Next, we make a change of variables and reparameterize the trajectory $(\gamma_{(0,y_0),u}, \lambda_{z,u})$ in the basis $B_\alpha$ defined in (2.18). For $\alpha \in S^1$, we introduce the input $u^\alpha = R_\alpha u$, the vector fields of $T_1M_2$ $f_\alpha = \cos(\alpha)f - \sin(\alpha)h$ and $h_\alpha = \sin(\alpha)f + \cos(\alpha)h$, the switching vector $\varphi^\alpha = R_\alpha\varphi$ and finally the switching functions $(\varphi_4^\alpha, \varphi_5^\alpha)^T = R_\alpha(\varphi_4, \varphi_5)^T$. We get that $\gamma_{(0,y_0),u}$ is the trajectory of $\mathscr{S}_R$ corresponding to $u^\alpha$ where $\mathscr{S}_R$ is now defined by

$$(4.19) \quad \dot{X} = v_1F_1^\alpha(X) + v_2F_2^\alpha(X).$$

In addition the equations (4.8)–(4.12) are transformed to

$$(4.20) \quad \dot{\varphi}_1^\alpha = -u_2^\alpha K\varphi_3,$$

$$(4.21) \quad \dot{\varphi}_2^\alpha = u_1^\alpha K\varphi_3,$$

$$(4.22) \quad \dot{\varphi}_3 = -u_2^\alpha\varphi_4^\alpha + u_1^\alpha\varphi_5^\alpha,$$

$$(4.23) \quad \dot{\varphi}_4^\alpha = -u_2^\alpha K\varphi_3,$$

$$(4.24) \quad \dot{\varphi}_5^\alpha = u_1^\alpha K\varphi_3.$$

Notice that for every $t \in [0,1]$, $\|\varphi^\alpha(t)\| = \|\varphi(t)\|$ and $\|u^\alpha(t)\| = \|u(t)\|$ and then for every $t, t' \in [0,1]$, $\|\varphi^\alpha\|_{[t,t']} = \|\varphi\|_{[t,t']}$ and $\|u^\alpha\|_{[t,t']} = \|u\|_{[t,t']}$. Therefore all the hypothesis of the lemma apply to $\varphi^\alpha$, $u^\alpha$. At $t = 1$, we have

$$\varphi_4^\alpha(1) = \cos(\alpha)\varphi_4(1) - \sin(\alpha)\varphi_5(1) \quad \text{and} \quad \varphi_5^\alpha(1) = \sin(\alpha)\varphi_4(1) + \cos(\alpha)\varphi_5(1),$$

and we deduce from the previous equation that

$$(4.25) \quad \varphi_5^\alpha(1) - \varphi_2^\alpha(1) = \sin(\alpha)(\varphi_4(1) - \varphi_1(1)) + \cos(\alpha)(\varphi_5(1) - \varphi_2(1)).$$

Since $\|z\| = 1$, $1 - 2C_1\rho^2 \le |\varphi_4(1) - \varphi_1(1)| \le 1 + 2C_1\rho^2$. From (4.25), we choose $\alpha$ such that $\varphi_5^\alpha(1) - \varphi_2^\alpha(1) = 0$, *i.e.*,

$$(4.26) \quad \alpha = \arctan\left(\frac{\varphi_5(1) - \varphi_2(1)}{\varphi_1(1) - \varphi_4(1)}\right),$$

which implies that $|\alpha| \le 3C_1\rho^2$. Integrating (4.9) and (4.12) together with (4.26) imply that the switching functions $\varphi_2$ and $\varphi_5$ are equal on $[0,1]$. Moreover, as a consequence of the hypothesis (i) of Lemma (4) and equation (4.16), we have for every $t \in [0,1]$ that $1 - 3C_1\rho^2 \le |\varphi_4(t)| \le 1 + 3C_1\rho^2$. We are then allowed to write that for every $t \in [0,1]$

$$(4.27) \quad u_2^\alpha(t) = \frac{u_1^\alpha \varphi_2^\alpha - \dot{\varphi}_3}{\varphi_4^\alpha}.$$

Adopting the notations of the chronological calculus, cf. [1], (vector fields and diffeomorphisms act on the right), the projections on $T_1M_2$ of trajectories of $\mathscr{S}_R$ can be written as

$$(4.28) \quad y = w\exp(U_1^\alpha(t)f_\alpha),$$

with $U_1^\alpha(t) = \int_0^t u_1^\alpha$. From (4.19), we get the dynamics of $w$:

$$\dot{w} = u_2^\alpha w\exp(U_1^\alpha(t)f_\alpha)h_\alpha\exp(-U_1^\alpha(t)f_\alpha) = u_2^\alpha wh_\alpha\exp(U_1^\alpha(t)\,\mathrm{ad}\,f_\alpha).$$

Using (4.27), the dynamics of the projections on $T_1M_2$ of $\gamma_{(0,y_0),u^\alpha}$ becomes

$$(4.29) \quad \dot{w} = (u_1^\alpha\varphi_2^\alpha - \dot{\varphi}_3)wH^\alpha(t),$$

where $H^\alpha(t)$ is the time-varying vector field of $T_1M_2$ given by

$$H^\alpha(t) = \frac{h_\alpha\exp(U_1^\alpha(t)\,\mathrm{ad}\,f_\alpha)}{\varphi_4^\alpha}.$$

Assume first that $H^\alpha(t)$ is bounded over $[0,1]$ by some $C_H \gg 1$, independent of $u^\alpha$ and $\rho$. Let $V(w) = d_2^2(w,G)$ and $V$ its evaluation along the trajectory given by (4.29). The time derivative of $V$ can be written

$$\dot{V} = (u_1^\alpha\varphi_2^\alpha - \dot{\varphi}_3)W(t),$$

where $W(t)$ is regular enough for all our computations and bounded over $[0,1]$ independently of $u^\alpha$ and $\rho$. Moreover the time derivative of $W$ can be written $u_1^\alpha(t)W_1(t) + u_2^\alpha(t)W_2(t)$ where the $W_i$'s are again bounded over $[0,1]$ independently of $u^\alpha$ and $\rho$. Integrating by part twice the last equation between any $t_0 < t_1$ in $[0,1]$ leads to

$$(4.30) \quad [V]_{t_0}^{t_1} = \int_{t_0}^{t_1} u_1^\alpha(t)\varphi_2^\alpha(t)W(t) - \left[\varphi_3 W + \frac{-\varphi_2^\alpha W_1 + \varphi_1^\alpha W_2}{K}\right]_{t_0}^{t_1}$$

$$+ \sum_{i,j=1}^{2}\int_{t_0}^{t_1} u_i^\alpha(t)\varphi_j^\alpha(t)W_{ij}'(t),$$

where the $W'_{ij}$ are bounded over $[0,1]$ independently of $u^\alpha$ and $\rho$. By using Cauchy-Schwarz and (4.18), we deduce that for every $t \in [0,1]$, $d_2(w(t), G) \leq 2\sqrt{C_1}\rho$. By using the geodesic coordinates in some fixed neighborhood $N_\mu(G)$, we can see that the integral curves of $f_\alpha$ are integral curves of $f$ modulo a change of coordinates for the $\psi$-component which associates $\psi$ to $\psi + \alpha$. Recall that $|\alpha| \leq 3C_1\rho^2$ and since $M_2$ satisfies condition (C), we just have to adjust $C_1$ small enough but independently of $u^\alpha$ and $\rho$ to conclude.

It remains to treat the case where $H^\alpha(t)$ takes values larger than $C_H \gg 1$ on $[0,1]$. Recall that $\|H^\alpha(1)\| = 1/|\varphi_4^\alpha(1)|$. Then there exists $t_0 < 1$ so that $\|H^\alpha(t)\| \leq C_H$ on $[t_0, 1]$ and $\|H^\alpha(t_0)\| = C_H$. Replacing $V$ in (4.30) by $\|H^\alpha\|^2$ and integrating between $t_0$ and 1, we easily contradict (4.18). □

## References


[1] Agrachev, A. A. and Gamkrelidze, R. V.: The exponential representation of flows and chronological calculus. Matem. Sbornik **109** (1978), 467–532

[2] Agrachev, A. A. and Sachkov, Y. L.: An intrinsic approach to the control of rolling bodies. In: IEEE Proc. Conf. Dec. Contr. volume 1, pp. 431–435. Phoenix, 1999

[3] Allgower, E. L. and Georg, K.: Continuation and path following. Acta Numerica pp. 1–64, 1992

[4] Bryant, R. L., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., and Griffiths, P. A.: Exterior Differential Systems. Springer Verlag, 1991

[5] Bryant, R. L. and Hsu, L. S.: Rigidity of integral curves of rank-2 distributions. Invent. Math. **114** (1993), 435–461

[6] Chelouah, A.: Equivalence and liouvillian systems; submitted

[7] Chelouah, A.: Extension of differential flat field and liouvillian systems. In: Proc. IEEE Conf. Dec. Contr. volume 5, pp. 4268–4273. San Diego, 1997

[8] Chitour, Y.: Applied and Theoretical Aspects of the Controllability of Nonholonomic Systems. PhD thesis, Rutgers University, 1996

[9] Fliess, M., Lévine, J., Martin, P., and Rouchon, P.: Linéarisation par bouclage dynamique et transformation de Lie-Bäcklund. C. R. Acad. Sci. Paris **I-317** (1993), 981–986

[10] Fliess, M., Lévine, J., Martin, P., and Rouchon, P.: Flatness and defect of nonlinear systems: introductory theory and examples. Internat. J. Control **61** (1995), 1327–1361

[11] Fliess, M., Lévine, J., Martin, P., and Rouchon, R.: Deux applications de la géométrie locales des diffiétés. Ann. Inst. Henri Poincaré **66** (1997), 275–292

[12] Fliess, M., Lévine, J., Martin, P., and Rouchon, P.: Nonlinear control and diffieties, with an application to physics. Contemp. Math. **219** (1998), 81–92

[13] Fliess, M., Lévine, J., Martin, P., and Rouchon, P.: A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. IEEE Trans. Automat. Control **44** (1999), 922–937

[14] Gardner, R.: Invariants of pfaffian systems. Trans. Amer. Math. Soc. **126** (1967), 514–533

[15] Jurdjevic, V.: The geometry of the plate-ball problem. Arch. Rational Mech. Anal. **124** (1993), 305–328

[16] Jurdjevic, V.: Geometric control theory. Cambridge Stud. Adv. Math. Camb. Univ. Press, 1997

[17] Kiss, B., Lévine, J., and Lantos, B.: On motion planning for robotic manipulation with permanent rolling contacts. to appear in International Journal of Robotics Research, 2002

[18] Klingenberg, W.: Riemannian Geometry. De Gruyter, 1982

[19] Li, Z. and Canny, J.: Motion of two rigid bodies with rolling constraint. IEEE Trans. Automat. Control **6** (1990), 62–72

[20] Liu, W. and Sussmann, H. J.: Shortest paths for sub-riemannian metrics on ran-two distributions. Mem. Amer. Math. Soc. 118. Providence, RI 1995

[21] Marigo, A. and Bicchi, A.: Rolling bodies with regular surfaces: controllability theory and applications. IEEE Trans. Automat. Control **45** (2000), 1586–1599

[22] Martin, P. and Rouchon, P.: Feedback linearization and driftless systems. Mathematics of Control, Signals and Systems **7** (1994), 235–254

[23] Murray, R. M.: Nilpotent bases for a class of non-integrable distributions with applications to trajectory generation of nonholonomic systems. Mathematics of Control, Signals and System **7** (1994), 58–75

[24] Murray, R. M., Li, Z., and Sastry, S. S.: Robotics Manipulation. CRC Press, 1994

[25] Sakai, T.: Riemannian Geometry. Transl. Math. Monographs 149. Amer. Math. Soc., 1996.

[26] Steenrod, N.: The Topology of Fibre Bundles. Princeton University Press, 1951

[27] Sussmann, H. J.: A continuation method for nonholonomic path-finding problems. In: Proc. IEEE Conf. Dec. Contr. pp. 1345–1348. San Antonio, 1992

[28] Sussmann, H. J.: New differential geometric methods in nonholonomic path finding. In: (A. Isidori and T. J. Tarn, editors) Systems, Models and Feedback. Birkhäuser, 1992

[29] Wazewski, T.: Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes. Ann. Soc. Pol. Math., 20, 1947.

[30] Zharinov, V. V.: Geometrical aspect of partial differential equations. World Scientific, Singapour 1992

Received January 20, 2001; in final form July 5, 2002

A. Chelouah, Laboratoire des Signaux et Systèmes, CNRS-Supelec-Université de Paris-Sud, 3 rue Joliot Curie, 91192 Gif sur Yvette Cedex—France
chelouah@lss.supelec.fr.

Y. Chitour, Département de Mathématiques, Université de Paris-Sud, F-91405 Orsay—France
Yacine.Chitour@math.u-psud.fr.