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ON THE MINIMUM TIME PROBLEM FOR DRIFTLESS LEFT-INVARIANT CONTROL SYSTEMS ON $SO(3)$

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Abstract. In this paper, we investigate the structure of time-optimal trajectories for a driftless control system on $SO(3)$ of the type $\dot{x} = x(u_1 f_1 + u_2 f_2)$, $|u_1|, |u_2| \leq 1$, where $f_1, f_2 \in so(3)$ define two linearly independent left-invariant vector fields on $SO(3)$. We show that every time-optimal trajectory is a finite concatenation of at most five (bang or singular) arcs. More precisely, a time-optimal trajectory is, on the one hand, bang-bang with at most either two consecutive switchings relative to the same input or three switchings alternating between two inputs, or, on the other hand, a concatenation of at most two bangs followed by a singular arc and then two other bangs. We end up finding a finite number of three-parameters trajectory types that are sufficient for time-optimality.

1. **Introduction and Statement of the Main Results.** We consider the time-optimal problem of the control system Σ given by:

$$\dot{x} = x(u_1 f_1 + u_2 f_2), \tag{1}$$

where $x \in SO(3)$, $|u_i| \leq 1$ and $f_i \in so(3)$ give rise to left invariant vector fields (acting on the right) on $SO(3)$. The control system Σ can be seen as a model for reorienting a satellite: the vector fields f_i represent infinitesimal rotations of the satellite about two non collinear fixed axis and the inputs u_i are scale multiples of the angular velocities of the rotors rotations. The problem consists of orienting the satellite from a given position to another one in minimum time.

The control system Σ is given as $\Big(SO(3), [-1,1]^2, Ad(\Sigma), (f_1, f_2)\Big)$ where $SO(3)$ is the state space, $[-1,1]^2$ is the control set, $Ad(\Sigma)$ is the control space (defined next) and f_1, f_2 are 3×3 skew-symmetric matrices, defining the dynamics of Σ by (1). The control space $Ad(\Sigma)$ is the set of admissible controls $u = (u_1, u_2)$ i.e. the set of measurable functions $u = (u_1, u_2) : [a, b] \to [-1,1]^2$, where a, b depend on u in general (see Jurdjevic [1] for a more general definition of control systems). A trajectory γ of Σ is an absolutely continuous curve $\gamma : J \to SO(3)$ with $J = [a, b]$

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