Manipulation of Polyhedral Parts by Rolling*

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Abstract
The nonholonomy exhibited by kinematic systems consisting of bodies rolling on top of each other can be used to the purpose of building dexterous mechanisms with a minimum hardware complication. Previous work concentrated on manipulation of objects possessing a regular surface. On the other hand, industrial parts are most often irregular, possessing vertices and edges. In this paper we present some results on the description of the set of positions and orientations that polyhedral objects can reach when manipulated by rolling without slipping. An algorithm for planning the manipulation of a polyhedral part from a given configuration to another reachable one, is also presented.

1 Introduction
Manipulation of industrial parts has been one of the core problems of robotics since its beginnings, and it still attracts large attention. Solutions have been proposed that vary in philosophy according to the different application domain. Thus multifingered hands apply where flexibility is at a premium, while more factory-oriented solutions privilege simplicity of the manipulator and use regrasping ([15], [8]) and/or pushing and tilting actions ([13], [7]) in conjunction with such simple end-effectors as parallel-jaw grippers. In this paper, we focus on tasks requiring much flexibility, but where the hardware complexity of the end-effector is to be minimized, in the interest of weight, unreliability, and cost reduction.

The nonholonomic behaviour of some systems has been exploited for achieving dexterous manipulation by means of simple mechanical design. Montana[11], and Li and Canny[9] used tools from differential geometry and nonlinear control to model manipulation by rolling and discuss its geometry. Bicchi and Sorrentino [5] designed and implemented a dexterous hand exploiting rolling, which used only three motors. Such hand is able to arbitrarily change the position and orientation of the manipulated object, provided that its surface complies with some assumptions (see [6]), including regularity and convexity. In order to approach genuine industrial problems, in this paper we consider a similar style of manipulation as applied to polyhedral parts.

The rolling of a polyhedron on a plane is itself a nonholonomic phenomenon, although a wider defin-
2 Problem Formulation

Consider a simple manipulator as the one in fig. 2, consisting of two plates one of which is fixed, while the other can translate remaining parallel to the first. A part is put between the plates, which are covered by compliant high-friction pads. By coordinated motion of the jaws, the object can be made to roll from a face to another adjacent one through the connecting edge. The goal of manipulation is to bring the part from a given initial configuration (a point in \( SE(3) \)) to another desired one. Without loss of generality, we only consider here different configurations modulo a rigid vertical translation of the whole mechanism (i.e., we restrict to \( \mathbb{R}^2 \times SO(3) \)).

Manipulated parts are considered that have a piece-wise flat, closed surface, comprised of a finite number of faces, edges, and vertices. Observe that actual parts need not be convex, in general. However, the finger plates being assumed to be large w.r.t. the diameter of parts, we only need to be concerned with the convex hull of parts themselves.

Several kinds of motions for a polyhedron on a plane are possible, as e.g. by sliding on a face, pivoting about a vertex or tumbling about an edge. In this paper, however, we rule out the former two possibilities, and only consider sequences of rotations about one of the edges in contact, by the amount that exactly brings another face to ground. This action on the parts, which will be referred to as an elementary tumble (ET for short), appears to be more reliable than slipping or pivoting, as it will be discussed later on.

Let \( P \) be a convex polyhedral rolling on a plane \( P \), and let \( V = \{ V_1, \ldots, V_n \} \) be the set of vertices, \( E = \{ E_1, \ldots, E_b \} \) the set of edges, and \( F = \{ F_1, \ldots, F_l \} \) the set of faces of \( P \). The configuration space \( \hat{M} \) of the system is given by the set of points of type \(( p, \theta, i ) \) where \( i \) is the index of the face in contact with the plane \( P \), \( p \) is the projection onto \( P \) of some point \( c \) fixed on the part (e.g., its center of gravity), and \( \theta \) is the angle between two reference systems fixed respectively on face \( F_i \) and on \( P \). Briefly, \( \hat{M} \) is the union of \( l \) copies of \( SE(2) \), i.e.

\[
\hat{M} = \mathbb{R}^2 \times S^1 \times \hat{F}.
\]  

In this terms, the problem of deciding whether the polyhedron \( P \) can be dextrously manipulated is solved by studying the subset of reachable configurations \( \mathcal{R}_1 \subset \hat{M} \), given by all configurations \(( p_f, \theta_f, i_f ) \) such that there exists some sequence of ET's bringing \( P \) from a given initial configuration \(( p_0, \theta_0, i_0 ) \) to \(( p_f, \theta_f, i_f ) \). Such sequence of ET's will be referred to as a "walk" and will be described by the sequence of faces brought successively in contact with \( P \), \( \{ F_{S_n} \} \), where \( \{ S_n \}_{n \in \mathbb{N}}, I \subset \mathbb{N}, S_n \in \{ 1, \ldots, l \} \) is a sequence of face indices. Thus \( \{ F_{S_n} \} \) represents a walk if \( F_{S_k} \) and \( F_{S_{k+1}} \), \( \forall k \in I \), are adjacent faces. Let then \( \mathcal{S} \) be the set of all the sequences \( \{ S_n \} \) such that \( \{ F_{S_n} \} \) represent a "walk" on \( \hat{M} \) for a walk \( \{ F_{S_n} \} \) steering configuration \(( p_0, \theta_0, i_0 ) \in \hat{M} \) in configuration \(( p_f, \theta_f, i_f ) \in \hat{M} \) we will use the notation

\[
\{ F_{S_n} \} : ( p_0, \theta_0, i_0 ) \mapsto ( p_f, \theta_f, i_f ) \]  

or briefly, when we are not interested in position and orientation, as \( \{ F_{S_n} \} : F_{I_0} \mapsto F_{I_f} \). While in analysing the rolling motions of regular surfaces the central role was played by the surface metric, curvature, and torsion forms ([11]), to study the structure of the reachable set of a polyhedron it is instrumental to refer to the geometrical quantities introduced below:

**Definition 1** Let \( D_k \) denote the length of the edge incident to vertices \( V_i \) and \( V_k \). Also, denote \( \alpha_{ij} \) the angles of face \( F_j \) at vertex \( V_i \). The defect angle \( \beta_i \) at vertex \( V_i \) is defined as the complement to \( 2 \pi \) of the sum of angles \( \alpha_{ij} \) for all \( j \) such that face \( F_j \) is adjacent to \( V_i \).

Being convex polyhedral parts topological spheres, their total curvature is \( 4 \pi \). Because faces and edges of a polyhedron have zero Gauss curvature, all curvature is concentrated at vertices. In fact, the defect angle represents how much of the curvature of the object is concentrated at \( V_i \), and clearly we have \( \sum \beta_i = 4 \pi \). The fact that all the curvature of a polyhedral part (and hence, sensitivity to rolling) is concentrated at its vertices, along with the fact that such vertices are never perfectly sharp in real-world parts, suggests that pivoting about vertices may be much less robust a means of manipulation for polyhedral parts, than that of tumbling about edges.

The main theoretical results reported in this paper concern the structure of the set of configurations reachable by rolling a given polyhedron. We will say that such set is dense w.r.t. positions if for any desired position \( p_f \) in the plane, and any given tolerance \( \delta_p \), there exists a walk such that the polyhedron reaches a position closer to \( p_f \) than \( \delta_p \). Analogously, the reachable set is dense w.r.t. orientations if, for any \( \theta_f \) and \( \delta_\theta \), there exists a walk leading to an orientation closer to \( \theta_f \) than \( \delta_\theta \). The reachable set will be called dense in \( \hat{M} \), or dense tout-court, if the polyhedron can be brought arbitrarily close to any desired position with an orientation arbitrarily close to any desired orientation. The term discrete will be used for the negation of dense. Our results are as follows:

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Theorem 1 The set of configurations reachable by a polyhedron is dense in $\mathcal{M}$ if and only if there exists a vertex $V_i$ whose defect angle is irrational with $\pi$, i.e.,

$$\exists \beta_i : \frac{\beta_i}{\pi} \not\in \mathbb{Q}.$$ 

Theorem 2 The reachable set is discrete in both positions and orientations if and only if either of these conditions hold:

i) all angles of all faces (hence all defect angles) are $\pi/2$, and all lengths of the edges are rational w.r.t. each other (i.e., $D_{ij}/D_{kl} \in \mathbb{Q}, \forall i, j, k, l \in \{1, ..., m\}$;

ii) all angles of all faces (hence all defect angles) are integer multiples of $\pi/3$, and all lengths of the edges are rational w.r.t. each other (in other words, $\alpha_{ij} \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}, \forall i \in \{1, ..., m\}$ and $\forall j \in \{1, ..., l\}$, and $D_{ij}/D_{kl} \in \mathbb{Q}, \forall i, j, k, l \in \{1, ..., m\}$;

Theorem 3 The reachable set is dense in positions and discrete in orientations if and only if the defect angles are all rational w.r.t. $\pi$, and neither conditions i) or ii) of theorem 2 apply.

Remark 1. Polyhedra satisfying condition i) of theorem 2 are rectangular parallelepipeds, as e.g. a cube or a sum of cubes which is convex. Polyhedra as in condition ii) are those whose surface can be covered by a tessellation of equilateral triangles, as e.g. any Platonic solid except the dodecahedron.

Before giving a sketch of the proof of these results in the next section, it is perhaps the case to underline that they concern theoretical idealizations of real-world polyhedral parts. No part can be measured with accuracy fine enough to say whether the characteristic ratios above are rational or irrational. However, from the very machinery developed for the proof, an intuition of what goes on in the real case and an useful planning algorithm will result.

3 Mathematical development

We first reduce our problem of studying the set $\mathcal{R}_1$ reachable from a configuration with face $F_1$ in contact with $P$ to the study of one of its subsets. In fact, the density of the subset of reachable configurations with face $F_1$ in contact with $P$ is the same for every face: the problem does not depend on the initial configuration. Thus if through some sequence $\{F_{S_a}\} = S$, that will be referred to as “transit walk”, we bring face $F_1$ in contact with $P$ then $F_1$ can be brought in any position and orientation if and only if this is true for $F_1$. Therefore, what is actually to be studied is the set of reachable configurations with face $F_1$ in contact with $P$. By doing this, it is clear that such subset is the orbit of the initial configuration under the action of the subset of all walks bringing face $F_1$ back in contact with $P$.

The stereographic projection $\pi_N:S\setminus \{N\}$ maps a tiling $X'$ on the plane $P$ with one infinite component (corresponding to the cell of $S$ containing $N$) and $l-1$ bounded connected components. The graph naturally associated to such tiling $X'$ is the so-called Schlegel map of the polyhedron. The Schlegel map has the same number $k$ of edges and $m$ of vertices (nodes) as the original polyhedron. Furthermore, since we consider convex polyhedra for which the Euler relation $l+m-k=2$ holds, the map is line-crossing free ([11]).

3.1 A set of canonical movements for the motion of the polyhedron

By the assumption of convexity, parts are topological spheres, i.e. they can be continuously deformed onto spheres. As an example of such process, consider "blowing up" a polyhedron $\mathcal{P}$ onto a sphere $S$ large enough to encompass all of it (see fig. 3) by projecting the surface of $\mathcal{P}$ on the sphere from a point inside the polyhedron. Consider the tiling $X$ induced on $S$ by the image of the edges and the vertices of $\mathcal{P}$, i.e. the covering of its surface by a number of connected components (cells) which are the image on the sphere of the faces of the polyhedron. The Schlegel map ([11]) of $\mathcal{P}$ associates to the polyhedron a graph on the plane built by the stereographic projection (fig.4)

$$\pi_N : S \setminus \{N\} \rightarrow \Pi$$

from a point $N \in S$ (the "north" pole) onto a plane $\Pi$, the latter being tangent to $S$ through a point $S$ (the "south" pole) provided that both poles do not belong to the projection of any edge or vertex of the polyhedron on the sphere.

The stereographic projection $\pi_N|_X$ produces a tiling $X'$ on the plane $P$ with one infinite component (corresponding to the cell of $S$ containing $N$) and $l-1$ bounded connected components. The graph naturally associated to such tiling $X'$ is the so-called Schlegel map of the polyhedron. The Schlegel map has the same number $k$ of edges and $m$ of vertices (nodes) as the original polyhedron. Furthermore, since we consider convex polyhedra for which the Euler relation $l+m-k=2$ holds, the map is line-crossing free ([11]).

1. take new vertices as interior points of the cells of
In order to describe such generators, and hence the group of walks eventually bringing face \( F_j \) to \( F_j \) (see fig.6). Concatenate with \( t_j \) a loop \( L_1 \) that touches all and only the nodes on the \( i \)-th cell boundary. Finally return to the base point \( F_1 \) by following \( t_j \) in the inverse sense. By repeating the procedure for each bounded cell, we obtain \( m - 1 \) independent paths on the graph of type \( R_i = t_j \circ L_1 \circ t_j^{-1} \), which form a set of generators of all possible paths on the graph. Recalling that nodes and cells of the dual Schlegel map correspond to faces and vertices of the polyhedron, respectively, we can view each generator as an operator acting on \( \overline{M} \):

\[
R_i : (p_0, \theta_0, 1) \mapsto (p_j, \theta_j, 1),
\]

Such a map can be explicitly calculated based on the geometry of the polyhedron. Consider a plane development of the polyhedron based at the position of face 1 (i.e., glue face 1 on the plane, then "spread" the polyhedron cutting along its edges when necessary so as to bring all faces on the plane while leaving the surface connected, see fig.7). Let \( F_j \) be the image on the plane of face \( F_j \), and let \( V_k \) be the image of vertex \( V_k \) on the boundary of \( F_j \). Recalling the geometrical meaning of the defect angle \( \beta_i \), therefore, the action of the generators is described as

\[
R_i (p_0, \theta_0, 1) = (p_0 + (V_k - p_0) e^{i \beta_i}, \beta_i + \theta_0, 1).
\]

### 3.2 Proof of theorems (sketch)

Observe first that the action on \( S^1 \) of the generators \( R_i \) is transitive. Thus the structure of the projection of the reachable set on \( S^1 \), i.e., the orientation part of the three theorems, is proved at once: the set of reachable orientations is in fact given by all \( \theta \) such that the Diophantine equation

\[
\sum \alpha_i \beta_i = \theta + 2k \pi,
\]

has a solution with \( \alpha_i, i = 1, \ldots, m - 1, \) and \( k \) integers. If all \( \beta_i \) are rational w.r.t. \( \pi \), the set of such solutions is discrete (actually, finite modulo \( 2\pi \)) and easily characterized as the integer multiples of the greatest common divisor of the \( \beta_i \)'s, denoted by \( \beta \).

Concerning the structure of the projection of the reachable set in \( \mathbb{R}^2 \), i.e. displacements of the polyhedra, for the case of existence of a defect angle irrational w.r.t. \( \pi \), we recall the proof of theorem 1 given in [3].

If all \( \beta_i \) are irrational w.r.t. \( \pi \), it is possible to focus on the subgroup \( T \) of the translations of \( L_1 \), i.e., the set of all walks of type \( (p_0, \theta_0, 1) \mapsto (p_f, \theta_f, 1) \). In fact, the density of \( T \) implies and is implied by the density of the whole set of reachable positions in the plane. Again, being \( T \) a transitive subgroup of the group generated by the \( R_i \)'s, the description of a complete set of generators is sufficient to fully characterize its action. To describe such a set of generators, a "virtual rotation" \( R_V \) is introduced which is comprised of a composition of rotations \( R_i \) such that the total rotation is \( \beta \), the G.C.D. of the
defect angles. Any solution set $\alpha_i$, $i = 1, \ldots, m - 1$ of (3) with $\theta = \beta$ is taken as such a virtual rotation. Denote by $\hat{C}$ the point in the plane about which $R_{\hat{V}}$ occurs (this is computed easily given the $\alpha_i$'s and the polyhedron geometric parameters). The generators of $T$ can be written as translations in the direction of the vectors $g_{hj} = (V_j - \hat{C})(e^\beta_j - 1)e^{\hat{h}h\hat{\delta}}$, where $j = 1, \ldots, m - 1$, $h = 1, \ldots, \tilde{h}$, where $\tilde{h}$ is the smallest integer s.t. $\tilde{h}\hat{\beta} \equiv 0 \pmod{2\pi}$ (for a detailed calculation of these generators, see [10]).

The generators $g_{hj}$ of $T$ thus evaluated can be represented as vectors in $P$ originating from $p_0$. The set of reachable positions is the locus of points in the plane that can be reached by summing such vectors, i.e., the set of all points $p$ such that the 2-dimensional Diophantine equation

$$\sum_{h=1}^{\tilde{h}} \sum_{j=1}^{m-1} \gamma_{hj} g_{hj} = p$$

has a solution with integers $\gamma_{hj}$.

If there exist no two generators such that any other generator can be written as a combination over the rationals of the two, than the reachable set is clearly dense. Otherwise, it is always possible to find two new vectors in the plane (the so-called "greatest common divisors" of the set of generators, see [12]) such that any reachable position can be written as an integer combination of the g.c.d. vectors. The set of reachable positions is discrete in the latter case, and furthermore it lies on a lattice whose description is given completely by the above analysis (see fig.8). Using this algebraic description of the generators, and the geometric properties of the polyhedra, the proof of theorems 2 and 3 easily follow.

4 Planning Algorithm

The theoretical analysis above summarized allows one to design a practical algorithm for planning manipulation of polyhedral parts by rolling. As already mentioned, however, some caution has to be taken in applying the results. In particular, although from the theory the discreteness of the reachable set appears to be an exception, this is the only practically relevant case. A first reason in fact is that any representation of the angles $\beta_k$ and of the generators $g_{hj}$ is forcedly rational in a digital computer with finite precision. Secondly, and more stringently, as the description of the polyhedral part comes from a physical process of measurement or machining, it can only be known to within a tolerance. The numeric representation of such data has therefore to be chosen with comparable accuracy (usually much less than that available in modern computers).

These considerations imply that the only reasonable specification of a planning problem in this context is to give a desired face, position, and orientation, along with a tolerance for the latter two (see fig.8). Deciding whether reaching the goal within the tolerance is possible for the given part description and associated accuracy should be considered as an important part of any planning algorithm.

In this setting, an algorithm for finding a sequence of elementary tumbles that steers the polyhedron from configuration $(p_0, \theta_0, i_0) \rightarrow (p_f, \theta_f, 1)$ within a tolerance $\epsilon_p$ on positions and $\epsilon_\theta$ on orientations, can be given as follows.

1. Measure the polyhedron parameters $D_k$ and $\alpha_k$, and provide their continued fraction expansion with reasonable accuracy;

2. Take face 1 on the plane, i.e., find a transit walk $t_i$ such that $t_i^{-1} : (p_0, \theta_0, i_0) \rightarrow (p_1, \theta_1, 1)$;

3. Compute angles $\beta_1$ and their g.c.d. $\hat{\beta}$; the virtual rotation $R_{\hat{V}}$ and $\hat{C}$; the plane development of the polyhedron on the plane and the $V_ij$'s;

4. Verify that the tolerance on orientations is admissible, i.e., that $\epsilon_\theta > \frac{|\hat{\beta}|}{2}$;

5. Let $\delta \theta = \theta_f - \theta_0$ and $k$ be the smallest integer s.t. $|k \beta_f - \delta \theta| < \epsilon_\theta$; then apply $k$ times the virtual rotation, i.e. $R_{\hat{V}}^k : (p_1, \theta_1, 1) \rightarrow (p_2, \theta_2, 1)$;

6. Compute the generators $g_{kj}$, and their g.c.d. $\hat{g}_1$, $\hat{g}_2$ using a lattice reduction algorithm (see [12]);

7. Verify that the tolerance on positions is admissible, i.e., that $\epsilon_p > \min \left\{ \frac{||\hat{g}_1 + \hat{g}_2||}{2}, \frac{||\hat{g}_1 - \hat{g}_2||}{2} \right\}$;

8. Let $\delta p = p_f - p_2$, and find the smallest integers $k_1$, $k_2$ such that $||k_1 \hat{g}_1 + k_2 \hat{g}_2 - p_2|| \leq \epsilon_p$;

9. Invert the g.c.d. algorithm to obtain integers $k_{hj}$ such that $\sum_h \sum_j k_{hj} g_{hj} = k_1 \hat{g}_1 + k_2 \hat{g}_2$;

If the admissibility checks hold true, the algorithm finds a solution to the planning problem in the form of a concatenation of walks, $k_{h_{m-1}} \hat{g}_{m-1} \ldots k_{h_{1}} \hat{g}_{1} R_{\hat{V}}^k \circ \hat{V}_j^{-1} : (p_0, \theta_0, i_0) \rightarrow (p_3, \theta_2, 1)$, with $p_3$ and $\theta_2$ within the prescribed tolerance. The resulting path may be rather complex. Once converted in terms of the sequence of faces, positions, and orientations to be followed by the polyhedron, the length of the walk can often be trimmed by deleting the largest subsequence comprised within two equal configurations. To further reduce the complication of manipulation maneuvers, the sequence resulting from the algorithm can be used as a feasible starting solution of a branch-and-bound algorithm for discrete optimization. As an example of application, manipulation by rolling of a 32-pin connector is reported in fig.9.

5 Conclusions

In this paper we have investigated the structure of the reachable set of a polyhedron rolling on a plane, and deduced a complete algorithm for planning such motions. Experimental work is under implementation to show the practicality of manipulation by rolling.
parts. The interest of the concepts and tools developed however may not be confined to robot manipulation. In fact, the problem appears to have multiple cousins in the scientific literature at large: for instance in ergodic theory, automata theory, discrete gravitation theory (Regge's calculus), and billiards.

References