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Finite Gain l_p Stabilization of Discrete-Time Linear Systems Subject to Actuator Saturation: the Case of $p = 1$

†Yacine Chitour ‡Zongli Lin

†Département de mathématiques, Université de Paris-Sud, Orsay-91405, France.
Email: Yacine.Chitour@math.u-psud.fr

‡Department of Electrical and Computer Engineering, University of Virginia, P.O. Box 400743
Charlottesville, VA 22904-4743, U.S.A. Email: zlin@virginia.edu

***Abstract*— It has been established by Bao, Lin and Sontag (2000) that, for neutrally stable discrete-time linear systems subject to actuator saturation, finite gain l_p stabilization can be achieved by linear output feedback, for every $p \in (1, \infty]$ except $p = 1$. An explicit construction of the corresponding feedback laws was given. The feedback laws constructed also resulted in a closed-loop system that is globally asymptotically stable. This note complements the results of Bao, Lin and Sontag (2000) by showing that they also hold for the case of $p = 1$.**

I. INTRODUCTION

This short note revisits the problem of simultaneous global asymptotic stabilization and finite gain L_p (l_p) stabilization of a linear system in the presence of actuator saturation and measurement and actuator noises (see [8], [1] and the references therein). More specifically, we would like to construct a controller $\mathcal{C}$ so that the operator $(u_1, u_2) \mapsto (y_1, y_2)$ as defined by the following standard systems interconnection (see Fig. 1)

$$\begin{cases} y_1 = \mathcal{P}(u_1 + y_2) \\ y_2 = \mathcal{C}(u_2 + y_1) \end{cases} \quad \text{(I. 1)}$$

is well-defined and finite gain stable. This problem was

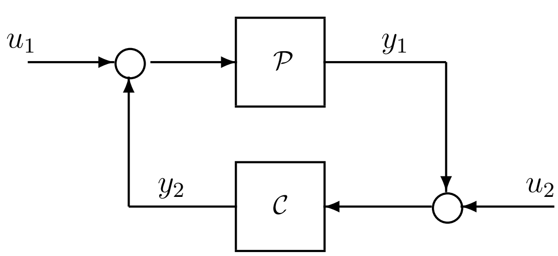

Fig. 1. Standard closed-loop connection.

first studied for continuous-time systems and various results have been established for such systems. It was shown in [8] that, for neutrally stable open loop systems, linear feedback laws can be used to achieve finite gain L_p stabilization, for all $p \in [1, \infty]$. Various continuity and incremental-gain properties of the resulting closed-loop system were discussed in detail in [4]. For a neutrally stable system, all open loop poles are located in the closed left-half plane, with those on the $j\omega$ axis having Jordan blocks of size one. In the case that full state is available for feedback (i.e., $y_1 = x$ and $u_2 = 0$), it was shown in [7] that, if the external input signal is uniformly bounded, then finite-gain L_p-stabilization, for any $p \in (1, \infty]$ except $p = 1$, and local asymptotic stabilization can always be achieved simultaneously by linear feedback, no matter where the poles of the open loop system are. The uniform boundedness condition of [7] was later removed for the case $p = 2$ by resorting to nonlinear feedback [6]. More recently, the problem of L_p stabilization for a double integrator system subject to input saturation feedback and disturbances that are not additive was investigated in [3], where it is considered the control system (DI), $\ddot{x} = \sigma(-x - \dot{x} + u) + v$, with $x \in I\!R$ and disturbances (u, v). For $v = 0$ and zero initial conditions, it was established, among other results, that the L_2-gain from u to the output of the saturation nonlinearity was finite. This partially solved Problem 36 as defined in [2]. For nonzero v, an L_∞-bound is of course necessary to get any positive result regarding L_p-stabilization. It was shown that (DI) is L_p-stable (see [3] for the precise definition of L_p-stability) if and only if $p \leq 2$. In other words, for $p > 2$, one can construct a disturbance v with finite L_p-norm and arbitrarily small L_∞-norm that results in an unbounded trajectory of (DI). Examples of other works related to the topic are [5], [9], [10] and the references therein. The extension of the results of [8] to discrete-time systems was made in [1]. In particular, it was shown in [1] that, for neutrally stable discrete-time linear systems subject to actuator saturation, finite gain l_p stabilization can be achieved by linear output feedback, for every $p \in (1, \infty]$ except $p = 1$. An explicit construction of the corresponding feedback laws was given. The feedback laws constructed also result in a closed-loop system that is globally asymptotically stable. The objective of this note is to complement the results of [1] by showing that they also hold true for the case of $p = 1$.

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II. Statement and Proof of the Result

We consider a discrete-time linear system subject to actuator saturation and input additive measurement and actuator noises,

$$\mathcal{P}: \begin{cases} x^+ = Ax + B\sigma(u+u_1), & x \in I\!R^n,\ u \in I\!R^m \\ y = Cx + u_2, & y \in I\!R^r \end{cases} \tag{II. 1}$$

(we use the notation x^+ to indicate a forward shift, that is, for a function x and an integer t, $x^+(t)$ is $x(t+1)$), where $u_1 \in I\!R^m$ is the actuator disturbance, $u_2 \in I\!R^r$ is the sensor noise, and $\sigma : I\!R^m \to I\!R^m$ represents the vector valued actuator saturation, i.e., $\sigma(s) = [\,\sigma_1(s_1) \quad \sigma_2(s_2) \quad \cdots \quad \sigma_m(s_m)\,]'$ with σ_i being a standard saturation function

$$\sigma_i(\cdot) = \text{sgn }(\cdot)\min\{1, |\cdot|\}.$$

Moreover, the pair (A,B) is assumed to be stabilizable. To state the results, we first need to recall some notation. For a vector $X \in I\!R^\ell$, X' denotes the transpose of X, $\|X\|$ its Euclidean norm and, for a matrix $X \in I\!R^{m\times n}$, the induced operator norm. We use I_n to denote the identity matrix of $I\!R^{n\times n}$. We write l_1^n for the set of all sequences $\{x(t)\}_{t=0}^{\infty}$, where $x \in I\!R^n$, such that $\sum_{t=0}^{\infty}\|x(t)\| < \infty$, and the l_1-norm of $x \in l_1^n$ is defined as $\|x\|_{l_1} = \sum_{t=0}^{\infty}\|x(t)\|$. It is easy to verify that, for every $\xi, \eta \in I\!R^m$

$$\|\sigma(\xi) - \sigma(\eta)\| \le \inf(1, \|\xi - \eta\|), \tag{II. 2}$$

and,

$$\|\sigma(\xi) - \xi\| \le \xi'\sigma(\xi). \tag{II. 3}$$

The main result of this note is stated in the following theorem.

Theorem II. 1 *Consider a system (II. 1). Let A be neutrally stable, i.e., all the eigenvalues of A are inside or on the unit circle, with those on the unit circle having all Jordan blocks of size one. Also assume that (A,B) is stabilizable and (A,C) is detectable. Then, there exits a linear observer-based output feedback law of the form*

$$\begin{cases} \hat{x}^+ = A\hat{x} + B\sigma(F\hat{x}) - L(y - C\hat{x}) \\ u = F\hat{x} \end{cases} \tag{II. 4}$$

such that the resulting closed-loop system has the following properties:

1) *It is finite gain l_1-stable, i.e., there exists a $\gamma_1 > 0$ such that*

$$\|x\|_{l_1} \le \gamma_1\left[\|u_1\|_{l_1} + \|u_2\|_{l_1}\right], \quad \forall u_1 \in l_1^m,\ u_2 \in l_1^r, \quad \text{and } x(0) = 0,\ \hat{x}(0) = 0\,. \tag{II. 5}$$

2) *In the absence of actuator and sensor noises u_1 and u_2, the equilibrium $(x,\hat{x}) = (0,0)$ is globally asymptotically stable.*

Remark II. 2 *Theorem II. 1 establishes simultaneous global asymptotic stabilizability and finite gain l_1 stabilizability for system (II. 1). Theorem 1 of [1] establishes simultaneous global asymptotic stabilizability and finite gain l_p stabilizability for every $p \in (1,\infty]$ except $p = 1$. Thus, Theorem II. 1 complements the results of [1].*

Following the lines of reasoning in [1], the proof of Theorem II. 1 reduces to the following proposition.

Proposition II. 3 *Let A be orthogonal (i.e., $A'A = I$), and suppose that the pair (A,B) is controllable. Then, the system*

$$x^+ = Ax - B\sigma(\kappa B'Ax + u), \quad x \in I\!R^n,\ u \in I\!R^m \tag{II. 6}$$

is finite gain l_1-stable, for sufficiently small $\kappa > 0$. Moreover, there exist a real γ_1, a $\kappa^* \in (0,1]$, and a class-$\mathcal{K}$ function θ_1 such that, for all $\kappa \in (0,\kappa^*]$,

$$\|x\|_{l_1} \le \gamma_1\|u\|_{l_1} + \theta_1(|x(0)|) \tag{II. 7}$$

for all inputs $u \in l_1^m$ and all initial states $x(0)$.

Proof of Proposition II. 3. If $W : I\!R^n \to I\!R$, then W^+ and ΔW denote respectively $W(x^+(t))$ and the increment $W(x^+(t)) - W(x(t))$ along any given trajectory of (II. 6). In the sequel, C_i and $C_i(\kappa)$, $i = 1,2,\cdots$ denote constants that are independent and dependent of κ, respectively. As in [1], the proof of Proposition II. 3 consists of constructing a Lyapunov function V for (II. 6) verifying the following dissipative inequality: for every $\kappa \in (0,\kappa_0]$, there exist $C_1(\kappa), C_2(\kappa) > 0$ such that along trajectories of (II. 6), we have

$$\Delta V \le -C_1(\kappa)\|x\| + C_2(\kappa)\|u\|, \quad \forall \kappa \in (0,\kappa^*], \tag{II. 8}$$

for some $\kappa^* > 0$. For $\kappa > 0$ such that $\kappa B'B < 2I_n$, set $\bar{A}(\kappa) := A - \kappa BB'A$ and $P(\kappa)$ the unique symmetric positive definite solution of the Lyapunov equation

$$\bar{A}(\kappa)'P\bar{A}(\kappa) - P = -I_n. \tag{II. 9}$$

It was proved in [1] that there exists a $\kappa_1^* > 0$ and $C_3, C_4 > 0$ such that for every $\kappa \in (0,\kappa_1^*]$,

$$\frac{C_3}{\kappa}I_n \le P(\kappa) \le \frac{C_4}{\kappa}I_n. \tag{II. 10}$$

For $x \in I\!R^n$, consider $V_1(x) = x'P(\kappa)x$ and $V_0(x) = \|x\|^2$. As in [1], the Lyapunov function candidate V for (II. 8) will be chosen equal to $V_1^{1/2} + rV_0$ for an appropriate choice of $r > 0$. We rewrite the dynamics of (II. 6) as

$$x^+ = \bar{A}(\kappa)x - B[\tilde{x} - \sigma(\tilde{x}) - u], \tag{II. 11}$$

where $\tilde{x} := \kappa B'Ax + u$. By using the fact that $x^+ = Ax - B\sigma(\tilde{x})$, we have

$$\Delta V_0 = -\frac{2}{\kappa}\tilde{x}'\sigma(\tilde{x}) + \frac{2}{\kappa}u'\sigma(\tilde{x}) + \|B\sigma(\tilde{x})\|^2. \tag{II. 12}$$

Moreover, for every $x \in I\!R^n$ and $u \in I\!R^m$ we have,

$$\begin{aligned} V_1^+ &= (x^+)'P(\kappa)x^+ \\ &= (x'A' - \sigma(\tilde{x})'B')P(\kappa)(Ax - B\sigma(\tilde{x})) \\ &\le \frac{C_5}{\kappa}(\|x\|^2 + \|u\|^2), \end{aligned} \tag{II. 13}$$

which simply follows from (II. 10) and (II. 2). In the same manner, we also have that, for every $x \in I\!R^n$ and $u \in I\!R^m$,

$$V_0^+ = (x'A' - \sigma(\tilde{x})'B')(Ax - B\sigma(\tilde{x})) \leq C_6(\|x\|^2 + \|u\|^2). \tag{II. 14}$$

Equation (II. 13) clearly implies after taking square roots on both sides of it and elementary calculus that

$$(V_1^{1/2})^+ \leq \frac{C_7}{\sqrt{\kappa}}(\|x\| + \|u\|). \tag{II. 15}$$

We will show that (II. 8) holds by dividing the argument in two cases, namely whether $\|x\| \leq \|u\|$ or not. Assume first that $\|x\| \leq \|u\|$. We clearly have

$$\Delta(V_1^{1/2}) \leq (V_1^{1/2})^+ \leq \frac{C_8}{\sqrt{\kappa}}\|u\|, \tag{II. 16}$$

by using (II. 15) and the hypothesis of the case under study. Recall that $\|\sigma(\xi)\| \leq \inf(1, \|\xi\|)$ for every $\xi \in I\!R^m$. Then, from (II. 12), we get

$$\Delta V_0 \leq C_9\|u\|. \tag{II. 17}$$

Adding (II. 16) and (II. 17), we get

$$\Delta V \leq C_{10}(\kappa)\|u\|.$$

We deduce that

$$\Delta V \leq C_{10}(\kappa)\|u\| + (\|u\| - \|x\|) = -\|x\| + C_{11}(\kappa)\|u\|,$$

which is exactly (II. 8). We now treat the case where $\|x\| > \|u\|$. From (II. 15), we get

$$(V_1^{1/2})^+ \leq \frac{C_{12}}{\sqrt{\kappa}}\|x\|, \tag{II. 18}$$

which implies that

$$\frac{C_{13}}{\sqrt{\kappa}}\|x\| \leq V_1^{1/2} \leq V_1^{1/2} + (V_1^{1/2})^+ \leq \frac{C_{14}}{\sqrt{\kappa}}\|x\|. \tag{II. 19}$$

Recall that $\Delta V = \Delta(V_1^{1/2}) + r\Delta V_0$. Let us consider first ΔV_0 as given by (II. 12). The right-hand side of the equality is the sum of three terms. For the second one, we use the fact that σ is bounded and for the third one, we recall that, for every $\xi \in I\!R^m$, $\|\sigma(\xi)\| \leq \inf(1, \|\xi\|)$. After elementary calculus, we get

$$\Delta V_0 \leq -\frac{C_{15}}{\kappa}\tilde{x}'\sigma(\tilde{x}) + \frac{C_{16}}{\kappa}\|u\|. \tag{II. 20}$$

For $\Delta(V_1^{1/2})$, we simply write it as

$$\Delta(V_1^{1/2}) = \frac{\Delta V_1}{V_1^{1/2} + (V_1^{1/2})^+}. \tag{II. 21}$$

We next compute ΔV_1 from (II. 11) and get

$$\begin{aligned} \Delta V_1 = & \Big(x'\bar{A}(\kappa)' - [\tilde{x} - \sigma(\tilde{x}) + u]'\Big)P(\kappa) \\ & \times\Big(\bar{A}(\kappa)x - B[\tilde{x} - \sigma(\tilde{x}) - u]\Big) - x'P(\kappa)x \\ = & \; x'(\bar{A}(\kappa)'P(\kappa)\bar{A}(\kappa) - P(\kappa))x \\ & -2x'\bar{A}(\kappa)'P(\kappa)B[\tilde{x} - \sigma(\tilde{x}) - u] \\ & +[\tilde{x} - \sigma(\tilde{x}) - u]'B'P(\kappa)B[\tilde{x} - \sigma(\tilde{x}) - u]. \end{aligned} \tag{II. 22}$$

Using (II. 9) and (II. 3), we deduce from the previous equation that

$$\Delta V_1 \leq -\frac{1}{2}\|x\|^2 + \frac{C_{17}}{\kappa}\|x\|\tilde{x}'\sigma(\tilde{x}) + \frac{C_{18}}{\kappa^2}\|u\|^2, \quad \forall \kappa \in (0, \kappa^*], \tag{II. 23}$$

for some $\kappa^* \in (0, \kappa_1^*]$. Using (II. 19) and the fact that $\frac{\|u\|}{\|x\|} \leq 1$, we divide (II. 23) by $V_1^{1/2} + (V_1^{1/2})^+$ and get

$$\Delta(V_1^{1/2}) \leq -C_{19}\sqrt{\kappa}\|x\| + \frac{C_{20}}{\sqrt{\kappa}}\tilde{x}'\sigma(\tilde{x}) + \frac{C_{21}}{\kappa^{3/2}}\|u\|. \tag{II. 24}$$

Adding up (II. 20) and (II. 24) with $r := \frac{C_{20}}{C_{15}}\sqrt{\kappa}$, (II. 8) follows.

III. Conclusions

In this note, we established that a neutrally stable discrete-time linear system subject to actuator saturation can be rendered simultaneous finite gain l_1 stable and globally asymptotically stable by a simple linear feedback law. This complements and completes a recent work, where simultaneous finite gain l_p stabilizability and global asymptotic stabilizability for the same system was established for every $p \in (1, \infty]$ except $p = 1$.

IV. References

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