Common Polynomial Lyapunov Functions for Linear Switched Systems

Paolo Mason, Ugo Boscain,
SISSA-ISAS, Via Beirut 2-4, 34014 Trieste, Italy
Yacine Chitour
Université Paris XI, Département de Mathématiques, F-91405 Orsay, France

Abstract In this paper, we consider linear switched systems $\dot{x}(t) = A_{u(t)}x(t)$, $x \in \mathbb{R}^n$, $u \in U$, and the problem of asymptotic stability for arbitrary switching functions, uniform with respect to switching (**UAS** for short). We first prove that, given a **UAS** system, it is always possible to build a common polynomial Lyapunov function. Then our main result is that the degree of that common polynomial Lyapunov function is not uniformly bounded over all the **UAS** systems. This result answers a question raised by Dayawansa and Martin. A generalization to a class of piecewise-polynomial Lyapunov functions is given.

Keywords: Switched systems, Stability, Polynomial Lyapunov function.

AMS subject classifications: 93D20, 37N35.

PREPRINT SISSA 52/2004/M

1 Introduction

In recent years, the problem of stability and stabilizability of switched systems has attracted increasing attentions (see for instance [1, 4, 5, 7, 10, 12, 13, 16]), and still many questions remain unsolved.

In this paper, we address the problem of existence of common polynomial Lyapunov functions for linear switched systems.

By a switched system, we mean a family of continuous–time dynamical systems and a rule that determines at each time which dynamical system is responsible of the time evolution. More precisely, let $\{f_u : u \in U\}$ (where U is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$) be a finite or infinite set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{1}$$

The rule is given by assigning the so-called switching function, i.e. a measurable function $u(.) : [0, \infty[\to U \subset \mathbb{R}^m$. Here, we consider the situation in which the switching function is not known a priori and represents some phenomenon (e.g. a disturbance) that is not possible to control. Therefore, the dynamics defined in (1) also fits into the framework of uncertain systems (cf. for instance [9]).

In the sequel, we use the notations $u \in U$ to label a fixed individual system and $u(.)$ to indicate the switching function.

These kind of systems are sometimes called "n-modal systems", "dynamical polysystems", "polysystems", "input systems". The term "switched system" is often reserved to situations in which the set U is finite. For the purpose of this paper, we only require U to be measurable. For a discussion of various issues related to switched systems, we refer the reader to [5, 12, 13].

A typical problem for switched systems goes as follows. Assume that, for every $u \in U$, the dynamical system $\dot{x} = f_u(x)$ satisfies a given property (P). Then one can investigate conditions under which property (P) still holds for $\dot{x} = f_{u(t)}(x)$, where $u(.)$ is an arbitrary switching function.

In [1, 7, 10, 11], the case of switched linear systems was considered:

$$\dot{x}(t) = A_{u(t)}x(t), \quad x \in \mathbb{R}^n, \quad A_u \in \mathbb{R}^{n\times n}, \tag{2}$$

where n is a positive integer and $u(.) : [0, \infty[\to U$ is a (measurable) switching function. For these systems, the problem of asymptotic stability of the origin, uniformly with respect to switching functions, was investigated.

Next, we set $\mathbf{A} := \{A_u : u \in U\}$ and, to simplify the notation, we still call switching function the measurable matrix-valued map $A(.) := A_{u(.)}$. In this way, the switching system (2) reads:

$$\dot{x}(t) = A(t)x(t), \text{ where } x \in \mathbb{R}^n, \text{ and } A(.) : [0, \infty[\to \mathbf{A} \text{ is a measurable map.} \tag{3}$$

In the following, we assume that:

(H0) the set $\mathbf{A}$ is a compact subset of the set of $n \times n$ real matrices.

Moreover, the set of switching functions, denoted by $\mathcal{A}$, is the set of measurable functions $A(.) : [0, \infty[\to \mathbf{A}$. With our assumptions, for every switching function $A(.)$ and initial condition $x_0 \in \mathbb{R}^n$, the corresponding (Carathéodory) solution of (3) is defined for every $t \geq 0$. We use $\phi_t^{A(.)}(x_0)$ to denote the flow of (3) at time $t \geq 0$ corresponding to the switching function $A(.)$ and starting from x_0.

Let us recall usual notions of stability used for the system (3).

Definition 1 *Consider the switching system (3). We say that the origin is:*

(S) *stable, if for every $A(.) \in \mathcal{A}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\phi_t^{A(.)}(x_0)\| \leq \varepsilon$ for every $t \geq 0$, $\|x_0\| \leq \delta$.*

(US) *uniformly stable, if it is stable with δ not depending on $A(.)$.*

(U) *unstable if it is not stable (i.e. if there exists $A(.) \in \mathcal{A}$ s.t. the system $\dot{x}(t) = A(t)x(t)$ is unstable as a linear time-varying system.)*

(AS) *asymptotically stable, if it is stable and attractive (i.e. there exists $\delta' > 0$ so that, for every $A(.) \in \mathcal{A}$ and $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq \delta'$, we have $\lim_{t\to\infty} \|\phi_t^{A(.)}(x_0)\| = 0$).*

(UAS) *uniformly asymptotically stable if it is uniformly stable and if, for every $\varepsilon' > 0$ and $\delta' > 0$, there exists $T > 0$ such that for every switching function $A(.) \in \mathcal{A}$, $t \geq T$ and $\|x_0\| \leq \delta'$, we have $\|\phi_t^{A(.)}(x_0)\| \leq \varepsilon'$.*

(GUES) *globally uniformly exponentially stable, if there exist positive constants M, λ such that: $\|\phi_t^{A(.)}(x_0)\| \leq M\ e^{-\lambda t}\|x_0\|$, for every $x_0 \in \mathbb{R}^n$, $t > 0$, $A(.) \in \mathcal{A}$.*

Due to the fact that the dynamics is linear in the state variable, the local and global notions of stability are equivalent. More precisely, it was proved in [3] that, for system (3) subject to **H0**, the three notions **AS**, **UAS**, **GUES** and the notion of attractivity are all equivalent (see also [10, 13]). In addition, if the system is unstable, then there exists a switching function $A(.) \in \mathcal{A}$ and an initial condition x_0 such that $\lim_{t\to\infty} \|\phi_t^{A(.)}(x_0)\| \to \infty$. In the following, we just refer to the notions of stability, instability and **GUES**.

Remark **1** Since for the stability issue, a system of type (3), subject to **H0**, is uniquely determined by a compact set **A** of $n \times n$ real matrices, we identify **A** with the corresponding system for the rest of the paper. For instance, when we say that **A** is **GUES**, we mean that the corresponding system of type (3) is **GUES**. We will often consider the problem of determining whether a system, belonging to a certain class C of systems of type (3) subject to **H0**, is **GUES** or not. Notice that fixing such a class of systems means to fix a set of compact subsets of $\mathbb{R}^{n\times n}$ i.e. C can be identified with a subset of $\{\mathbf{A} \subset \mathbb{R}^{n\times n} : \ \mathbf{A} \text{ compact}\}$.

For a system (3) subject to **H0**, it is well known that the **GUES** property is a consequence of the existence of a common Lyapunov function.

Definition 2 *A common Lyapunov function (CLF for short) $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, for a switched system (S) of the type (3), is a continuous function such V is positive definite (i.e. $V(x) > 0$, $\forall x \neq 0$, $V(0) = 0$) and V is strictly decreasing along nonconstant trajectories of (S).*

Vice-versa, it is proved in [10] that, given a **GUES** system of the type (3) subject to **(H0)**, it is always possible to build a $\mathcal{C}^\infty$ common Lyapunov function.

Anyway, the problem of finding a CLF or proving the nonexistence of a CLF is in general a difficult task. Sometimes, it is even easier to prove directly that a system is **GUES** or unstable. An example is provided below by bidimensional switched systems.

1.1 Single-Input Bidimensional Switched Systems

Consider a single input bidimensional system of the type:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{4}$$

where $x \in \ \mathbb{R}^2$, A and B are two 2×2 real Hurwitz matrices and $u(.)$ is a measurable function defined on $\mathbb{R}^+$ and taking values in U equal either to $[0, 1]$ or $\{0, 1\}$. In the sequel, we call Ξ the class of bidimensional systems of the above form. This class is parameterized by couples of 2×2 real Hurwitz matrices.

Remark **2** Whether systems of type (4) are **GUES** or not is independent on the specific choice $U = [0, 1]$ or $U = \{0, 1\}$. In fact, this is a particular instance of a more general result stating that the stability properties of systems (3) subject to **H0** only depend on the convex hull of the set **A**, see Proposition 1 below and Appendix B.

In [15], the authors provide a necessary and sufficient condition on the pair (A, B) to share a quadratic CLF, but Dayawansa and Martin showed in [10] that there exist **GUES** linear bidimensional systems not admitting quadratic CLF. They also posed the problem of finding the minimal degree of a polynomial CLF. More precisely, the problem posed by Dayawansa and Martin is the following:

Problem P: Define $\Xi_{GUES} \subset \Xi$ as the set of **GUES** systems of the type (4). Find the minimal integer m such that every system of Ξ_{GUES} admits a polynomial CLF of degree less or equal than m.

Remark **3** In the problem posed by Dayawansa and Martin, it is implicitly assumed that a **GUES** system always admits a polynomial Lyapunov function and one of our results (our Theorem 1 below) indeed confirms that fact.

As for the **GUES** issue, it was completely resolved in [7], where a necessary and sufficient condition for a system of type (4) to be **GUES** was found directly, without looking for a CLF (see Section 3 and Appendix A for more details). This is a typical example in which it is easier to study directly the stability rather than looking for a CLF.

1.2 Sets of Functions Sufficient to Check GUES

The concept of Lyapunov function is useful for practical purposes when one can prove that, for a certain class of systems, if a CLF exists, then it is possible to find one of a certain type and possibly as simple as possible (e.g. polynomial with a bound on the degree, piecewise quadratic etc.).

More precisely, consider a class C of systems of type (3) in $\mathbb{R}^n$ subject to **H0**, in the sense of Remark 1. One would like to find a class of functions $\mathcal{S}_C$, identified by a finite number of parameters, which is sufficient to check **GUES** for systems belonging to C i.e., if a system of C admits a CLF, then it admits one in $\mathcal{S}_C$. Once such a class of functions is identified, then in order to verify **GUES** one could use numerical algorithms to check (by varying the parameters) whether a CLF exists (in which case the system is **GUES**) or not (meaning that the system is not **GUES**).

For instance, a remarkable result for a given class C of systems in $\mathbb{R}^n$ could be the following:

Claim: there exists a positive integer m (depending on n) such that, whenever a system of C admits a CLF, then it admits one that is polynomial of degree less than or equal to m. In other words, the class of polynomials of degree at most m is sufficient to check **GUES** for the class C.

If this result were true, one could use numerical algorithm to check, among all polynomial of degree m (varying the coefficients), if there is one that is a CLF. Unfortunately, this claim is not true, even for the simplest non trivial case of class of systems in $\mathbb{R}^2$, namely systems of type Ξ (cf. Equation (4)).

The next definition formalizes the idea of class of functions sufficient to check **GUES**.

Definition 3 *We say that a subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is finitely (or m-finitely) parameterized if there exist $\Omega \subset \mathbb{R}^m$ for a positive integer m and a bijective map $\Psi : \Omega \subseteq \mathbb{R}^m \to \mathcal{S} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$. A subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is said to be sufficient to check* **GUES** *for a class C of systems of type (3) in $\mathbb{R}^n$ (***SSF** *for short), if every* **GUES** *system of C admits a CLF in $\mathcal{S}$. A subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is said to be an (m-parameters) finite set of functions sufficient to check* **GUES** *for a class C of systems of type (3) in $\mathbb{R}^n$ (finite-***SSF** *for short), if $\mathcal{S}$ is m-finitely parameterized and is an* **SSF** *for C.*

If a subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is not finitely parameterizable but is an **SSF** *for a class of systems C, we call $\mathcal{S}$ an ∞-***SSF***.*

Remark **4** In [6], a concept similar to those introduced in the previous definition was provided and it was called "universal class of Lyapunov functions".

Using the previous definitions, the results and the problem formulated in [10] can be rephrased in the following way:

R1 for systems (3) subject to **H0**, the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is an ∞-**SSF**;

R2 for linear bidimensional systems of the class Ξ (cf. Formula (4)), the set of quadratic functions is not a **SSF**;

P let $\mathcal{P}^m$ be the set of polynomial functions of two variables with degree at most m. What is the minimal m such that $\mathcal{P}^m$ is a finite-**SSF** for the linear bidimensional systems of the class Ξ?

Remark **5** Notice that, to check numerically the existence of a CLF using the concept of finite-**SSF**, one needs some regularity properties of the functions of the family, with respect to the parameters (at least continuity). Anyway, this discussion is out of the purpose of this paper.

1.3 Main Results

We first prove that the implicit assumption of Dayawansa and Martin (i.e. that a linear **GUES** switched system always admits a polynomial CLF, cf. Remark 3) is correct in $\mathbb{R}^n$ (and in particular for bidimensional systems of type (4)).

Theorem 1 *If the origin is a* **GUES** *equilibrium for the switched system (3) subject to* **H0***, then there exists a polynomial CLF.*

The above result can be stated equivalently as follows.

Theorem 1bis *The set of polynomials from $\mathbb{R}^n$ to $\mathbb{R}$, is an ∞-***SSF** *for linear switched systems (3) subject to* **H0**.

The proof of Theorem 1 is given in Section 2 and the starting point is the construction of a homogeneous and convex CLF W, following the corresponding argument of [10]. The main idea is then to seek for a (homogeneous) polynomial $\tilde{W}$ whose level sets approximate, in some suitable sense, those of W and, finally to show that $\tilde{W}$ is also a CLF.
A related result has also been obtained in [6], using an intermediate approximation with polyhedral functions (this step is implicit in our proof) and starting from the case of discrete approximating systems (so-called Euler approximating systems).

The core of the paper consists of showing that problem **P** does not have a solution, i.e. the minimum degree of a polynomial CLF cannot be uniformly bounded over the set of all **GUES** systems of the form (4). More precisely, we have the following:

Theorem 2 *Let $\Xi_{GUES} \subset \Xi$ be the set of all* **GUES** *systems of the type (4). If (A, B) is a pair of 2×2 real matrices giving rise to a system of Ξ_{GUES}, let $m(A, B)$ be the minimum value of the degree of any polynomial CLF associated to that system. Then $m(A, B)$ cannot be bounded uniformly over Ξ_{GUES}.*

The above result can be stated equivalently as follows.

Theorem 2bis *Let $\mathcal{P}^m$ the set of polynomial functions from $\mathbb{R}^2 \to \mathbb{R}$ of degree at most m. Then $\mathcal{P}^m$ is not a finite-***SSF** *for Ξ.*

The proof, given in Section 4, is based on ideas developed in [7], where necessary and sufficient conditions for **GUES** of systems (4) are provided. We build a sequence of **GUES** systems corresponding to a sequence of pairs of matrices (A_i, B_i), $i \geq 1$. The sequence of systems is chosen in such a way that the limit system is uniformly stable but not attractive. In particular, that limit system admits a nontrivial periodic trajectory whose support Γ is a $\mathcal{C}^1$ but not a $\mathcal{C}^2$ submanifold of the plane. To each **GUES** system of the sequence, one considers any polynomial CLF $V_{\bar{A}_i,\bar{B}_i}$ whose degree is at most m. We prove that a subsequence of $(V_{\bar{A}_i,\bar{B}_i})$ converges to a non zero polynomial function V (of degree at most m) which admits Γ as a level set. Since Γ is not analytic, a contradiction is reached.

Remark **6** The result given by Theorem 2 generalizes to dimensions higher than 2 as follows. Let (A_i, B_i), $i \geq 1$, be a sequence of 2×2 matrices such that **i)** the corresponding systems of type (4) are **GUES**, **ii)** the limit is uniformly stable but not attractive. As explained above, for this sequence of systems it is not possible to build a sequence of polynomial CLF of uniformly bounded degree. Consider now the sequence of systems in $\mathbb{R}^n$, $n \geq 2$, of the form $\dot{\bar{x}} = u\bar{A}_i\bar{x} + (1-u)\bar{B}_i\bar{x}$ corresponding to the matrices:

$$\bar{A}_i = \left(\begin{array}{c|cccc} A_i & & 0 & & \\ \hline & -1 & \dots & \dots & \vdots \\ 0 & 0 & -1 & \dots & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & \dots & \dots & -1 \end{array}\right), \quad \bar{B}_i = \left(\begin{array}{c|cccc} B_i & & 0 & & \\ \hline & -1 & \dots & \dots & \vdots \\ 0 & 0 & -1 & \dots & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & \dots & \dots & -1 \end{array}\right). \tag{5}$$

Each system of the sequence is **GUES** but the limit system is not (it is just uniformly stable). Now, if $V_{\bar{A}_i,\bar{B}_i}$, $i \geq 1$, are the corresponding polynomial CLFs, then they cannot be polynomials of uniformly bounded degree since this is not true for the restriction of $V_{\bar{A}_i,\bar{B}_i}$ to the first two variables.

Remark **7 (extension to piecewise polynomial functions (PPF)).** Another class of functions commonly used to check **GUES** is that of piecewise quadratic functions or more generally piecewise polynomial functions (PPF for short). Here, by a PPF, we mean a continuous function $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ together with a finite number q of cones K_j, $1 \leq j \leq q$, based at zero and partitioning $\mathbb{R}^n$ so that V is a polynomial function of degree d_j on K_j, $1 \leq j \leq q$. We refer to $m(V) := \max\{q, d_1, ..., d_q\}$ as the total degree of V.

It is tempting to state a version of problem **P** by replacing polynomial functions of degree at most m with PPFs of total degree at most m.

Again, the PPF version of problem **P** does not have a solution for $n = 2$, i.e. the minimum total degree of a piecewise polynomial CLF cannot be uniformly bounded over the set of all **GUES** system of the form (4). The argument is a simple extension of the proof of Theorem 2 and it is briefly mentioned in Remark 13.

The last results of the paper concern the existence and the characterization of a finite-**SSF** for systems of the type (3) subject to **H0**.

Let us define the convex semicone generated by a set $D \subset \mathbb{R}^n$ as the set of points λx with $\lambda > 0$ and $x \in co(D)$, where $co(D)$ denotes the convex hull of the set D. With this definition, the point $x = 0$ does not belong to the convex semicone generated by a set D, if $0 \notin co(D)$.

First of all, we prove the following (see Appendix B for the argument).

Proposition 1 *For every compact subset* $\mathbf{A}$ *of* $\mathbb{R}^{n\times n}$ *(i.e. verifying* **H0***), let* $S_{\mathbf{A}}$ *be the system of the type (3) associated to* $\mathbf{A}$*. Then, for* $\mathbf{A}$ *and* $\mathbf{A}'$ *verifying* **H0** *and generating the same convex semicone,* $S_{\mathbf{A}}$ *is* **GUES** *(resp. uniformly stable) if and only if* $S_{\mathbf{A}'}$ *is* **GUES** *(resp. uniformly stable).*

Based on converse Lyapunov theorems, one can deduce some trivial existence results for finite-**SSF**s. For instance, consider a class C of systems of type (3) in $\mathbb{R}^n$ subject to **H0** and satisfying the following property: for every $\mathbf{A} \in C$, the convex hull of $\mathbf{A}$ is generated by at most k matrices $n \times n$, where k is a positive integer. One can build a finite-**SSF** for the class C as follows. If C does not admit any **GUES** system, the positive definite function $x \mapsto \|x\|^2$ will detect the instability of every element of C. In this case, the subset of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ reduced to $x \mapsto \|x\|^2$ is a finite-**SSF** for C. Otherwise, C admits at least one **GUES** system. Assume first that a finite-**SSF** $\mathcal{S}$ can be provided for the **GUES** systems of C and let $V \in \mathcal{S}$ be a CLF for a fixed **GUES** system of C. By associating V to every unstable system of C, it is clear that $\mathcal{S}$ becomes a finite-**SSF** for the whole class C.

Therefore, we may simply assume that C is made of **GUES** systems. Thanks to Proposition 1, the class C can be parameterized by k-tuples of $n \times n$ matrices, defined up to their norm. In this way, a $k(n^2 - 1)$)-parameters finite-**SSF** is provided for the class C. For instance, the class Ξ of two-dimensional systems of type (4) admits a 6-parameters finite-**SSF**.

The above construction is not explicit and therefore is not useful to check **GUES**. Similarly to Lyapunov functions, it is then clear that the real challenge for finite-**SSF**s concerns their explicit characterization. For classes of systems of type (3) in $\mathbb{R}^n$, $n \geq 3$, that issue is completely open in general. In dimension two, using the necessary and sufficient conditions for **GUES** given in [7], we provide an explicit 5-parameters finite-**SSF** for Ξ. This is the content of Section 5.

Clearly, Ξ can be parameterized by the pairs (A, B) of 2×2 real Hurwitz matrices, where both A and B are defined up to their norm. The construction of the explicit finite-**SSF** goes as follows. As done previously, we may assume that the pair (A, B) gives rise to a **GUES** system. By taking advantage of the complete characterization of **GUES** systems of the class Ξ given in [7], one can explicitly associate to every **GUES** pair (A, B) a CLF as explained next. We start by defining, from (A, B), a pair $(\tilde{A}, \tilde{B})$ giving rise to a system of Ξ which is uniformly stable but not attractive. Such a system admits a closed trajectory whose support Γ is a simple Jordan closed curve (cf. Sections 3, 4). We then construct a homogeneous positive definite function V whose level set 1 is Γ. We finally show that V is a CLF for (A, B). Since the set of $(\tilde{A}, \tilde{B})$ built from the **GUES** pairs (A, B) can be parameterized by using five parameters, we end up with a five-parameters finite-**SSF**.

1.4 Structure of the Paper

Section 2 contains the proof of Theorem 1. In Section 3, we recall the main ideas from [7] needed for the rest of the paper. For sake of completeness, we provide in Appendix A the full statement of the main result of [7]. The proof of Theorem 2 is given in Section 4 and the explicit construction of a five-parameters finite-**SSF** for systems of type (4) is provided in Section 5. Finally, in Appendix B, we prove Proposition 1.

2 Existence of Common Polynomial Lyapunov Functions

In this section, we prove Theorem 1. The starting point of the argument follows the first part of the proof of an analogous result in [10].

We define the function $V : \mathbb{R}^n \to \mathbb{R}^+$ by:

$$V(x) = \sup_{A(.)\in \mathcal{A}} \int_0^{+\infty} \|\phi_t^{A(.)}(x)\|^2 \mathrm{d}t \ .$$

The function V is well defined since there exist positive constants C, μ such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$:

$$\|\phi_t^{A(.)}(x)\| \leq C\, e^{-\mu t}\|x\|.$$

Note that V is homogeneous of degree 2 and continuous. In addition, we next show that V is strictly convex. That fact will be crucial later in the argument. Fix $x, y \in \mathbb{R}^n$ and $x \neq y$. Let $A(.)$ be a switching function. The function $x \mapsto \|\phi_t^{A(.)}(x)\|^2$ is strictly convex. Moreover, for every $\lambda \in]0, 1[$, by compactness of $\mathbf{A}$, the expression:

$$\lambda\, \|\phi_t^{A(.)}(x)\|^2 + (1 - \lambda)\, \|\phi_t^{A(.)}(y)\|^2 - \|\phi_t^{A(.)}(\lambda x + (1 - \lambda)y)\|^2,$$

is nonnegative for every $t \geq 0$ and is bounded from below by a positive constant on some interval $[0, \bar{t}]$, uniformly with respect to $A(.)$.
Therefore, dividing the integration interval into the two intervals $[0, \bar{t}]$ and $[\bar{t}, +\infty]$ and taking a maximizing sequence of switching functions for $V(\lambda\, x + (1 - \lambda)\, y)$, we have:

$$V(\lambda\, x + (1 - \lambda)\, y) < \lambda\, V(x) + (1 - \lambda)\, V(y), \quad \forall \lambda \in]0, 1[, \quad \forall x, y \in \mathbb{R}^n \ .$$

It is shown in [10] that V is a CLF. Nevertheless, we need to consider at least $\mathcal{C}^1$ Lyapunov functions, therefore we define:

$$\tilde{V}(x) = \int_{SO(n)} f(R)\, V(Rx)\, \mathrm{d}R \, .$$

where $f : SO(n) \longrightarrow [0, +\infty[$ is a smooth function with support on a small neighborhood of the identity matrix and $\int_{SO(n)} f(R)\, \mathrm{d}R = 1$.
In [10], it is also shown that $\tilde{V}$ is a smooth CLF except at the origin. Moreover, since V is homogeneous of degree 2 and strictly convex, it follows that $\tilde{V}$ also satisfies such properties.

We consider now the function $W(x) = \sqrt{\tilde{V}(x)}$, which is a continuous, positively homogeneous CLF. Therefore, $W^{-1}(1)$ is a compact set. Using the fact that the set $\{x : W(x) < 1\}$ is strictly convex, we construct a polynomial CLF $\tilde{W}$ by approximating the level sets of W. For this purpose, we need the following preliminary result which describes a continuity property of the function $\nabla W(y) \cdot Dx$ with respect to x, y, D.

Lemma 1 *Let us set:*

$$M := \min_{x \in W^{-1}(1)\ D \in \mathbf{A}} \big[-\nabla W(x) \cdot Dx \big].$$

Then, for every $\varepsilon \in (0, M)$, there exists $\delta \in (0,1)$ such that, for every $x, y \in W^{-1}(1)$ with $\nabla W(y) \cdot x > 1 - \delta$ and every $D \in \mathbf{A}$, one has:

$$\nabla W(y) \cdot Dx < -\varepsilon.$$

Proof of the Lemma. First of all, notice that M is well defined since it is the infimum of a continuous function over a compact set. Moreover, $M > 0$ because W is a CLF.
Since, by homogeneity, $\nabla W(y) \cdot y = W(y) = 1$, we have:

$$\nabla W(y) \cdot x = 1 - \nabla W(y) \cdot (y - x),$$

and then the hypothesis is equivalent to $\nabla W(y) \cdot (y - x) < \delta$.
Reasoning by contradiction, assume that there exists a sequence (x_j, y_j, D_j) such that $\nabla W(y_j) \cdot D_j x_j \geq -\varepsilon$ and $\nabla W(y_j) \cdot (y_j - x_j)$ converges to 0 as j goes to infinity. By compactness, we can find a subsequence of (x_j, y_j, D_j) converging to $(\bar{x}, \bar{y}, \bar{D})$ and therefore, by continuity, $\nabla W(\bar{y}) \cdot \bar{D}\bar{x} \geq -\varepsilon$ and $\nabla W(\bar{y}) \cdot (\bar{y} - \bar{x}) = 0$. Therefore $\bar{y} - \bar{x}$ belongs to the tangent space at $\bar{y}$ of the strictly convex set $W^{-1}([0,1])$. Since $\bar{x}$ also belongs to the boundary of that set, it must be $\bar{y} = \bar{x}$. It implies $\nabla W(\bar{y}) \cdot \bar{D}\bar{x} = \nabla W(\bar{x}) \cdot \bar{D}\bar{x} \leq -M$ and we reach a contradiction. □

Remark **8** Taking $-x$ instead of x, one obtains that for every $x, y \in W^{-1}(1)$ and every $D \in \mathbf{A}$, then $\nabla W(y) \cdot x < -1 + \delta \Longrightarrow \nabla W(y) \cdot Dx > \varepsilon$.

To conclude the proof of the theorem, we take $\delta \in (0,1)$ corresponding to some ε as in the lemma above, and for every $y \in W^{-1}(1)$ we consider the open sets $B_y = \big\{x \in \mathbb{R}^n : \nabla W(y) \cdot x > 1 - \delta/2\big\}$. Since $y \in B_y$, we have that $\{B_y\}_{y \in W^{-1}(1)}$ is an open covering of the compact set $W^{-1}(1)$, and therefore we can find $y_1, \ldots, y_N$ points of $W^{-1}(1)$ such that the union of B_{y_k}, $k = 1, \ldots, N$, covers $W^{-1}(1)$.
Let us define:

$$\tilde{W}(x) := \sum_{k=1}^{N} (\nabla W(y_k) \cdot x)^{2p} .$$

We claim that, for an integer p large enough, $\tilde{W}$ is a polynomial CLF. For $D \in \mathbf{A}$ and $x \in \mathbb{R}^n$, $x \neq 0$, we have:

$$\nabla \tilde{W}(x) \cdot Dx = 2p \sum_{k=1}^{N} \big(\nabla W(y_k) \cdot x\big)^{2p-1} \nabla W(y_k) \cdot Dx, \tag{6}$$

and we want to show that $\nabla \tilde{W}(x) \cdot Dx < 0$. By homogeneity, it is enough to do it for $x \in W^{-1}(1)$. Set:

$$K := \max_{x, y \in W^{-1}(1),\, D \in \mathbf{A}} \nabla W(y) \cdot Dx.$$

If, for some index k in $\{1, ..., N\}$, one has $|\nabla W(y_k) \cdot x| \leq 1 - \delta$. Then:

$$|\big(\nabla W(y_k) \cdot x\big)^{2p-1} \nabla W(y_k) \cdot Dx| \leq (1 - \delta)^{2p-1} K.$$

Otherwise, if the inequalities $1 - \delta/2 \geq |\nabla W(y_k) \cdot x| > 1 - \delta$ hold, then, by the previous lemma and remark, one has that the corresponding term in the summation must be negative.
Finally, since by the definition of the points y_k, there exist at least two distinct indices k_1 and k_2 such that $x \in B_{y_{k_1}}$ and $-x \in B_{y_{k_2}}$ we have that:

$$\big(\nabla W(y_{k_i}) \cdot x\big)^{2p-1} \nabla W(y_{k_i}) \cdot Dx < -(1 - \delta/2)^{2p-1} \varepsilon.$$

Summing up, we deduce that:

$$\nabla\tilde{W}(x)\cdot Dx < 2p\Big(-2(1-\delta/2)^{2p-1}\varepsilon+(N-2)(1-\delta)^{2p-1}K\Big) = -4p(1-\delta/2)^{2p-1}\varepsilon\Big(1-\frac{K(N-2)}{2\varepsilon}\Big(\frac{1-\delta}{1-\delta/2}\Big)^{2p-1}\Big).$$

For p large enough, the right-hand side of previous expression is negative, uniformly with respect to $D \in \mathbf{A}$ and $x \in W^{-1}(1)$. The theorem is proved.

Remark **9** One can also check that the level set $\tilde{W}^{-1}(1)$ approximates, as p tends to $+\infty$, the corresponding level set of the function $\max_{k=1,\ldots,N} |\nabla W(y_k)\cdot x|$ (which is a polytope) and, therefore, the latter is a CLF as well (cf. [6]).

3 Necessary and Sufficient Conditions for GUES of Bidimensional Systems

Consider the following property:

($\mathcal{P}$) The bi-dimensional switched system given by:

$$\dot{x}(t) = u(t)Ax(t) + (1-u(t))Bx(t), \text{ where } u(.) : [0,\infty[\to [0,1], \tag{7}$$

is **GUES** at the origin.

In this section, together with Appendix A, we recall the main ideas from [7], to get a necessary and sufficient condition on A and B under which ($\mathcal{P}$) holds, or under which we have at least uniform stability. The full statement of the Theorem is reported in Appendix A.1.

Remark **10** Recall that, by Proposition 1 (proved in Appendix (B)), the necessary and sufficient condition for stability of the system (7) are the same if we assume $u(.)$ taking values in $\{0,1\}$ or in $[0,1]$, or if we multiply A and B by two arbitrary positive constants.

Set $M(u) := uA + (1-u)B$, $u \in [0,1]$. In the class of constant functions the asymptotic stability of the origin of the system (7) occurs if and only if the matrix $M(u)$ has eigenvalues with strictly negative real part for each $u \in [0,1]$. So this is a necessary condition for **GUES**. On the other hand it is know that if $[A,B] = 0$ then the system (7) is **GUES**. So, in what follows, we always assume the conditions:

H1: Let λ_1, λ_2 (resp. λ_3, λ_4) be the eigenvalues of A (resp. B). Then $Re(\lambda_1)$, $Re(\lambda_2)$, $Re(\lambda_3)$, $Re(\lambda_4) < 0$.

H2: $[A,B] \neq 0$ (that implies that neither A nor B is proportional to the identity).

For simplicity we will also assume:

H3: A and B are diagonalizable in $\mathbb{C}$ (notice that if **H2** and **H3** hold then $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$).

H4: Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{C}P^1$ (resp. $\mathbf{V}_3, \mathbf{V}_4 \in \mathbb{C}P^1$) be the eigenvectors of A (resp. B). Then $\mathbf{V}_i \neq \mathbf{V}_j$ for $i \in \{1,2\}$, $j \in \{3,4\}$ (notice that, from **H2** and **H3**, the V_i's are uniquely defined, $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$, and **H4** can be violated only when both A and B have real eigenvalues).

All the other cases in which **H1** and **H2** hold are the following:

- A or B are not diagonalizable. This case (in which ($\mathcal{P}$) can be true or false) can be treated with techniques entirely similar to the ones of [7].
- A or B are diagonalizable, but one eigenvector of A coincides with one eigenvector of B. In this case, using arguments similar to those of [7], it possible to conclude that ($\mathcal{P}$) is true.

We will call respectively **(CC)** the case where both matrices have non-real eigenvalues, **(RR)** the case where both matrices have real eigenvalues and finally **(RC)** the case where one matrix has real eigenvalues and and the other non-real eigenvalues.

Theorem 3, reported in Appendix A.1, gives necessary and sufficient conditions for the stability of the system (7) in terms of three (coordinates invariant) parameters given below in Definition 4. The first two parameters, ρ_A and ρ_B, depend on the eigenvalues of A and B respectively, and the third parameter $\mathcal{K}$ depends on $Tr(AB)$, which is a Killing-type pseudo-scalar product in the space of 2×2 matrices. As explained in [7], the parameter $\mathcal{K}$ contains the inter-relation between the two systems $\dot{x} = Ax$ and $\dot{x} = Bx$, and it has a precise geometric meaning. It is in 1–1 correspondence with the cross ratio of the four points in the projective line $\mathbb{C}P^1$ that corresponds to the four eigenvectors of A and B.

Definition 4 *Let A and B be two 2×2 real matrices and suppose that* **H1**, **H2**, **H3** *and* **H4** *hold. Moreover choose the labels $(_1)$ and $(_2)$ (resp. $(_3)$ and $(_4)$) so that $|\lambda_2| > |\lambda_1|$ (resp. $|\lambda_4| > |\lambda_3|$) if they are real or $Im(\lambda_2) < 0$ (resp. $Im(\lambda_4) < 0$) if they are complex. Define:*

$$\rho_A := -i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}; \qquad \rho_B := -i\frac{\lambda_3 + \lambda_4}{\lambda_3 - \lambda_4}; \qquad \mathcal{K} := 2\frac{Tr(AB) - \frac{1}{2}Tr(A)Tr(B)}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}.$$

Moreover, define the following function of $\rho_A, \rho_B, \mathcal{K}$:

$$\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1 + \rho_A^2 + \rho_B^2). \tag{8}$$

Notice that ρ_A is a positive real number if and only if A has non-real eigenvalues and $\rho_A \in i\mathbb{R}$, $\rho_A/i > 1$ if and only if A has real eigenvalues. The same holds for B. Moreover $\mathcal{D} \in \mathbb{R}$.

Remark **11** Under hypotheses **H1** to **H4**, using a suitable 3-parameter changes of coordinates, it is always possible to put the matrices A and B, up the their norm, (cf. Remark 10) in the normal forms given in Appendix A.2, where $\rho_A, \rho_B, \mathcal{K}$ appear explicitly (see [7] for more details).

The parameter $\mathcal{K}$ contains important information about the matrices A and B. They are stated in the following Proposition that can be easily proved using the normal forms given in Appendix A.2.

Proposition 2 *Let A and B be as in Definition 4. Then:* **i)** *if A and B have both complex eigenvalues, then $\mathcal{K} \in \mathbb{R}$ and $|\mathcal{K}| > 1$;* **ii)** *if A and B have both real eigenvalues, then $\mathcal{K} \in \mathbb{R} \setminus \{\pm 1\}$;* **iii)** *$A$ and B have one complex and the other real eigenvalues if and only if $\mathcal{K} \in i\mathbb{R}$.*

Theorem 3, stated in Appendix A.1, is the main result of [7], and gives necessary and sufficient conditions for $(\mathcal{P})$ holding true. We next describe the main idea of the proof. All details can be found in [7].

We build the "worst trajectory" γ_{x_0} i.e. the trajectory (based at x_0) having the following property. At each time t, $\dot{\gamma}_{x_0}(t)$ forms the smallest angle (in absolute value) with the (exiting) radial direction (Figure 1 A).

Then the system (7) is **GUES** if and only if, for each $x_0 \in \mathbb{R}^2$, the "worst trajectory" γ_{x_0} tends to the origin. The worst trajectory is constructed as follows. We study the locus $Q^{-1}(0)$ (where $Q(x) := \det(Ax, Bx)$) where the two vector fields Ax and Bx are collinear. The quantity $\mathcal{D}$, defined in Definition 4, is proportional to the discriminant of the quadratic form Q. We have several cases:

- If $Q^{-1}(0)$ contains only the origin then, in the **(CC)** and **(RC)** case, one vector field points always on the same side of the other and the worst trajectory is a trajectory of a fixed vector field (either Ax or Bx). In that case, the system is **GUES** (case **(CC.1)** and **(RC.1)** of Theorem 3), see Figure 1, case B. The situation is similar in case **(RR.1)** (the worst trajectory tends to the origin).

- If $Q^{-1}(0)$ does not contain only the origin then it is the union of two lines passing through the origin (since Q is a quadratic form). If at each point of $Q^{-1}(0)$, the two vector fields have opposite direction, then there exists a trajectory going to infinity corresponding to a constant switching function (see Figure 1, case C). This correspond to cases **(CC.2.1)**, **(RC.2.1)** and **(RR.2.1)** of Theorem 3. In that situation, there exists $u \in [0, 1]$ such that the matrix $M(u) := uA + (1 - u)B$, $u \in [0, 1]$ admits an eigenvalue

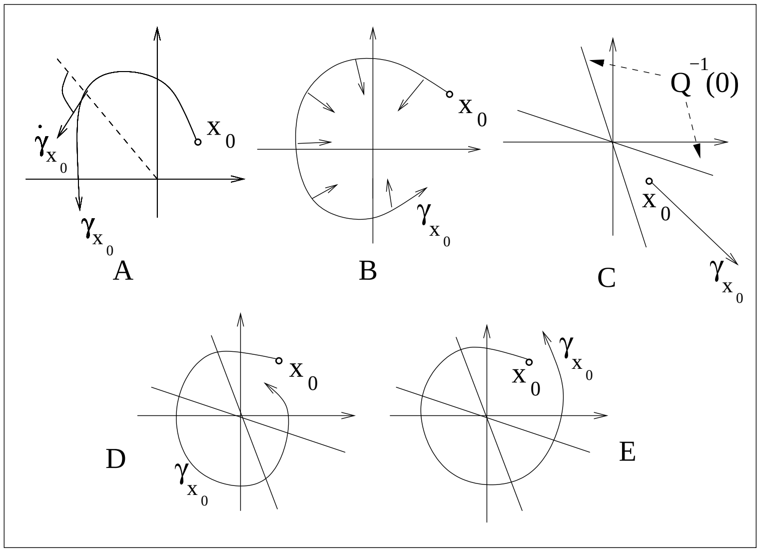

Figure 1: Proof of the stability conditions

with positive real part. If at each point of $Q^{-1}(0)$, the two vector fields have the same direction, then the system is **GUES** if and only if the worst trajectory turns around the origin and after one turn the distance from the origin is decreased. (see Figure 1, cases D and E). The quantities $\rho_{CC}, \rho_{RC}, \rho_{RR}$ defined in Theorem 3 (for the three cases **(CC)**, **(RC)**, **(RR)** resp.) represent the distance from the origin of the worst trajectory (that at time zero is at distance 1), after one half turn. This correspond to cases **(CC.2.2)**, **(RC.2.2)** and **(RR.2.2)** of Theorem 3.

- Finally **(CC.3)**, **(RC.3)** and **(RR.3)** are the degenerate cases in which the two straight lines coincide.

4 Non Existence of a Uniform Bound on the Minimal Degree of Polynomial Lyapunov Functions

In this section, we prove Theorem 2. The starting point of the argument is to consider a pair of matrices A and B having both non real eigenvalues (**(CC)** case) and satisfying:

$$\mathcal{D} > 0, \quad \mathcal{K} > 1, \quad \rho_{CC} = 1. \tag{9}$$

Such a pair exists. Indeed, Figure 2 translates graphically the contents of Theorem 3 for a fixed $\mathcal{K} > 1$, in the region of the (ρ_A, ρ_B)–plane where $\rho_A, \rho_B > 0$. The open shadowed region corresponds to values of the parameters ρ_A, ρ_B for which the system is GUES. We denote by S^+ the open subset of the shadowed region where $\mathcal{D} > 0$. The curve C represents the limit case where $\rho_{CC} = 1$. To each internal point of that curve, it is associated a system verifying (9), since $\mathcal{D} > 0$. A system corresponding to such a limit case is not asymptotically stable but just stable. Moreover, the worst trajectory is a periodic curve, whose support is of class $\mathcal{C}^1$ but not of class $\mathcal{C}^2$ (recall that the switchings occur on $Q^{-1}(0)$, i.e. when the linear vector fields corresponding to A and B are parallel).

Fix a point $(\rho_A, \rho_B) \in C$ corresponding to (A, B). From the picture, it is clear that there exists a sequence of points $(\rho_{Ak}, \rho_{Bk}) \in S^+$, for $k \geq 1$, converging to (ρ_A, ρ_B). This exactly means that there exists a sequence of **GUES** pairs (A_k, B_k), $k \geq 1$, such that (A_k, B_k) tends to (A, B) as k goes to ∞.

Let $x = (x_1, x_2)$. For every $k \geq 1$, consider a polynomial CLF $V_k = \sum_{1 \leq i+j \leq m_k} a_{ij}^{(k)} x_1^i x_2^j$ of degree at most m_k. Arguing by contradiction, we assume that the sequence (m_k) is bounded by a positive integer m.

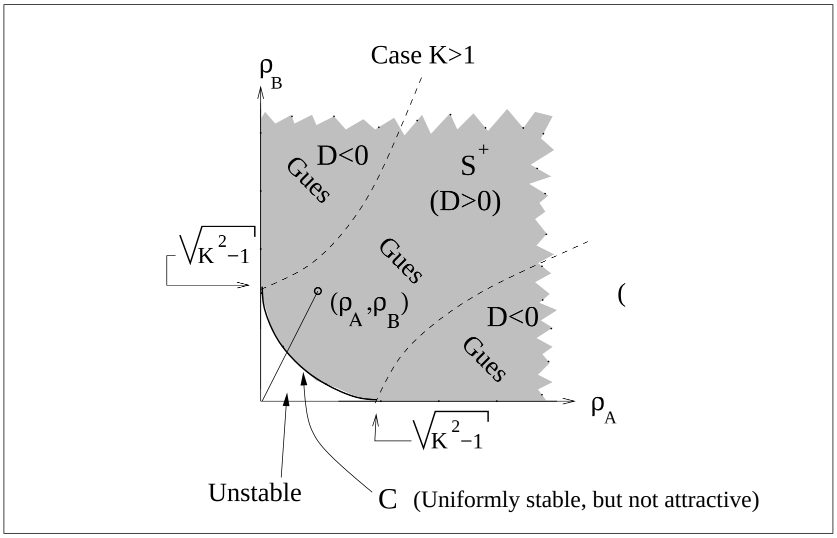

Figure 2: **GUES** property in the space of parameters and explicit construction of a 5-parameters **SSF** for systems of type (4). S^+ is the region in the (ρ_A, ρ_B)–plane in which the system is **GUES** and $\mathcal{D} > 0$.

Up to multiplication by a constant, we can choose $\sum_{1\leq i+j\leq m_k} |a_{ij}^{(k)}| = 1$. By compactness, there exists a subsequence of (V_k) (still denoted by (V_k)) which converges (uniformly on compact subsets of $\mathbb{R}^2$) to some non-zero polynomial V with degree at most m. Note that $V(0) = 0$ since the V_k's are CLFs.

Fix $x_0 \in \mathbb{R}^2$, $x_0 \neq 0$. Let $T > 0$ be the period of the worst trajectory γ_{x_0} corresponding to the pair (A, B), and starting at x_0 . Note that T is independent of x_0. The curve $\gamma_{x_0} : [0, T] \to \mathbb{R}^2$ can be seen as the concatenation of at most five arcs of integral curves of $\dot{x} = A\,x$ and $\dot{x} = B\,x$ (see Figure 3) and satisfies the Cauchy problem:

$$\begin{cases} \dot{x} = C(t)\,x, \\ x(0) = x_0, \end{cases}$$

where $C(t)$ is equal to A or B on subintervals of $[0, T]$.
For $k \geq 1$, consider the Cauchy problem:

$$\begin{cases} \dot{x} = C_k(t)\,x, \\ x(0) = x_0, \end{cases}$$

where $C_k(t) = A_k$ if $C(t) = A$ and $C_k(t) = B_k$ if $C(t) = B$. Then, γ_k is a trajectory of the switched system of the type (4) associated to (A_k, B_k). Since, the right-hand side of the previous equation is Lipschitz continuous in x and piecewise continuous in t, then the solutions γ_k converge uniformly to γ_{x_0} on $[0, T]$.
We next show that V remains constant on γ_{x_0}. For $k \geq 1$ and $t \in [0, T]$, one has:

$$\|V_k \circ \gamma_k(t) - V \circ \gamma_{x_0}(t)\| \leq \|V_k \circ \gamma_k(t) - V \circ \gamma_k(t)\| + \|V \circ \gamma_k(t) - V \circ \gamma_{x_0}(t)\|.$$

By uniform convergence of V_k to V and of γ_k to γ_{x_0}, and by continuity of V, we deduce that $V_k \circ \gamma_k(t)$ converges to $V \circ \gamma_{x_0}(t)$ for every fixed t.
Since, for every $k \geq 1$, V_k is a CLF for the switched system of the type (4) associated to (A_k, B_k), then $V_k \circ \gamma_k$ is a decreasing function and, hence, $V \circ \gamma_{x_0}$ is non-increasing. Moreover $V \circ \gamma_{x_0}(T) = V \circ \gamma_{x_0}(0)$. Therefore, $V \circ \gamma_{x_0}$ must be constant. It implies that there exists $t_1 > 0$ such that either $V(e^{At}x_0)$ or $V(e^{Bt}x_0)$ is constant on $[0, t_1]$. With no loss of generality, assume the first alternative. Since the map $t \mapsto V(e^{At}x_0)$ is real analytic, it follows that $V(e^{At}x_0)$ is constant over the whole real line. By letting t go to $+\infty$, since $e^{At}x_0 \to 0$, we deduce that $V(x_0) = V(0) = 0$. Since x_0 is an arbitrary non zero point of $\mathbb{R}^2$, we get that $V \equiv 0$, which is not possible.

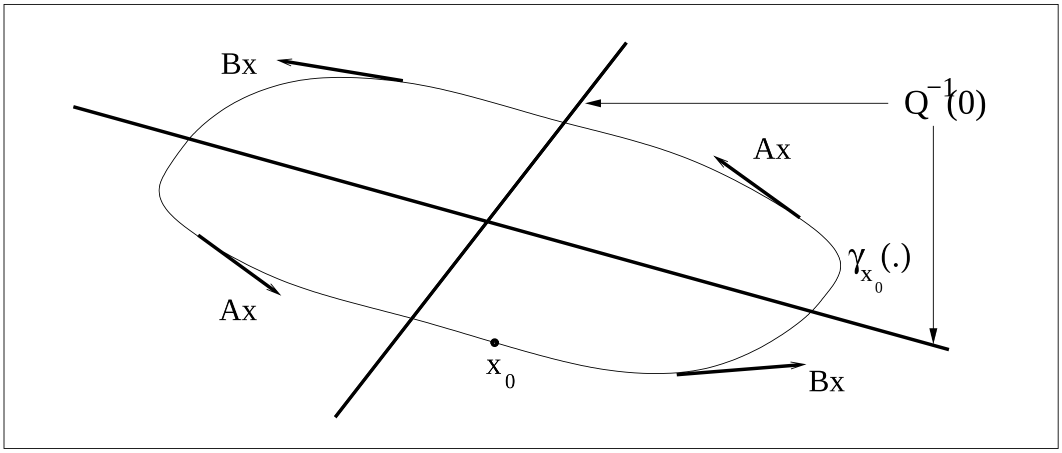

Figure 3: The "worst trajectory"

Remark **12** The construction of the sequence (A_i, B_i) with unbounded degree for polynomial CLF was performed for matrices having both non real eigenvalues (that corresponds to the **(CC)** case). The same construction can be reproduced for the **(RC)** and **(RR)** cases.

Remark **13** . For the PPF case (see Remark 7), the above argument can be easily modified to get that the minimum total degree of a piecewise polynomial CLF cannot be uniformly bounded over the set of all **GUES** systems of the form Ξ (cf. Formula 4). Indeed, let V_k be the sequence of PPFs taking the value $V_k^l(x, y) = \sum_{1\leq i+j\leq m} a_{ijl}^{(k)} x_1^i x_2^j$ in the cone K_l^k, for $1 \leq l \leq m$. Here, to simplify the notation, we assume without loss of generality, that, for each element of the sequence, the number of cones and the degree of $V_k^l(x, y)$ is always m.

Each cone can be identified by a couple of angles with the x-direction. Therefore to each function V_k we can associate a m-uple of angles $(\alpha_1^k, \ldots, \alpha_m^k)$ such that the cone K_l^k coincides with the region between the lines corresponding to α_l^k and α_{l+1}^k. In particular, up to subsequences, we can assume that the numbers α_l^k converge to α_l. Similarly to the case above, we can normalize the coefficients of the CLFs V_k by $\sum_{l=1}^m \sum_{1\leq i+j\leq m} |a_{ijl}^{(k)}| = 1$ and consider a subsequence of the coefficients converging to a_{ijl}. Then, if we define V as the PPF such that $V(x, y) = V^l(x, y) = \sum_{1\leq i+j\leq m} a_{ijl} x_1^i x_2^j$ on the cone K_l defined by the angles α_l and α_{l+1}, it is easy to verify that $V_k(x, y)$ converges uniformly on compact subsets of $\mathbb{R}^2$ to $V(x, y)$. We can conclude the proof as before showing that $V_l(x_0) = V(0) = 0$ for arbitrary x_0, which leads to a contradiction.

5 Explicit Construction of a finite-SSF for Systems of Type (4)

In this section, we provide a 5-parameters finite-**SSF** for the class Ξ of bidimensional systems of type (4). Recall that for what concern the stability issue, Ξ can be parameterized by the 6-parameters family provided by the pairs (A, B) (of 2×2 matrices) defined up to their norm.

As explained in the introduction, it is enough to construct a CLF for a pair (A, B) giving rise to a **GUES** system of Ξ. We only treat the **(CC.2.2)** case since, in all the other cases, the construction is entirely similar.

In the **(CC)** case, after a three-parameters change of coordinates, the normal form for the pair (A, B) is given by (cf. Appendix A.2):

$$A = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix}, \qquad E > 0.$$

Moreover, in the **(CC.2.2)** case, we have $\mathcal{K} > 1$, $\mathcal{D} > 0$ and $\rho_{CC} < 1$, where $\mathcal{K} := 1/2(E + 1/E)$, $\mathcal{D}$ and ρ_{CC} being respectively defined in (4) and (11). Recall that, for fixed $\mathcal{K} > 1$ (i.e fixed $E > 1$), Figure 2 describes, in the (ρ_A, ρ_B)-plane, the status of each point with respect to the **GUES** issue.

We now associate, to every **GUES** pair (A,B), a pair $(\tilde{A},\tilde{B})$ corresponding to a system of the type (4) uniformly stable but not attractive. Consider in Figure 2 the line segment joining the point $(0,0)$ to (ρ_A,ρ_B) in the S^+ region. That segment intersects the curve C in a point $(\tilde{\rho}_A,\tilde{\rho}_B)$. That results from the Jordan separation theorem and the fact that C connects the points $(\sqrt{\mathcal{K}^2-1},0)$ and $(0,\sqrt{\mathcal{K}^2-1})$. Therefore, there exists a $\zeta\in(0,1)$ such that, for the system given by:

$$\tilde{A}=\begin{pmatrix} -\tilde{\rho}_A & -1/E \\ E & -\tilde{\rho}_A \end{pmatrix}, \qquad \tilde{B}=\begin{pmatrix} -\tilde{\rho}_B & -1 \\ 1 & -\tilde{\rho}_B \end{pmatrix}, \qquad \tilde{\rho}_A=\rho_A\zeta, \qquad \tilde{\rho}_B=\rho_B\zeta,$$

the worst trajectories γ_{x_0} are closed curves, i.e. $\rho_{CC}=1$.
Moreover, one can easily compute:

$$\det(\tilde{A}x,Ax)=\rho_A(1-\zeta)\left({x_1}^2E+\frac{{x_2}^2}{E}\right)>0\ ,$$

$$\det(\tilde{B}x,Bx)=\rho_B(1-\zeta)|x|^2>0\ .$$

Therefore the vector fields Ax, Bx point inside the area delimited by a fixed worst trajectory (that is closed curve) of the modified switched system and so, passing to angular coordinates, the function:

$$V(r,\alpha)=\frac{r}{\tilde{r}(\alpha)}\ , \tag{10}$$

where $\tilde{r}(\alpha)$ is a parameterization of the fixed worst trajectory, is a CLF for the system defined by (A,B).
Hence, we have provided a 5-parameter **SSF** in the **(CC.2.2)** case. The five parameters are: $\mathcal{K}$, the ratio ρ_B/ρ_A, and the three parameters involved in the change of coordinates to get the normal forms (15), (16).

<u>*Remark*</u> **14** Notice that, in the cases **(CC.1)** and **(CC.2.1)** (cf. Section 3 and Theorem 3 in Appendix A.1), one can choose as **SSF** the set of quadratic polynomials, which actually is parameterized by two parameters.

<u>*Remark*</u> **15** Let us come back to the general system (3), subject to **H0**. Notice that the question of finding the smallest m such that there exists a m-parameters finite-**SSF**, for a certain class C of systems, has no real meaning if one does not require suitable conditions on the map Ψ in Definition 3. Indeed, it is always possible to build a countable **SSF** for the class of systems of type (3) in $\mathbb{R}^n$ subject to **H0**.

A Stability Conditions for bidimensional Systems

A.1 Statement of the Stability Conditions

Theorem 3 *Let A and B be two real matrices such that* **H1**, **H2**, **H3** *and* **H4**, *given in Section 3, hold and define $\rho_A,\rho_B,\mathcal{K},\mathcal{D}$ as in Definition 4. We have the following stability conditions:*

Case (CC) *If A and B have both complex eigenvalues then:*

Case (CC.1) *if $\mathcal{D}<0$ then $(\mathcal{P})$ is true;*

Case (CC.2) *if $\mathcal{D}>0$ then:*

Case (CC.2.1) *if $\mathcal{K}<-1$ then $(\mathcal{P})$ is false;*

Case (CC.2.2) *if $\mathcal{K}>1$ then $(\mathcal{P})$ is true if and only if it holds the following condition:*

$$\begin{aligned}\rho_{CC} \ :=\ & \exp\left[-\rho_A\arctan\left(\frac{-\rho_A\mathcal{K}+\rho_B}{\sqrt{\mathcal{D}}}\right)-\right. \\ & \left.\rho_B\arctan\left(\frac{\rho_A-\rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right)-\frac{\pi}{2}(\rho_A+\rho_B)\right]\times \\ & \times\sqrt{\frac{(\rho_A\rho_B+\mathcal{K})+\sqrt{\mathcal{D}}}{(\rho_A\rho_B+\mathcal{K})-\sqrt{\mathcal{D}}}}<1\end{aligned} \tag{11}$$

Case (CC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true or false whether $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC) *If A and B have one of them complex and the other real eigenvalues, define $\chi := \rho_A \mathcal{K} - \rho_B$, where ρ_A and ρ_B are chosen in such a way $\rho_A \in i\mathbb{R},\ \rho_B \in \mathbb{R}$. Then:*

Case (RC.1) *if $\mathcal{D} > 0$ then $(\mathcal{P})$ is true;*

Case (RC.2) *if $\mathcal{D} < 0$ then $\chi \neq 0$ and we have:*

Case (RC.2.1) *if $\chi > 0$ then (P) is false. Moreover in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2) *if $\chi < 0$, then:*

Case (RC2.2.A) *if $\mathcal{K}/i \leq 0$ then $(\mathcal{P})$ is true;*

Case (RC2.2.B) *if $\mathcal{K}/i > 0$ then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\rho_{RC} := \Big(\frac{m^+}{m^-}\Big)^{-\frac{1}{2}(\rho_A/i-1)} e^{-\rho_B \bar{t}} \Big(\sqrt{1-\mathcal{K}^2}\, m^- \, \sin \bar{t} - (\cos \bar{t} - \frac{\mathcal{K}}{i} \sin \bar{t})\Big) < 1 \tag{12}$$

$$\textit{where:} \qquad m^{\pm} := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\mathcal{K}/i}$$

$$\bar{t} = \arccos \frac{-\rho_A/i + \rho_B \mathcal{K}/i}{\sqrt{(1-\mathcal{K}^2)(1+\rho_B^2)}}$$

Case (RC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true whether $\chi < 0$ or $\chi > 0$.*

Case (RR) *If A and B have both real eigenvalues then:*

Case (RR.1) *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true. Moreover we have $|\mathcal{K}| > 1$;*

Case (RR.2) *if $\mathcal{D} > 0$ then $\mathcal{K} \neq -\rho_A\rho_B$ (notice that $-\rho_A\rho_B > 1$) and :*

Case (RR.2.1) *if $\mathcal{K} > -\rho_A\rho_B$ then (P) is false*

Case (RR.2.2) *if $\mathcal{K} < -\rho_A\rho_B$ then:*

Case (RR.2.2.A) *if $\mathcal{K} > -1$ then (P) is true;*

Case (RR.2.2.B) *if $\mathcal{K} < -1$ then (P) is true iff the following condition holds:*

$$\begin{aligned}\rho_{RR} := -f^{sym}(\rho_A, \rho_B, \mathcal{K}) f^{asym}(\rho_A, \rho_B, \mathcal{K}) \times \\ f^{asym}(\rho_B, \rho_A, \mathcal{K}) < 1,\end{aligned} \tag{13}$$

where:

$$\begin{aligned} f^{sym}(\rho_A, \rho_B, \mathcal{K}) &:= \frac{1 + \rho_A/i + \rho_B/i + \mathcal{K} - \sqrt{\mathcal{D}}}{1 + \rho_A/i + \rho_B/i + \mathcal{K} + \sqrt{\mathcal{D}}}; \\ f^{aym}(\rho_A, \rho_B, \mathcal{K}) &:= \left(\frac{\rho_B/i - \mathcal{K}\rho_A/i - \sqrt{\mathcal{D}}}{\rho_B/i - \mathcal{K}\rho_A/i + \sqrt{\mathcal{D}}}\right)^{\frac{1}{2}(\rho_A/i-1)} . \end{aligned}$$

Case (RR.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true or false whether $\mathcal{K} < -\rho_A\rho_B$ or $\mathcal{K} > -\rho_A\rho_B$.*

Finally, if $(\mathcal{P})$ does not hold true, then in case **CC.2.2** *with $\rho_{CC} = 1$, case* **(RC.2.2.B)***, with $\rho_{RC} = 1$, case* **(RR.2.2.B)***, with $\rho_{RR} = 1$, case* **(CC.3)** *with $\mathcal{K} < -1$, case* **(RC.3)** *with $\chi > 0$ and case* **(RR.3)** *with $\mathcal{K} > -\rho_A\rho_B$, the origin is just stable. In the other cases, the system is unstable.*

Remark **16** Formula (13) is a corrected version of Formula (6), p.93, of [7] and it is proved in Appendix A.3.

A.2 Normal Forms of 2×2 Matrices

Proposition 3 *Let A, B be two 2×2 real matrices satisfying conditions* **H1**, **H2**, **H3** *and* **H4** *given in Section 3. In the case in which one of the two matrices has real and the other non-real eigenvalues (i.e. the* **(RC)** *case), assume that A is the one having real eigenvalues. Then there exists a 3-parameter change of coordinates and two constant $\alpha_A, \alpha_B > 0$ such that the matrices A/α_A and B/α_B (still denoted below by A and B) are in the following normal forms:*

Case in which A and B have both non-real eigenvalues ((CC) case):

$$A = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix}, \tag{14}$$

where $\rho_A, \rho_B > 0$, $|E| > 1$. In this case, $\mathcal{K} = \frac{1}{2}(E + \frac{1}{E})$. Moreover, the eigenvalues of A and B are respectively $-\rho_A \pm i$ and $-\rho_B \pm i$.

Case in which A has real and B non-real eigenvalues ((RC) case):

$$A = \begin{pmatrix} -\rho_A/i + 1 & 0 \\ 0 & -\rho_A/i - 1 \end{pmatrix}, \tag{15}$$

$$B = \begin{pmatrix} -\rho_B - \mathcal{K}/i & -\sqrt{1-\mathcal{K}^2} \\ \sqrt{1-\mathcal{K}^2} & -\rho_B + \mathcal{K}/i \end{pmatrix}, \tag{16}$$

where $\rho_B > 0$, $\rho_A/i > 1$, $\mathcal{K} \in i\mathbb{R}$. In this case, the eigenvalues of A and B are respectively $-\rho_A/i \pm 1$ and $-\rho_B \pm i$.

Case in which A and B have both real eigenvalues ((RR) case):

$$A = \begin{pmatrix} -\rho_A/i + 1 & 0 \\ 0 & -\rho_A/i - 1 \end{pmatrix}, \tag{17}$$

$$B = \begin{pmatrix} \mathcal{K} - \rho_B/i & 1 - \mathcal{K} \\ 1 + \mathcal{K} & -\mathcal{K} - \rho_B/i \end{pmatrix}, \tag{18}$$

$$\tag{19}$$

where $\rho_A/i, \rho_B/i > 1$ and $\mathcal{K} \in \mathbb{R} \setminus \{\pm 1\}$. In this case, the eigenvalues of A and B are respectively $-\rho_A/i \pm 1$ and $-\rho_B/i \pm 1$.

A.3 Proof of Formula (12)

In this paragraph, we prove Formula (12), i.e. in the **(RC.2.2.B)** case, we determine an inequality defining the set of parameters ρ_A, ρ_B, $\mathcal{K}$ such that the property $(\mathcal{P})$, stated in Section 3, holds.

Thanks to Proposition 3 (see also [7], Appendix B, p.110), we can find a coordinate transformation such that (up to a rescaling of the matrices) A and B are given by equations (15), (16). In the case **(RC.2.2.B)**, we have $\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1 + \rho_A^2 + \rho_B^2) < 0$, $\chi := \rho_A\mathcal{K} - \rho_B < 0$, $\mathcal{K}/i > 0$. Moreover, the set $Q^{-1}(0)$ is the union of two lines passing from the origin and, at each point of $Q^{-1}(0)$, the two vector fields point in the same direction. One easily checks that the slope of the two lines defining $Q^{-1}(0)$ is:

$$m^{\pm} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\sqrt{1-\mathcal{K}^2}}.$$

Notice that, in our case we have $m^{\pm} < 0$ and $m^{+} < m^{-}$.

In this case, the worst trajectories are concatenations of arcs of integral curves of the vector fields Ax, Bx and rotate counterclockwise around the origin. More precisely, they are integral curves of Ax from the line $x_2 = m^+x_1$ to the line $x_2 = m^-x_1$, and integral curves of Bx otherwise.

Therefore, starting from the point $\begin{pmatrix} 1 \\ m^+ \end{pmatrix}$ (with the field Ax), we follow the worst trajectory until it touches again the line $x_2 = m^+ x_1$. Property $(\mathcal{P})$ is then satisfied if and only if $\rho_{RC} < 1$, where ρ_{RC} is the absolute value of the first coordinate of the final point.

One can easily compute that the first switching time is $t_1 = \frac{1}{2}\log\frac{m^+}{m^-}$, which is positive since $\frac{m^+}{m^-} > 1$. Moreover, the integral curve of Bx starting from the point $\begin{pmatrix} 1 \\ m^- \end{pmatrix}$ is:

$$e^{-\rho_B t}\begin{pmatrix} -\sqrt{1-\mathcal{K}^2}\, m^- \,\sin t + (\cos t \,-\frac{\mathcal{K}}{i}\sin t) \\ \sqrt{1-\mathcal{K}^2}\,\sin t \,+\, m^-(\cos t \,+\frac{\mathcal{K}}{i}\sin t) \end{pmatrix},$$

and, setting the ratio between the second coordinate and the first one equal to m^+, one obtains that the second switching time is $t_2 = \arccos\frac{-\rho_A/i + \rho_B\mathcal{K}/i}{\sqrt{(1-\mathcal{K}^2)(1+\rho_B^2)}}$. Notice that t_2 is well defined if and only if $\mathcal{D} < 0$ (condition which is satisfied in our case). Moreover, t_2 is positive and less than π. Finally, the inequality we was seeking for is:

$$\rho_{RC} = \Big(\frac{m^+}{m^-}\Big)^{-\frac{1}{2}(\rho_A/i-1)} e^{-\rho_B t_2}\Big(\sqrt{1-\mathcal{K}^2}\, m^- \,\sin t_2 \,-(\cos t_2 \,-\frac{\mathcal{K}}{i}\sin t_2)\Big) < 1\,.$$

B The Stability Properties of (3) Only Depend on the Convex Hull of A

We provide here the proof of Proposition 1. First, let us show the following:

Claim Consider the switched system (3), under **H0**, and let $\mathbf{A}'$ be a measurable subset of $\mathbf{A}$ such that the convex hull of $\mathbf{A}'$ contains $\mathbf{A}$. Then the following two conditions are equivalent:

i) the system is **GUES** (resp. uniformly stable), with $A(.)$ measurable, taking values in $\mathbf{A}$,

ii) the system is **GUES** (resp. uniformly stable), with $A(.)$ measurable, taking values in $\mathbf{A}'$.

Proof of the Claim. Let $\mathcal{A}$ (resp. $\mathcal{A}'$) be the set of measurable functions $A(.) : [0,\infty[\to \mathbf{A}$ (resp. $A(.) : [0,\infty[\to \mathbf{A}'$). Since $\mathbf{A}'$ is contained in $\mathbf{A}$ then the implication **i)** $\Rightarrow$ **ii)** is obvious.

Let us prove the other implication (that is strictly related to the classical approximability theorems in control theory). We start considering uniform stability. By contradiction, assume that we can find $\epsilon > 0$ satisfying the following. There exists a sequence of points (x_l) tending to zero and a sequence of controls $A_l(.) \in \mathcal{A}$ such that the corresponding trajectory γ_l starting at x_l exits the interior of the ball of radius ϵ for some time t_l. Using classical approximability results (see for instance [2]), the trajectory γ_l can be approximated in the L^∞-norm on $[0, t_l]$ by a trajectory γ_l' corresponding to a switching function $A_l'(.) \in \mathcal{A}'$ and starting at x_l. Hence γ_l' exits the interior of the ball of radius $\epsilon/2$ at time t_l. We reached a contradiction.

Now we want to prove that **GUES** holds in the case $A(.) \in \mathcal{A}'$ implies **GUES** holds in the case $A(.) \in \mathcal{A}$. Since $\mathbf{A}$ is compact, we know (see Definition 1 and below) that attractivity and **GUES** are equivalent for the corresponding switched system. Therefore, proceeding by contradiction, we can assume that there is a trajectory $\gamma(.)$ of the switched system corresponding to $A(.) \in \mathcal{A}$ not converging to zero. That means that there exist $\epsilon > 0$ and a sequence t_n of times tending to infinity such that $|\gamma(t_n)| > \epsilon$. As before, we can approximate $\gamma(.)$ on the interval $[0, t_n]$ with a trajectory $\gamma_n(.)$ corresponding to controls taking values in $\mathbf{A}'$, in such a way that $|\gamma_n(t_n)| > \epsilon/2$. But this is impossible since we have assumed **GUES** for the switched system with $A(.) \in \mathcal{A}'$.

Notice that one can provide an alternative argument for the **GUES** part of Proposition 1, by using CLFs. □

Then one immediately extend to semicones observing that the stability properties of the system (3), subject to the compactness hypothesis **H0**, depend only on the shape of the trajectories and not on the way in which they are parameterized.

References

[1] A. A. Agrachev and D. Liberzon, *Lie-algebraic stability criteria for switched systems*, SIAM Journal on Control and Optimization, no. 1, 253–269 (2001).

[2] A.A. Agrachev, Yu.L. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopedia of Mathematical Sciences, Vol.87, Springer (2004).

[3] D. Angeli, *A Note on Stability of Arbitrarily Switched Homogeneous Systems*, Systems and Control Letters, to appear.

[4] D. Angeli, D. Liberzon, *A note on uniform global asymptotic stability of nonlinear switched systems in triangular form*, Proc. 14th Int Symp. on Mathematical Theory of Networks and Systems (MTNS), (2000).

[5] A. Bacciotti, F. Ceragioli, *Closed loop Stabilization of Switched Systems*, preprint.

[6] Blanchini, S. Miani, *A New Class of Universal Lyapunov Functions for the Control of Uncertain Linear Systems* IEEE Transactions on Automatic Control, Vol. 44, no. 4, pp. 641-647, (1999).

[7] Ugo Boscain: *Stability of Planar Switched Systems: the Linear Single Input Case*, SIAM J. Control Optim. no.41, pp.89-112 (2002).

[8] Boscain U. and Piccoli B., *Optimal Syntheses for Control Systems on 2-D Manifolds*, Springer SMAI, VOL.43, (2004)

[9] Bubnicki, Z. *Analysis and decision making in uncertain systems*, Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, (2004).

[10] W. P. Dayawansa, C. F. Martin, *A Converse Lyapunov Theorem for a Class of Dynamical Systems which Undergo Switching*, IEEE Transactions on Automatic Control, Vol. 44, no. 4, pp. 751-760, (1999).

[11] D. Liberzon, J. P. Hespanha and A. S. Morse, *Stability of switched systems: a Lie-algebraic condition*, Systems and Control Letters, vol. 37, no. 3, pp. 117-122, (1999).

[12] D. Liberzon and A. S. Morse, *Basic problems in stability and design of switched systems*, IEEE Control Systems Magazine, vol. 19, no. 5, pp. 59-70, (1999).

[13] D. Liberzon, *Switching in Systems and Control*, Volume in series Systems and Control: Foundations and Applications. Birkhauser, Boston, MA, (2003).

[14] H. J. Sussmann, *The structure of time–optimal trajectories for single-input systems in the plane: the general real–analytic case*, SIAM J. Control and Opt., 25, pp. 868-904 (1987).

[15] R. Shorten and K. Narendra, *Necessary and Sufficient Conditions for the Existence of a Common Quadratic Lyapunov Function for Two Stable Second Order Linear Time–Invariant Systems*, Proceeding 1999, American Control Conf. pp. 1410-1414 (1999).

[16] X. Xu and P.J. Antsaklis, *Stabilization of Second Order LTI Switched Systems*, Internat. J. Control 73 no. 14, 1261–1279, (2000).