Stabilization of two-dimensional persistently excited linear control systems with arbitrary rate of convergence

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Abstract

We consider the control system $\dot{x} = Ax + \alpha(t)bu$ where the pair (A,b) is controllable, $x \in \mathbb{R}^2$, $u \in \mathbb{R}$ is a scalar control and the unknown signal $\alpha : \mathbb{R}_+ \to [0,1]$ is (T,μ)-persistently exciting (PE), i.e., for all $t \in \mathbb{R}_+$, $\int_t^{t+T} \alpha(s)ds \geq \mu$ for two constants $T \geq \mu > 0$. We study the stabilization of this system by a linear state feedback $u = -Kx$. In this paper, we positively answer a question asked in [6] and prove the following: Assume that the class of (T,μ)-PE signals is restricted to those which are M-Lipschitz, where M is a positive constant. Then, given any $C > 0$, there exists a linear state feedback $u = -Kx$ where K only depends on (A,b) and T,μ,M so that, for every M-Lipschitz (T,μ)-persistently exciting signal α, the rate of exponential decay of the time-varying system $\dot{x} = (A - \alpha(t)bK)x$ is larger than C.

1 Introduction

The aim of this paper is to continue the study of persistently excited (PE) linear control systems of [5, 6]. Consider a system in the form

$$\dot{x} = Ax + \alpha(t)Bu, \tag{1.1}$$

where $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$ is the control and $\alpha : \mathbb{R}_+ \to [0,1]$ is a scalar measurable signal. In the control system (1.1), the signal α determines when the control signal u is activated. We suppose that α is not precisely known and the only information on α we have is that it belongs to a certain class of functions $\mathcal{G}$.

We consider the problem of exponential stabilization to the origin, by means of a linear state feedback $u = -Kx$, of (1.1), where the uncontrolled dynamics $\dot{x} = Ax$ can be unstable. Taking into account the nature of the signal α, this stabilization must be *uniform* with respect to α, i.e., K may depend on class of functions $\mathcal{G}$, but it should not depend on a particular signal $\alpha \in \mathcal{G}$. If this is shown to be possible, then the next major issue is that of stabilization with an arbitrary rate of exponential decay, still with a linear state feedback and uniformly with respect to $\alpha \in \mathcal{G}$.

The question that arises consists in determining classes of functions $\mathcal{G}$ for which the above mentioned stabilization problem has a positive answer. For instance, it is obviously not suitable to choose $\mathcal{G} = L^\infty(\mathbb{R}_+,[0,1])$ since the trajectories of the non-controlled system $\dot{x} = Ax$ would be admissible. We thus look for a class of functions α that are "active" often enough. For this purpose, [5, 6] consider the case where α is a persistently exciting signal (PE signal for short), that is, there exist two positive constants $T \geq \mu$ such that, for every $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{1.2}$$

A signal verifying (1.2) is called a (T,μ)-PE signal and we denote by $\mathcal{G}(T,\mu)$ the class of all (T,μ)-PE signals. Condition (1.2) is known as the *persistent excitation condition* and appears in the context of identification and adaptive control (see [14]).

Throughout this paper, we consider only the case where the control u is scalar, and thus the matrix B is actually a column vector $b \in \mathbb{R}^d$. System (1.1) becomes then

$$\dot{x} = Ax + \alpha(t)bu, \tag{1.3}$$

where $x \in \mathbb{R}^d$, $u \in \mathbb{R}$ and $\alpha \in \mathcal{G}(T,\mu)$.

Since $\alpha \equiv 1$ belongs to any class $\mathcal{G}(T,\mu)$, then a necessary condition for the uniform stabilization of (1.3) is that the pair (A,b) is stabilizable. Let us describe briefly the intuition guiding the choice of a stabilzer for System (1.3). For that purpose, recall the following result obtained in [8]: for every $\rho > 0$ it is possible to choose a linear feedback $u = -Kx$ that stabilizes System (1.3) uniformly with respect to $\alpha \in L^\infty(\mathbb{R}_+,[\rho,1])$. Their argument can also be adapted, using a high-gain technique, to show that the rate of convergence can be made arbitrarily large when (A,b) is controllable. Consider now $\rho > 0$ small enough with respect to μ/T. Then Equation (1.2) shows that $\alpha(t) \geq \rho$ for a total time that is lower bounded by a positive constant on every time window of length T, uniformly with respect to $\alpha \in \mathcal{G}(T,\mu)$. For further simplicity of the exposition, assume that the (T,μ)-PE signal α is piecewise constant. Then we know how to stabilize exponentially System (1.3) on the "good" time intervals where $\alpha \geq \rho$. Thus, in order to stabilize the system, we seek a linear feedback $u = -Kx$ providing enough convergence in the "good" time intervals, so that it compensates the possible blow-up behavior of the solution in the "bad" time intervals (i.e., those on which $\alpha < \rho$).

This intuition was partially validated in [6], where it is shown that exponential stabilization to the origin of System (1.3) is possible if (A,b) is a controllable pair and every eigenvalue of A has non-positive real part (cf. Theorem 2.6 below).

In this paper we address the question of exponential stabilization at an arbitrary rate, i.e., given any $C > 0$, we want to choose a feedback $u = -Kx$ such that every solution of $\dot{x} = (A - \alpha(t)bK)x$ converges to 0 exponentially at a rate which is larger than C, uniformly with respect to $\alpha \in \mathcal{G}(T,\mu)$. A necessary condition is clearly that the pair (A,b) is controllable, as it follows from the Pole Shifting theorem.

It turns out that the above described intuition guiding the choice of the stabilizer can be shown to be false when applied to the problem of exponential stabilization at an arbitrary rate: in dimension $d = 2$, it was proved in [6] that there exists $\rho_\star$ so that, if $\frac{\mu}{T} \in (0,\rho_\star)$, then the maximal rate of convergence of System (1.3) is finite.

The contradiction to the intuitive idea lies in the overshoot phenomenon. One can choose K such that the solution of $\dot{x} = (A - bK)x$ stabilizes fast enough, but its norm may increase in a small time interval $[0,t]$ before exponentially decreasing with the desired convergence rate. Then, if

$\alpha = 1$ on a short period of time only, it is actually the overshoot phenomenon, and not the exponential stabilization, that dominates the behavior of the solution of (1.3). By switching fast enough between $\alpha = 1$ and $\alpha = 0$ on a fixed window of time, we can repeat several times the overshoot phenomenon and still satisfy Condition (1.2). For $\frac{\mu}{T}$ small enough, one can then construct for any given K a signal $\alpha \in \mathcal{G}(T,\mu)$ such that the overshoot phenomenon dominates the exponential stabilization provided by K. Notice that the regularity of $\alpha \in \mathcal{G}(T,\mu)$ is not an issue here since one can replace faster and faster switchings (between $\alpha = 1$ and $\alpha = 0$) by faster and faster oscillations as the norm of K increases in the above construction.

It was then proposed in [6] to restrict the class $\mathcal{G}(T,\mu)$ of PE signals in order to recover stabilization by a linear state feedback at an arbitrary rate of convergence for System (1.3). More precisely, the stabilization at an arbitrary rate of convergence for System (1.3) is conjectured to hold true for the subclass $\mathcal{D}(T,\mu,M)$ of $\mathcal{G}(T,\mu)$ of PE signals that are M-Lipschitz (cf. [6, Open Problem 5]). The goal of this paper is to bring a positive answer to Open Problem 5 in the case of planar control systems (1.3).

Before presenting the plan of the paper, let us briefly describe the strategy of the argument. We first decompose the time range into two classes of intervals, $\mathcal{I}_+$, the "good" intervals, where an auxiliary signal γ (obtained from α) is larger than a certain positive number, and $\mathcal{I}_-$, the "bad" intervals, where γ is small, retrieving thus the idea of "good" and "bad" intervals mentioned above. Estimations on "good" intervals are performed by integrating the dynamics of the control system written in polar coordinates: if we take the feedback gain K large enough, we can show that the solution rotates around the origin in "good" intervals, and the growth of the norm is estimated using the polar angle as new time. A different approach is needed in the "bad" intervals: we resort to optimal control in order to find the "worst trajectory", a particular solution of the system yielding the largest possible growth rate on a "bad" interval (in the spirit of [2, 7, 12, 13]). The final part of the proof consists in merging the two types of estimates.

The plan of the paper is the following. In Section 2, we provide the notations and definitions used throughout the paper as well as previous results on linear persistently excited control systems. We then turn in Section 3 to the core of the paper, where a precise statement of the main result is provided together with its proof.

2 Notations, definitions and previous results

2.1 Notations and definitions

In this paper, $\mathcal{M}_{d,m}(\mathbb{R})$ denotes the set of $d \times m$ matrices with real coefficients. When $m = d$, this set is denoted simply by $\mathcal{M}_d(\mathbb{R})$. As usual, we identify column matrices in $\mathcal{M}_{d,1}(\mathbb{R})$ with vectors in $\mathbb{R}^d$. The Euclidean norm of an element $x \in \mathbb{R}^d$ is denoted by $\|x\|$, and the associate matrix norm of a matrix $A \in \mathcal{M}_d(\mathbb{R})$ is also denoted by $\|A\|$, whereas the symbol $|a|$ is reserved for the absolute value of a real or complex number a. The real and imaginary parts of a complex number z are denoted by $\Re(z)$ and $\Im(z)$ respectively.

We shall consider control systems of the form

$$\dot{x} = Ax + \alpha(t)Bu \tag{2.1}$$

where $x \in \mathbb{R}^d$, $A \in \mathcal{M}_d(\mathbb{R})$, $u \in \mathbb{R}^m$ is the control, $B \in \mathcal{M}_{d,m}(\mathbb{R})$ and α belongs to the class of persistently exciting signals, defined below.

Definition 2.1 (PE signal and (T,μ)-signal)**.** Let T, μ be two positive constants with $T \geq \mu$. We say that a measurable function $\alpha : \mathbb{R}_+ \to [0,1]$ is a (T,μ)-*signal* if, for every $t \in \mathbb{R}_+$, one has

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{2.2}$$

The set of (T,μ)-signals is denoted by $\mathcal{G}(T,\mu)$. We say that a measurable function $\alpha : \mathbb{R}_+ \to [0,1]$ is a *persistently exciting signal* (or simply *PE signal*) if it is a (T,μ)-signal for certain positive constants T and μ with $T \geq \mu$.

We shall use later a restriction of this class, namely that of Lipschitz (T,μ)-signals, which we define below.

Definition 2.2 ((T,μ,M)-signal)**.** Let T, μ and M be positive constants with $T \geq \mu$. We say that a measurable function $\alpha : \mathbb{R}_+ \to [0,1]$ is a (T,μ,M)-*signal* if it is a (T,μ)-signal and, in addition, α is globally M-Lipschitz, that is, for every $t,s \in \mathbb{R}_+$,

$$|\alpha(t) - \alpha(s)| \leq M\,|t-s|.$$

The set of (T,μ,M)-signals is denoted by $\mathcal{D}(T,\mu,M)$.

We can now define the object of our study.

Definition 2.3 (PE and PEL systems)**.** Given a pair $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ and two positive constants T and μ (resp. three positive constants T, μ and M) with $T \geq \mu$, we say that the family of linear control systems

$$\dot{x} = Ax + \alpha Bu, \qquad \alpha \in \mathcal{G}(T,\mu) \quad (\text{resp. } \alpha \in \mathcal{D}(T,\mu,M)) \tag{2.3}$$

is the *PE system* associated with A, B, T and μ (resp. the *PEL system* associated with A, B, T, μ and M).

The main problem we are interested in is the question of *uniform* stabilization of System (2.3) by a linear state feedback of the form $u = -Kx$ with $K \in \mathcal{M}_{m,d}(\mathbb{R})$, which makes System (2.3) take the form

$$\dot{x} = (A - \alpha(t)BK)x. \tag{2.4}$$

The problem is thus the choice of K such that the origin of the linear system (2.4) is globally asymptotically stable. With this in mind, we can introduce the following notion of stabilizer.

Definition 2.4 (Stabilizer)**.** Let T and μ (resp. T, μ and M) be positive constants with $T \geq \mu$. We say that $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T,μ)-*stabilizer* (resp. (T,μ,M)-*stabilizer*) for System (2.3) if, for every $\alpha \in \mathcal{G}(T,\mu)$ (resp. $\alpha \in \mathcal{D}(T,\mu,M)$), System (2.4) is globally asymptotically stable.

We remark that K may depend on T, μ and M, but it cannot depend on the particular signal $\alpha \in \mathcal{G}(T,\mu)$ or $\alpha \in \mathcal{D}(T,\mu,M)$. We also remark that a (T,μ)-stabilizer is also a (T,μ,M)-stabilizer for every $M > 0$.

The question we are interested in is not only to stabilize a PE or PEL system, but also to stabilize it with an arbitrary rate of convergence. In order to rigorously define this notion, we introduce some concepts.

Definition 2.5. Let $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$, $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $T \geq \mu > 0$, $M > 0$, and consider System (2.4). Fix $\alpha \in \mathcal{G}(T,\mu)$ (resp. $\alpha \in \mathcal{D}(T,\mu,M)$). We denote by $x(t;x_0)$ the solution of System (2.4) with initial condition $x(0;x_0) = x_0$.

- The *maximal Lyapunov exponent* $\lambda^+(\alpha,K)$ associated with (2.4) is defined as

$$\lambda^+(\alpha,K) = \sup_{\|x_0\|=1} \limsup_{t\to+\infty} \frac{\ln\|x(t;x_0)\|}{t}.$$

- The *rate of convergence* associated with the systems $\dot{x} = (A - \alpha(t)BK)x$, $\alpha \in \mathcal{G}(T,\mu)$ (resp. $\alpha \in \mathcal{D}(T,\mu,M)$) is defined as

$$\mathrm{rc}_{\mathcal{G}}(T,\mu,K) = -\sup_{\alpha\in\mathcal{G}(T,\mu)} \lambda^+(\alpha,K) \ \text{(resp. } \mathrm{rc}_{\mathcal{D}}(T,\mu,M,K) = -\sup_{\alpha\in\mathcal{D}(T,\mu,M)} \lambda^+(\alpha,K)\text{)}.$$

- The *maximal rate of convergence* associated with System (2.3) is defined as

$$\mathrm{RC}_{\mathcal{G}}(T,\mu) = \sup_{K\in\mathcal{M}_{m,d}(\mathbb{R})} \mathrm{rc}_{\mathcal{G}}(T,\mu,K) \ \text{(resp. } \mathrm{RC}_{\mathcal{D}}(T,\mu,M) = \sup_{K\in\mathcal{M}_{m,d}(\mathbb{R})} \mathrm{rc}_{\mathcal{D}}(T,\mu,M,K)\text{)}.$$

The stabilization of System (2.3) at an arbitrary rate of convergence corresponds thus to the equality $\mathrm{RC}_{\mathcal{G}}(T,\mu) = +\infty$ or $\mathrm{RC}_{\mathcal{D}}(T,\mu,M) = +\infty$.

The fact that we are interested in the maximal rate of convergence explains why we consider only the case where the pair $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ is controllable.

2.2 Previous results

The first stabilization problem is the case of a neutrally stable system, that is, a system in the form (2.1) such that every eigenvalue of A has non-positive real part, and those with real part zero have trivial Jordan blocks. Under such hypothesis on A, and assuming that $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ is stabilizable, it is proved in [1, 5] that there exists a matrix $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for every $T \geq \mu > 0$, K is a (T,μ)-stabilizer for the PE system

$$\dot{x} = Ax + \alpha(t)Bu, \qquad \alpha \in \mathcal{G}(T,\mu). \tag{2.5}$$

We remark that the gain K is independent of T and μ. Some extension of this result to the case where $\mathbb{R}^d$ is replaced by an infinite-dimensional Hilbert space is discussed in [9].

The next case that have been studied has been the double integrator ([5]), which has been generalized in [6] as follows.

Theorem 2.6. *Let $(A,b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$ be a controllable pair and assume that the eigenvalues of A have non-positive real part. Then for every T, μ with $T \geq \mu > 0$ there exists a (T,μ)-stabilizer for $\dot{x} = Ax + \alpha(t)bu$, $\alpha \in \mathcal{G}(T,\mu)$.*

In order to justify the analysis of this paper it is useful to recall briefly how the proof of Theorem 2.6 goes. To capture its main features, it is enough to consider the case of the double integrator, i.e., $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. System (2.5) is thus written as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \alpha(t)u. \end{cases} \tag{2.6}$$

For every $\nu > 0$, $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ is a (T,μ)-stabilizer of (2.6) if and only if $\begin{pmatrix} \nu^2 k_1 & \nu k_2 \end{pmatrix}$ is a $(T/\nu, \mu/\nu)$-stabilizer of (2.6), as it can be seen by considering the equation satisfied by

$$x_\nu(t) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix} x(\nu t).$$

The idea of the proof is thus to construct a $(T/\nu, \mu/\nu)$-stabilizer $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ for (2.6) for a certain ν large enough, and then the (T,μ)-stabilizer we seek for is $\begin{pmatrix} k_1/\nu^2 & k_2/\nu \end{pmatrix}$. The construction of such a K is based on a limit process: given a family of signals $\alpha_n \in \mathcal{G}(T/\nu_n, \mu/\nu_n)$ with $\lim_{n\to+\infty} \nu_n = +\infty$, by weak-$\star$ compactness of $L^\infty(\mathbb{R}_+, [0,1])$ there exists a subsequence weak-$\star$ converging in $L^\infty(\mathbb{R}_+, [0,1])$ to a certain limit $\alpha_\star$, which can be shown to satisfy $\alpha_\star(t) \geq \frac{\mu}{T}$ almost everywhere. We can thus study the limit system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \alpha_\star(t)u, \end{cases} \qquad \alpha_\star(t) \geq \frac{\mu}{T},$$

in order to obtain properties of System (2.6) by a limit process.

The result recalled in Theorem 2.6 left open, for (T,μ) given, the case where A has at least one eigenvalue with positive real part. That issue was somehow resolved by reformulating the question as a stabilization problem with arbitrary rate of convergence. Let us state the latter in terms of the maximal rates of convergence as the problem of determining whether $\mathrm{RC}_{\mathcal{G}}(T,\mu)$ and $\mathrm{RC}_{\mathcal{D}}(T,\mu,M)$ are finite or not. In this sense, [6] gives two results concerning the stabilization of PE systems and points out the importance played by the parameter $\frac{\mu}{T}$.

Theorem 2.7. *Let d be a positive integer. There exists $\rho^\star \in (0,1)$ such that, for every controllable pair $(A,b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$ and every positive T, μ satisfying $\rho^\star < \frac{\mu}{T} \leq 1$, one has $RC_{\mathcal{G}}(T,\mu) = +\infty$.*

This means that, at least for $\frac{\mu}{T}$ large enough, stabilization at an arbitrary rate of convergence is possible for a PE system with *any* controllable (A,b). Nevertheless, [6] also proves that the result is false for $\frac{\mu}{T}$ small, at least in dimension 2.

Theorem 2.8. *There exists $\rho_\star \in (0,1)$ such that, for every controllable pair $(A,b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$ and every positive T, μ satisfying $0 < \frac{\mu}{T} < \rho_\star$, one has $RC_{\mathcal{G}}(T,\mu) < +\infty$.*

As recalled in the introduction, the proof of Theorem 2.8 is based on the explicit construction of fast-oscillating controls. This motivates the conjecture that $\mathrm{RC}_{\mathcal{D}}(T,\mu,M) = +\infty$. The technique used in the proof of Theorem 2.6 (also recalled in the introduction) could not provide any help in this case: the direct study of a limit system comes from accelerating the dynamics of the system by a factor $\nu > 0$ and letting ν go to infinity. The signals appearing in the limit system do not provide any additional information with respect to the case without Lipschitz continuity constraints, since they are weak-$\star$ limits as $\nu \to \infty$ of signals in $\mathcal{D}(T/\nu, \mu/\nu, \nu M)$, that is, of signals with larger and larger Lipschitz constant.

3 Main result

The main result we want to prove concerns planar systems of the type (2.1). More specifically, we fix positive constants T, μ and M with $T \geq \mu$ and we study the PEL system

$$\dot{x} = Ax + \alpha(t)bu, \qquad \alpha \in \mathcal{D}(T,\mu,M), \tag{3.1}$$

where $x \in \mathbb{R}^2$, (A,b) is controllable. We get the following result.

Theorem 3.1. *Let T, μ and M be positive constants with $T \geq \mu$. Then for the PEL System* (3.1) *one has $RC_{\mathcal{D}}(T,\mu,M) = +\infty$.*

The rest of the section is devoted to the proof of Theorem 3.1.

We first perform a linear algebraic transformation on the control system. With no loss of generality, we can assume that (A,b) is under controllable form, i.e., $A = \begin{pmatrix} 0 & 1 \\ -d & \mathrm{Tr}(A) \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where $\mathrm{Tr}(A)$ is the trace of A. Moreover, if A is replaced by $A - \mathrm{Tr}(A)\mathrm{Id}_2$, then $\mathrm{RC}_{\mathcal{D}}(T,\mu,M)$ is simply translated by $-\mathrm{Tr}(A)$. It is therefore enough to prove the theorem assuming that $\mathrm{Tr}(A) = 0$.

The system can thus be written in the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -dx_1 + \alpha(t)u, \end{cases} \qquad \alpha \in \mathcal{D}(T,\mu,M). \tag{3.2}$$

From now on, we suppose that T, μ, M, d and λ are fixed. We prove Theorem 3.1 by explicitly constructing a gain K that satisfies $\lambda^+(\alpha,K) \leq -\lambda$ for every $\alpha \in \mathcal{D}(T,\mu,M)$. To do so, we write $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ and thus the feedback $u = -Kx$ leads to the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -(d + \alpha(t)k_1) & -\alpha(t)k_2 \end{pmatrix} x.$$

The variable x_1 satisfies the scalar equation $\ddot{x}_1 + k_2\alpha(t)\dot{x}_1 + (d + k_1\alpha(t))x_1 = 0$ and we have $x_2 = \dot{x}_1$.

We remark that the signal α constant and equal to 1 is in $\mathcal{D}(T,\mu,M)$, and thus a necessary condition for K to be a (T,μ,M)-stabilizer is that the matrix

$$A - bK = \begin{pmatrix} 0 & 1 \\ -d - k_1 & -k_2 \end{pmatrix}$$

is Hurwitz, which is the case if and only if $k_1 > -d$, $k_2 > 0$. In what follows, we restrict ourselves to search K in the form

$$K = \begin{pmatrix} k^2 & k \end{pmatrix}, \qquad k \text{ positive and large.} \tag{3.3}$$

The differential equation satisfied by x_1 is thus

$$\ddot{x}_1 + k\alpha(t)\dot{x}_1 + (d + k^2\alpha(t))x_1 = 0. \tag{3.4}$$

3.1 Strategy of the proof

Let us discuss briefly the strategy that we will use to prove Theorem 3.1. We start, in Section 3.2, by making a change of variables on (3.4) that makes the systems easier to handle. The new variable y is related to x by an exponential term $e^{-\frac{k}{2}\int_0^t \alpha(s)ds + t\sqrt{\frac{kM}{2}-d}}$ that converges to 0 as $t \to +\infty$ (see (3.6)). The problem is then to estimate the rate of exponential growth of y (Section 3.3). We start by proving that y turns around the origin infinitely many times (Section 3.3.2). On each complete turn the exponential growth of y is estimated either by direct integration when the exciting signal is “large" (Section 3.3.4) or by optimal control where it is “small" (Section 3.3.5).

3.2 Change of variables

In order to simplify the notations, we write $h = \sqrt{2kM - 4d}$, which is well defined for $k \geq \frac{2d}{M}$. We consider the system in a new variable $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^{\mathrm{T}}$ defined by the relations

$$\begin{cases} y_1 = x_1 e^{\frac{k}{2}\int_0^t \alpha(s)ds - \frac{h}{2}t}, \\ y_2 = \dot{y}_1 = \left(x_2 + \left(\dfrac{k}{2}\alpha(t) - \dfrac{h}{2}\right)x_1\right) e^{\frac{k}{2}\int_0^t \alpha(s)ds - \frac{h}{2}t}, \end{cases} \tag{3.5}$$

whose choice is justified at the end of this section. The variables x and y are thus related by

$$y = e^{\frac{k}{2}\int_0^t \alpha(s)ds - \frac{h}{2}t} \begin{pmatrix} 1 & 0 \\ \frac{k}{2}\alpha(t) - \frac{h}{2} & 1 \end{pmatrix} x, \quad x = e^{-\frac{k}{2}\int_0^t \alpha(s)ds + \frac{h}{2}t} \begin{pmatrix} 1 & 0 \\ \frac{h}{2} - \frac{k}{2}\alpha(t) & 1 \end{pmatrix} y \tag{3.6}$$

and y_1 satisfies the differential equation

$$\ddot{y}_1 + h\dot{y}_1 + k^2\gamma(t)y_1 = 0 \tag{3.7}$$

with

$$\gamma(t) = \beta(t) + \frac{M - \dot{\alpha}(t)}{2k}, \qquad \beta(t) = \alpha(t)\left(1 - \tfrac{1}{4}\alpha(t)\right). \tag{3.8}$$

The system satisfied by y is

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -k^2\gamma(t) & -h \end{pmatrix} y. \tag{3.9}$$

Since $\alpha(t) \in [0,1]$ for every $t \in \mathbb{R}_+$, we have $\beta(t) \in [0, 3/4]$. Furthermore, since α is M-Lipschitz, β is also Lipschitz continuous with the same Lipschitz constant, since

$$\begin{aligned} |\beta(t) - \beta(s)| &= \left|\alpha(t) - \alpha(s) - \tfrac{1}{4}\left(\alpha(t)^2 - \alpha(s)^2\right)\right| = |\alpha(t) - \alpha(s)| \left|1 - \tfrac{\alpha(t)+\alpha(s)}{4}\right| \\ &\leq |\alpha(t) - \alpha(s)| \leq M|t - s| \end{aligned}$$

for every $t,s \in \mathbb{R}_+$. Since α satisfies the PE condition (2.2), β satisfies

$$\int_t^{t+T} \beta(s)ds \geq \tfrac{3}{4}\mu. \tag{3.10}$$

Since $|\dot{\alpha}(t)| \leq M$ almost everywhere, γ can be bounded by

$$0 \leq \gamma(t) \leq \tfrac{3}{4} + \tfrac{M}{k}$$

almost everywhere. It also satisfies the PE condition

$$\int_t^{t+T} \gamma(s)ds \geq \tfrac{3}{4}\mu. \tag{3.11}$$

From now on, we suppose that

$$k \geq K_1(M) := \max\left(4M, \frac{2|d|}{M}\right), \tag{3.12}$$

so that $h \leq 2\sqrt{kM}$ and

$$0 \leq \gamma(t) \leq 1$$

for almost every $t \in \mathbb{R}_+$.

Let us discuss the change of variables (3.5). The term $e^{\frac{k}{2}\int_0^t \alpha(s)ds}$ corresponds to a classical change of variables in second-order scalar equations (see, for instance, [10]) that eliminates the term in $\dot{x}_1$ from (3.4), which is replaced by a new term $-\frac{1}{4}k^2\alpha(t)^2 - \frac{k}{2}\dot{\alpha}(t)$ multiplying y_1. However, if we took only this term in the change of variables, the resulting function γ would be $\gamma(t) = \beta(t) + \frac{2d/k - \dot{\alpha}(t)}{2k}$, which may be negative at certain times t. To apply the techniques of optimal control of Section 3.3.5, it is important to manipulate a positive function γ, and that is why we introduce the term $e^{\frac{h}{2}t}$ in the change of variables.

Another important feature of this change of variables is that $x(t)$ behaves like $e^{-\frac{k}{2}\int_0^t \alpha(s)ds + \frac{h}{2}t}y(t)$. Since $h \leq 2\sqrt{kM}$ and α is persistently exciting, this exponential factor is bounded by $e^{-c_1 kt}$ for large k, for a certain $c_1 > 0$. We have now to show that the exponential growth of y is bounded by $e^{c_2 k^s t}$ for large k, for some $c_2 > 0$ and $s < 1$.

This change of variables also justifies the choice of K in the form (3.3). Equation (3.7) is a linear second-order scalar differential equation and, in the case where its coefficients are constant, $h\dot{y}_1$ can be interpreted as a damping term and $k^2\gamma y_1$ as an oscillatory term. Such a system oscillates around the origin if $4k^2\gamma \geq h^2 = 2kM - 4d$, which is the case for k large enough. In the case where γ depends on time, the PE condition (3.11) still guarantees a certain oscillatory behavior for k large enough, which is used in order to prove Theorem 3.1.

3.3 Properties of the system in the new variables

3.3.1 Polar coordinates

We now wish to study System (3.9) and the corresponding differential equation (3.7). To do so, we first write this system in polar coordinates in the plan $(y_1, \dot{y}_1)$: we define the variables $r \in \mathbb{R}_+$

and $\theta \in \mathbb{R}$ (or $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, depending on the context) by the relations

$$y_1 = r\cos\theta,$$
$$\dot{y}_1 = r\sin\theta,$$

which leads to the equations

$$\dot{\theta} = -\sin^2\theta - k^2\gamma(t)\cos^2\theta - h\sin\theta\cos\theta, \tag{3.13a}$$
$$\dot{r} = r\sin\theta\cos\theta(1-k^2\gamma(t)) - hr\sin^2\theta. \tag{3.13b}$$

For nonzero solutions we can write (3.13b) as

$$\frac{d}{dt}\ln r = \sin\theta\cos\theta(1-k^2\gamma(t)) - h\sin^2\theta. \tag{3.13c}$$

3.3.2 Rotations around the origin

Let us consider Equation (3.13a). If $\sin\theta\cos\theta \geq 0$, then $\dot{\theta} \leq 0$, with the strict inequality being true for all times except when $\sin\theta = 0$ and $\gamma = 0$. The following lemma shows that in the general case it is still possible, for k large enough, to guarantee that y keeps on turning clockwise around the origin, even if, at certain points, it may go counterclockwise for a short period of time.

Lemma 3.2. *There exists $K_2(T,\mu,M)$ such that, for $k > K_2(T,\mu,M)$, the solution θ of* (3.13a) *satisfies* $\lim_{t\to+\infty}\theta(t) = -\infty$.

Proof. We start by fixing $t \in \mathbb{R}_+$ and the interval $I = [t, t+T]$. Equation (3.10) shows that there exists $t_\star \in I$ such that $\beta(t_\star) \geq \frac{3\mu}{4T}$. Since β is M-Lipschitz, we have $\beta(s) \geq \frac{\mu}{2T}$ if $|s-t_\star| \leq \frac{\mu}{4MT}$, and thus, since $\gamma(s) \geq \beta(s)$, we have $\gamma(s) \geq \frac{\mu}{2T}$ for $|s-t_\star| \leq \frac{\mu}{4MT}$. If we take

$$k \geq \max\left(1, \left(\frac{\mu}{2MT^2}\right)^4\right), \tag{3.14}$$

we have $\frac{\mu}{4MTk^{1/4}} \leq \frac{\mu}{4MT}$ and $\frac{\mu}{4MTk^{1/4}} \leq \frac{T}{2}$, which implies that at least one of the intervals $\left[t_\star - \frac{\mu}{4MTk^{1/4}}, t_\star\right]$ and $\left[t_\star, t_\star + \frac{\mu}{4MTk^{1/4}}\right]$ is contained in I; let us denote this interval by $J = [s_0, s_1]$, so that $s_1 - s_0 = \frac{\mu}{4MTk^{1/4}}$ and $\gamma(s) \geq \frac{\mu}{2T}$ for $s \in J$.

If $s \in J$, one can estimate $\dot{\theta}$ in (3.13a) by

$$-\dot{\theta}(s) \geq \sin^2\theta(s) + \frac{\mu k^2}{2T}\cos^2\theta(s) + h\sin\theta(s)\cos\theta(s) = \begin{pmatrix}\sin\theta(s) & \cos\theta(s)\end{pmatrix}\begin{pmatrix}1 & \frac{h}{2}\\ \frac{h}{2} & \frac{\mu k^2}{2T}\end{pmatrix}\begin{pmatrix}\sin\theta(s)\\ \cos\theta(s)\end{pmatrix}.$$

In particular, if

$$k > \frac{2MT}{\mu}, \tag{3.15}$$

then the matrix $\begin{pmatrix} 1 & \frac{h}{2} \\ \frac{h}{2} & \frac{\mu k^2}{2T} \end{pmatrix}$ is positive definite and thus $\dot\theta(s) < 0$ for every $s \in J$. Therefore θ is strictly decreasing on J and is a bijection between J and its image $\theta(J)$. One can write Equation (3.13a) on J as

$$\frac{\dot\theta}{\sin^2\theta + k^2\gamma\cos^2\theta + h\sin\theta\cos\theta} = -1 \tag{3.16}$$

and, by integrating from s_0 to s_1 and using the relation

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2\theta + a\cos^2\theta + b\sin\theta\cos\theta} = \frac{2\pi}{\sqrt{4a-b^2}}, \qquad a>0,\, b^2<4a$$

(which can be computed directly by the change of variables $\hat t = \tan\theta$), we obtain

$$\begin{aligned} \frac{\mu}{4MTk^{1/4}} = s_1 - s_0 &= -\int_{s_0}^{s_1} \frac{\dot\theta(s)}{\sin^2\theta(s) + k^2\gamma(s)\cos^2\theta(s) + h\sin\theta(s)\cos\theta(s)} ds \\ &\le \int_{\theta(s_1)}^{\theta(s_0)} \frac{d\theta}{\sin^2\theta + \frac{k^2\mu}{2T}\cos^2\theta + h\sin\theta\cos\theta} \\ &\le \int_{\theta(s_1)}^{\theta(s_1)+\pi(N+1)} \frac{d\theta}{\sin^2\theta + \frac{k^2\mu}{2T}\cos^2\theta + h\sin\theta\cos\theta} \\ &= \frac{2\pi(N+1)}{\sqrt{\frac{2k^2\mu}{T} - h^2}} \le \frac{2\pi(N+1)}{\sqrt{\frac{2\mu}{T}k^2 - 4Mk}}, \end{aligned} \tag{3.17}$$

where N is the number of rotations of angle π during the interval J, i.e., $N = \left\lfloor \frac{\theta(s_0)-\theta(s_1)}{\pi} \right\rfloor$. Therefore

$$\theta(s_0) - \theta(s_1) \ge \pi N \ge k^{3/4}\frac{\mu}{8MT}\sqrt{\frac{2\mu}{T} - \frac{4M}{k}} - \pi. \tag{3.18}$$

On the other hand, one can estimate $\dot\theta$ in (3.13a) for every $s \in I$ by $\dot\theta(s) \le h$, so that

$$\theta(s_0) - \theta(t) \le h(s_0 - t), \qquad \theta(t+T) - \theta(s_1) \le h(t+T-s_1). \tag{3.19}$$

Thus, by (3.18) and (3.19), we obtain

$$\theta(t+T) - \theta(t) \le 2\sqrt{kM}T - k^{3/4}\frac{\mu}{8MT}\sqrt{\frac{2\mu}{T} - \frac{4M}{k}} + \pi.$$

The expression on the right-hand side tends to $-\infty$ as $k \to +\infty$ and the parameters T, μ and M are fixed. Hence, there exists $K_\star(T,\mu,M)$ such that, if

$$k \ge K_\star(T,\mu,M), \tag{3.20}$$

then $2\sqrt{kM}T - k^{3/4}\frac{\mu}{8MT}\sqrt{\frac{2\mu}{T} - \frac{4M}{k}} + \pi \le -2\pi$ and thus $\theta(t+T) - \theta(t) \le -2\pi$. We group conditions (3.14), (3.15) and (3.20) in a single one by setting

$$K_2(T,\mu,M) = \max\left(1, \left(\frac{\mu}{2MT^2}\right)^4, \frac{2MT}{\mu}, K_\star(T,\mu,M)\right)$$

and asking that $k > K_2(T, \mu, M)$. Under this condition, the solution completes at least one entire clockwise rotation by the end of the interval $[t, t+T]$. This result being true for every $t \in \mathbb{R}_+$, the proof is completed. ■

3.3.3 Decomposition of the time in intervals $\mathcal{I}_+$ and $\mathcal{I}_-$

Using Lemma 3.2, we can decompose $\mathbb{R}_+$ in a sequence of intervals (depending on α) on which the solution rotates by an angle π around the origin. More precisely, we define the sequence $(t_n)_{n\in\mathbb{N}}$ by induction as

$$t_0 = \inf\{t \geq 0 \mid \tfrac{\theta(t)}{\pi} \in \mathbb{Z}\}, \qquad t_n = \inf\{t \geq t_{n-1} \mid \theta(t) = \theta(t_{n-1}) - \pi\}, \quad n \geq 1, \tag{3.21}$$

and the continuity of θ and Lemma 3.2 show that this sequence is well defined. We also define the sequence of intervals $(I_n)_{n\in\mathbb{N}}$ by $I_n = [t_{n-1}, t_n]$ for $n \geq 1$ and $I_0 = [0, t_0]$.

Let us show a first result about the behavior of θ on these intervals.

Lemma 3.3. *Let $n \geq 1$. Then for every $t \in I_n = [t_{n-1}, t_n]$ one has*

$$\theta(t_n) \leq \theta(t) \leq \theta(t_{n-1}). \tag{3.22}$$

Proof. The first inequality in (3.22) is a consequence of the definition of t_n: if there was $t \in I_n$ with $\theta(t) < \theta(t_n)$, then, by the continuity of θ, there would be $s \in [t_{n-1}, t]$ such that $\theta(s) = \theta(t_n) = \theta(t_{n-1}) - \pi$, leading to a contradiction.

The second inequality in (3.22) can also be proved by contradiction. Suppose that there exists $t \in I_n$ such that $\theta(t) > \theta(t_{n-1})$. Then, by continuity of θ, there exists $s_0, s_1 \in [t_{n-1}, t]$ such that $\theta(s_0) = \theta(t_{n-1})$, $\theta(s_1) > \theta(t_{n-1})$ and $\theta(s) \in [\theta(t_{n-1}), \theta(t_{n-1}) + \pi/2]$ for every $s \in [s_0, s_1]$. Since $\theta(t_{n-1}) = 0 \mod \pi$, however, then $\sin\vartheta\cos\vartheta \geq 0$ for $\vartheta \in [\theta(t_{n-1}), \theta(t_{n-1}) + \pi/2]$. Thus, by (3.13a), $\dot{\theta}(s) \leq 0$ for almost every $s \in [s_0, s_1]$, which contradicts the fact that $\theta(s_0) < \theta(s_1)$. ■

We now split the intervals of the sequence $(I_n)_{n\geq 1}$ into two classes, $\mathcal{I}_+$ and $\mathcal{I}_-$, according to the behavior of β on these intervals. We define

$$\mathcal{I}_+ = \{I_n \mid n \geq 1,\ \exists t \in I_n \text{ s.t. } \beta(t) \geq 2/\sqrt{k}\}, \qquad \mathcal{I}_- = \{I_n \mid n \geq 1,\ \forall t \in I_n,\ \beta(t) < 2/\sqrt{k}\}.$$

3.3.4 Estimations on intervals belonging to the family $\mathcal{I}_+$

We start by studying the intervals in the class $\mathcal{I}_+$. We first claim that, for k large enough, we have $\gamma(t) \geq 1/\sqrt{k}$ for almost every $t \in I$ and every $I \in \mathcal{I}_+$.

Lemma 3.4. *There exists $K_3(M)$ such that, for $k > K_3(M)$ and for every $I \in \mathcal{I}_+$, one has $\beta(t) \geq 1/\sqrt{k}$ for every $t \in I$ and $\gamma(t) \geq 1/\sqrt{k}$ for almost every $t \in I$.*

Proof. We fix an interval $I = [t_{n-1}, t_n] \in \mathcal{I}_+$ and we denote by $t_\star$ an element of I such that $\beta(t_\star) \geq 2/\sqrt{k}$. Since β is M-Lipschitz, for every t such that $|t - t_\star| \leq \frac{1}{M\sqrt{k}}$, we have $1/\sqrt{k} \leq \beta(t) \leq 3/\sqrt{k}$. In particular, since $\gamma(t) \geq \beta(t)$ on $\mathbb{R}_+$, we have $\gamma(t) \geq 1/\sqrt{k}$ for $|t - t_\star| \leq \frac{1}{M\sqrt{k}}$.

The idea is to show that, for k large enough, $I \subset \left[t_\star - \frac{1}{M\sqrt{k}}, t_\star + \frac{1}{M\sqrt{k}}\right]$. This is done by proving that, for k large enough, the number of rotations of angle π around the origin done on each of the intervals $\left[t_\star - \frac{1}{M\sqrt{k}}, t_\star\right]$ and $\left[t_\star, t_\star + \frac{1}{M\sqrt{k}}\right]$ is larger than 1.

We take $s_0, s_1 \in \left[t_\star - \frac{1}{M\sqrt{k}}, t_\star + \frac{1}{M\sqrt{k}}\right]$, $s_0 < s_1$. For every $s \in [s_0, s_1]$, we have

$$\begin{aligned}-\dot{\theta}(s) &\geq \sin^2\theta(s) + k^{3/2}\cos^2\theta(s) + h\sin\theta(s)\cos\theta(s) = \\ &= \begin{pmatrix}\sin\theta(s) & \cos\theta(s)\end{pmatrix} \begin{pmatrix}1 & \frac{h}{2} \\ \frac{h}{2} & k^{3/2}\end{pmatrix} \begin{pmatrix}\sin\theta(s) \\ \cos\theta(s)\end{pmatrix},\end{aligned}$$

and the matrix $\begin{pmatrix}1 & \frac{h}{2} \\ \frac{h}{2} & k^{3/2}\end{pmatrix}$ is positive definite if

$$k > M^2. \tag{3.23}$$

We take k satisfying (3.23). We can thus write Equation (3.13a) on $[s_0, s_1]$ as (3.16), and by integrating as in (3.17), we obtain

$$s_1 - s_0 \leq \int_{\theta(s_1)}^{\theta(s_1)+\pi(N(s_0,s_1)+1)} \frac{d\theta}{\sin^2\theta + k^{3/2}\cos^2\theta + h\sin\theta\cos\theta} \leq \frac{\pi(N(s_0,s_1)+1)}{k^{3/4}\sqrt{1-\frac{M}{k^{1/2}}}},$$

where $N(s_0,s_1) = \left\lfloor \frac{\theta(s_0)-\theta(s_1)}{\pi} \right\rfloor$ is the number of rotations of angle π around the origin done by the solution between s_0 and s_1. Hence

$$N(s_0,s_1) \geq k^{3/4}\frac{s_1 - s_0}{\pi}\sqrt{1-\frac{M}{k^{1/2}}} - 1,$$

and, in particular,

$$N\left(t_\star, t_\star + \frac{1}{M\sqrt{k}}\right) \geq \frac{k^{1/4}}{M\pi}\sqrt{1-\frac{M}{k^{1/2}}} - 1$$

and the same is true for $N\left(t_\star - \frac{1}{M\sqrt{k}}, t_\star\right)$. For M fixed we have $\frac{k^{1/4}}{M\pi}\sqrt{1-\frac{M}{k^{1/2}}} - 1 \xrightarrow[k\to+\infty]{} +\infty$ and thus there exists $K_\star(M)$ such that, for

$$k > K_\star(M), \tag{3.24}$$

one has $\frac{k^{1/4}}{M\pi}\sqrt{1-\frac{M}{k^{1/2}}} - 1 > 1$. Therefore both $N\left(t_\star - \frac{1}{M\sqrt{k}}, t_\star\right)$ and $N\left(t_\star, t_\star + \frac{1}{M\sqrt{k}}\right)$ are larger than 1, and then $\theta(t_\star) - \theta\left(t_\star + \frac{1}{M\sqrt{k}}\right)$ and $\theta\left(t_\star - \frac{1}{M\sqrt{k}}\right) - \theta(t_\star)$ are larger than π. By definition of I and thanks to Lemma 3.3,

$$t_\star - \frac{1}{M\sqrt{k}} < t_{n-1}, \qquad t_\star + \frac{1}{M\sqrt{k}} > t_n,$$

and then $I \subset \left[t_\star - \frac{1}{M\sqrt{k}}, t_\star + \frac{1}{M\sqrt{k}}\right]$. According to (3.23) and (3.24) the lemma is proved by setting $K_3(M) = \max\left(M^2, K_\star(M)\right)$. ∎

By using the previous result, we can estimate the divergence rate of the solutions of (3.13c) over the intervals belonging to $\mathcal{I}_+$.

Lemma 3.5. *There exists $K_4(M)$ such that, for every $k > K_4(M)$ and every $I = [t_{n-1}, t_n] \in \mathcal{I}_+$, the solution of* (3.13c) *satisfies*

$$r(t_n) \leq r(t_{n-1}) e^{4Mk^{1/2}(t_n - t_{n-1})}. \tag{3.25}$$

Proof. We start by taking

$$k > K_3(M), \tag{3.26}$$

so that we can apply Lemma 3.4 and obtain that, for almost every $t \in I$, $\beta(t), \gamma(t) \geq 1/\sqrt{k}$ and

$$\begin{aligned} -\dot{\theta}(t) &\geq \sin^2\theta(t) + k^{3/2}\cos^2\theta(t) + h\sin\theta(t)\cos\theta(t) = \\ &= \begin{pmatrix}\sin\theta(t) & \cos\theta(t)\end{pmatrix} \begin{pmatrix} 1 & \frac{h}{2} \\ \frac{h}{2} & k^{3/2} \end{pmatrix} \begin{pmatrix} \sin\theta(t) \\ \cos\theta(t) \end{pmatrix} > 0. \end{aligned}$$

Hence θ is a continuous bijection between $I = [t_{n-1}, t_n]$ and its image $[\theta(t_n), \theta(t_{n-1})]$. We note by τ the inverse of θ, defined on $[\theta(t_n), \theta(t_{n-1})]$, which satisfies

$$\frac{d\tau}{d\vartheta}(\vartheta) = \frac{1}{\dot{\theta}(\tau(\vartheta))} = -\frac{1}{\sin^2\vartheta + k^2\gamma(\tau(\vartheta))\cos^2\vartheta + h\sin\vartheta\cos\vartheta}. \tag{3.27}$$

Writing $\rho = r \circ \tau$ and using Equations (3.13c) and (3.27), we have

$$\frac{d}{d\vartheta}\ln\rho = -\frac{\sin\vartheta\cos\vartheta(1 - k^2\gamma\circ\tau(\vartheta)) - h\sin^2\vartheta}{\sin^2\vartheta + k^2\gamma\circ\tau(\vartheta)\cos^2\vartheta + h\sin\vartheta\cos\vartheta}.$$

We can integrate this expression from $\theta(t_n)$ to $\theta(t_{n-1}) = \theta(t_n) + \pi$, obtaining

$$\ln\frac{r(t_n)}{r(t_{n-1})} = \int_{\theta(t_n)}^{\theta(t_n)+\pi} F(\vartheta, \gamma\circ\tau(\vartheta))d\vartheta,$$

with $F(\vartheta,\gamma) = \frac{\sin\vartheta\cos\vartheta(1-k^2\gamma) - h\sin^2\vartheta}{\sin^2\vartheta + k^2\gamma\cos^2\vartheta + h\sin\vartheta\cos\vartheta}$. We claim that if $\gamma_0 \geq 1/\sqrt{k}$ is constant then

$$\int_{\theta(t_n)}^{\theta(t_n)+\pi} F(\vartheta, \gamma_0)d\vartheta \leq 0. \tag{3.28}$$

Indeed, by π-periodicity of F with respect to its first variable, $\int_{\theta(t_n)}^{\theta(t_n)+\pi} F(\vartheta,\gamma_0)d\vartheta = \int_{-\pi/2}^{\pi/2} F(\vartheta,\gamma_0)d\vartheta$. Moreover, thanks to the change of variables $\hat{t} = \tan\vartheta$,

$$\int_{-\pi/2}^{\pi/2} F(\vartheta,\gamma_0)d\vartheta = \int_{-\infty}^{+\infty} \frac{(1-k^2\gamma_0)\hat{t} - h\hat{t}^2}{(\hat{t}^2 + h\hat{t} + k^2\gamma_0)(\hat{t}^2+1)}d\hat{t} \leq \int_{-\infty}^{+\infty} \frac{(1-k^2\gamma_0)\hat{t}}{(a_0\hat{t}^2 + b_0)(\hat{t}^2+1)}d\hat{t} = 0,$$

where $a_0 = \frac{k^2\gamma_0 - h^2/4}{k^2\gamma_0 + h^2/4}$ and $b_0 = \frac{k^2\gamma_0}{2} - \frac{h^2}{8}$ are positive because $\gamma_0 \geq 1/\sqrt{k}$ and thanks to (3.23).

By (3.28) we have

$$\ln\frac{r(t_n)}{r(t_{n-1})} \leq \int_{\theta(t_n)}^{\theta(t_n)+\pi} \left[F(\vartheta,\gamma\circ\tau(\vartheta)) - F(\vartheta,\gamma_0)\right]d\vartheta. \tag{3.29}$$

We compute
$$\frac{\partial F}{\partial \gamma}(\vartheta,\gamma)=-\frac{k^2\sin\vartheta\cos\vartheta}{(\sin^2\vartheta+k^2\gamma\cos^2\vartheta+h\sin\vartheta\cos\vartheta)^2},$$
and thus, for $t\in I$,
$$\left|\frac{\partial F}{\partial \gamma}(\vartheta,\gamma(t))\right|\leq\frac{k^2\,|\sin\vartheta|\,|\cos\vartheta|}{(\sin^2\vartheta+k^{3/2}\cos^2\vartheta+h\sin\vartheta\cos\vartheta)^2}.$$
We now take $\gamma_0=\beta(t_{n-1})$ in (3.29), obtaining
$$\ln\frac{r(t_n)}{r(t_{n-1})}\leq\int_{\theta(t_n)}^{\theta(t_n)+\pi}\frac{k^2\,|\sin\vartheta|\,|\cos\vartheta|}{(\sin^2\vartheta+k^{3/2}\cos^2\vartheta+h\sin\vartheta\cos\vartheta)^2}\,|\gamma\circ\tau(\vartheta)-\beta(t_{n-1})|\,d\vartheta.\tag{3.30}$$
For almost every $t\in I$, one can estimate
$$|\gamma(t)-\beta(t_{n-1})|\leq|\beta(t)-\beta(t_{n-1})|+\left|\frac{\dot\alpha(t)}{2k}\right|\leq M(t_n-t_{n-1})+\frac{M}{2k}.\tag{3.31}$$
We take k satisfying (3.12), which means that $0\leq\gamma(t)\leq 1$ for almost every $t\in\mathbb{R}_+$, and thus, by integrating (3.16) from t_{n-1} to t_n, we obtain
$$\begin{aligned}t_n-t_{n-1}&=-\int_{t_{n-1}}^{t_n}\frac{\dot\theta(s)}{\sin^2\theta(s)+k^2\gamma(s)\cos^2\theta(s)+h\sin\theta(s)\cos\theta(s)}ds\geq\\&\geq\int_{\theta(t_n)}^{\theta(t_n)+\pi}\frac{d\theta}{\sin^2\theta+k^2\cos^2\theta+h\sin\theta\cos\theta}\geq\frac{\pi}{k}.\end{aligned}$$
Inequality (3.31) thus yields $|\gamma(t)-\beta(t_{n-1})|\leq M\left(1+\frac{1}{2\pi}\right)(t_n-t_{n-1})<2M(t_n-t_{n-1})$. We use this estimate in (3.30), which leads to
$$\ln\frac{r(t_n)}{r(t_{n-1})}\leq 2k^2M(t_n-t_{n-1})\int_{\theta(t_n)}^{\theta(t_n)+\pi}\frac{|\sin\vartheta|\,|\cos\vartheta|}{(\sin^2\vartheta+k^{3/2}\cos^2\vartheta+h\sin\vartheta\cos\vartheta)^2}d\vartheta.\tag{3.32}$$
Notice that, for any $a>0$ and b satisfying $b^2<4a$,
$$\int_{-\pi/2}^{\pi/2}\frac{|\sin\vartheta|\,|\cos\vartheta|}{(\sin^2\vartheta+a\cos^2\vartheta+b\sin\vartheta\cos\vartheta)^2}d\vartheta=\frac{1}{A}+\frac{B}{A^{3/2}}\arctan\left(\frac{B}{\sqrt{A}}\right)\leq\frac{1}{A}\left(1+\frac{\pi}{2}C\right),$$
with $A=a-{b^2}/{4}>0$, $B={b}/{2}$ and $C={B}/{\sqrt{A}}=\frac{b}{\sqrt{4a-b^2}}$. Applying this to (3.32) we have
$$\ln\frac{r(t_n)}{r(t_{n-1})}\leq\frac{2k^{1/2}M(t_n-t_{n-1})}{1-\frac{M}{k^{1/2}}}\left(1+\frac{\pi}{2}\sqrt{\frac{kM}{2k^{3/2}-2kM}}\right)$$
and, since $\frac{1}{1-\frac{M}{k^{1/2}}}\left(1+\frac{\pi}{2}\sqrt{\frac{kM}{2k^{3/2}-2kM}}\right)\xrightarrow[k\to+\infty]{}1$, there exists $K_\star(M)$ such that, if
$$k\geq K_\star(M),\tag{3.33}$$
then $\frac{1}{1-\frac{M}{k^{1/2}}}\left(1+\frac{\pi}{2}\sqrt{\frac{kM}{2k^{3/2}-2kM}}\right)\leq 2$, and thus $\ln\frac{r(t_n)}{r(t_{n-1})}\leq 4k^{1/2}M(t_n-t_{n-1})$. We collect (3.12), (3.26) and (3.33) by setting $K_4(M)=\max\left(K_1(M),K_3(M),K_\star(M)\right)$ and requiring that $k>K_4(M)$. Under this hypothesis, we obtain $r(t_n)\leq r(t_{n-1})e^{4Mk^{1/2}(t_n-t_{n-1})}$, as required. ■

3.3.5 Estimations on intervals belonging to the family $\mathcal{I}_-$

We wish here to obtain a result analogous to Lemma 3.5 for the intervals in the class $\mathcal{I}_-$. We start by characterizing the duration of these intervals and the behavior of γ on them.

Lemma 3.6. *There exists $K_5(T,\mu,M)$ such that, if $k > K_5(T,\mu,M)$, then for every $I = [t_{n-1},t_n] \in \mathcal{I}_-$ one has $\gamma(t) \leq 3/\sqrt{k}$ for almost every $t \in I$ and*

$$\frac{\pi}{1+h+3k^{3/2}} \leq t_n - t_{n-1} < T.$$

Proof. We fix $I = [t_{n-1},t_n] \in \mathcal{I}_-$. If

$$k \geq M^2, \tag{3.34}$$

then $0 \leq \gamma(t) - \beta(t) \leq \frac{M}{k} \leq \frac{1}{\sqrt{k}}$, and thus $\gamma(t) \leq 3/\sqrt{k}$ almost everywhere on I. In addition, if

$$k > \left(\frac{8T}{3\mu}\right)^2, \tag{3.35}$$

we have $\beta(t) < \frac{2}{\sqrt{k}} < \frac{3\mu}{4T}$, and thus, by the persistence of excitation (3.10) of β, we obtain that $t_n - t_{n-1} < T$. Furthermore, (3.13a) implies that $-\dot{\theta} \leq 1 + 3k^{3/2} + h$ almost everywhere on I, and then by integrating on I we get $t_n - t_{n-1} \geq \frac{\pi}{1+h+3k^{3/2}}$. The lemma is proved by taking

$$K_5(T,\mu,M) = \max\left(M^2, \left(\frac{8T}{3\mu}\right)^2\right),$$

which collects (3.34) and (3.35). ■

We suppose from now on that $k > K_5(T,\mu,M)$. We define the class

$$\mathfrak{D}(T,\mu,M,k) = \left\{\alpha\left(1-\tfrac{1}{4}\alpha\right) + \frac{M-\dot{\alpha}}{2k} \mid \alpha \in \mathcal{D}(T,\mu,M)\right\}$$

which contains γ. We fix $I = [t_{n-1},t_n] \in \mathcal{I}_-$ and we remark that, if $\gamma \in \mathfrak{D}(T,\mu,M,k)$, then, for every $t_0 \in \mathbb{R}_+$, the function $t \mapsto \gamma(t+t_0)$ is also in $\mathfrak{D}(T,\mu,M,k)$. Up to a translation in time, we can then suppose $I = [0,\tau]$ with $\tau = t_n - t_{n-1} \in \left[\frac{\pi}{1+h+3k^{3/2}}, T\right)$. The solution $r(\tau)$ of (3.13c) at time τ can be written as $r(\tau) = r(0)e^{\Lambda\tau}$ for a certain constant Λ. Our goal is to estimate Λ uniformly with respect to γ, i.e., to estimate the maximal value of $\frac{1}{\tau}\ln\frac{\|y(\tau)\|}{\|y(0)\|}$ over all $\gamma \in \mathfrak{D}(T,\mu,M,k)$ with $\|\gamma\|_{L^\infty(0,\tau)} \leq 3/\sqrt{k}$, where y is a solution of (3.9) with both $y(0)$ and $y(\tau)$ in the axis y_1.

By homogeneity reasons we can choose $y_1(0) = -1$. Thus, by enlarging the class where we take γ, Λ is upper-bounded by the solution of the problem

$$\begin{cases} \text{Find } \sup \dfrac{1}{\tau}\ln\|y(\tau)\| \text{ with} \\ \tau \in \left[\dfrac{\pi}{1+h+3k^{3/2}}, T\right], \quad \gamma \in L^\infty([0,\tau],[0,1]), \\ \dot{y} = \begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma(t) & -h \end{pmatrix} y, \quad y(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad y(\tau) \in \mathbb{R}_+ \times \{0\}. \end{cases} \tag{3.36}$$

The discussion above can be summarized by the following result.

Lemma 3.7. *Let $\mathbf{\Lambda}(T,M,k)$ be the solution of Problem* (3.36) *and take $K_5(T,\mu,M)$ as in Lemma 3.6. If $k > K_5(T,\mu,M)$, then, for every $\gamma \in \mathfrak{D}(T,\mu,M,k)$ and for every $I = [t_{n-1},t_n] \in \mathfrak{I}_-$, we have*

$$r(t_n) \leq r(t_{n-1})e^{\mathbf{\Lambda}(T,M,k)(t_n-t_{n-1})}. \tag{3.37}$$

We can now focus on the problem of solving the maximization problem (3.36). We start by proving that the sup is attained.

Lemma 3.8. *Let $k > K_5(T,\mu,M)$ where K_5 is defined as in Lemma 3.6 and let $\mathbf{\Lambda}(T,M,k)$ be the solution of Problem* (3.36). *Then there exist $\tau_\star \in \left[\frac{\pi}{1+h+3k^{3/2}}, T\right]$ and $\gamma_\star \in L^\infty([0,\tau_\star],[0,1])$ such that, if $y_\star$ satisfies*

$$\dot{y}_\star = \begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma_\star(t) & -h \end{pmatrix} y_\star, \quad y_\star(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

then $y_\star(\tau) \in \mathbb{R}_+ \times \{0\}$ and $\frac{1}{\tau_\star}\ln\|y_\star(\tau_\star)\| = \mathbf{\Lambda}(T,M,k)$.

Proof. We start by taking a sequence $(\tau_n,\gamma_n)_{n\in\mathbb{N}}$ with $\tau_n \in \left[\frac{\pi}{1+h+3k^{3/2}}, T\right]$ and $\gamma_n \in L^\infty([0,\tau_n],[0,1])$ such that, denoting by y_n the solution of

$$\dot{y}_n = \begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma_n(t) & -h \end{pmatrix} y_n, \qquad y_n(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \tag{3.38}$$

we have $\lim_{n\to+\infty}\frac{1}{\tau_n}\ln\|y_n(\tau_n)\| = \mathbf{\Lambda}(T,M,k)$. Up to extending γ_n by 0 outside $[0,\tau_n]$, we can suppose that γ_n belongs to $L^\infty(I,[0,1])$ where $I = [0,T]$. By weak-$\star$ compactness of this space and by compactness of $\left[\frac{\pi}{1+h+3k^{3/2}}, T\right]$, we can find a subsequence of $(\gamma_n)_{n\in\mathbb{N}}$ weak-$\star$ converging to a certain function $\gamma_\star \in L^\infty(I,[0,1])$ and such that the corresponding subsequence of $(\tau_n)_{n\in\mathbb{N}}$ converges to $\tau_\star \in \left[\frac{\pi}{1+h+3k^{3/2}}, T\right]$. To simplify the notation, we still write $(\gamma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ for these subsequences.

We denote by $y_\star$ the solution of

$$\dot{y}_\star = \begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma_\star(t) & -h \end{pmatrix} y_\star, \qquad y_\star(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{3.39}$$

By considering the solutions y_n of (3.38) to be defined on $[0,T]$ and up to extracting a subsequence, we have $\lim_{n\to+\infty} y_n = y_\star$ uniformly on $[0,T]$, as it follows from Gronwall's lemma (see [5, Proposition 21] for details). In particular, $y_\star(\tau_\star) \in \mathbb{R}_+ \times \{0\}$. Moreover, $\frac{1}{\tau_\star}\ln\|y_\star(\tau_\star)\| = \lim_{n\to+\infty}\frac{1}{\tau_n}\ln\|y_n(\tau_n)\| = \mathbf{\Lambda}(T,M,k)$, which completes the proof. ■

Since the sup in Problem (3.36) is attained, the Pontryagin Maximum Principle (PMP for short) can be used to characterize the maximizing trajectory $y_\star$. For a formulation of the PMP with boundary conditions as those used here, see, for instance, [4, Theorem 7.3].

Lemma 3.9. *Let $\tau_\star$, $\gamma_\star$ and $y_\star$ be as in the statement of Lemma 3.8. Then, up to a modification on a set of measure zero, $\gamma_\star(\cdot)$ is piecewise constant with values in $\{0,1\}$. Moreover, there exist $s_1,s_2 \in (0,\tau_\star)$ with $s_1 \leq s_2$ such that $\gamma_\star(t) = 1$ if $t \in [0,s_1) \cup (s_2,\tau_\star]$ and $\gamma_\star(t) = 0$ if $t \in (s_1,s_2)$. The trajectory $y_\star$ is contained in the quadrant $Q_2 = \{(y_1,y_2) \mid y_1 \leq 0, y_2 \geq 0\}$ during the interval $[0,s_1]$ and in the quadrant $Q_1 = \{(y_1,y_2) \mid y_1 \geq 0, y_2 \geq 0\}$ during $[s_2,\tau_\star]$.*

Proof. The adjoint vector $p = (p_1, p_2)$ whose existence is guaranteed by the PMP satisfies

$$\begin{cases} \dot{p}_1(t) = 3k^{3/2}\gamma_\star(t)p_2(t), \\ \dot{p}_2(t) = hp_2(t) - p_1(t). \end{cases} \tag{3.40}$$

The Hamiltonian is given by $p\begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma & -h \end{pmatrix} y = p_1 y_{2\star} - 3k^{3/2}\omega p_2 y_{1\star} - hp_2 y_{2\star}$, and so the maximization condition provided by the PMP writes

$$\gamma_\star(t)p_2(t)y_{1\star}(t) = \min_{\omega\in[0,1]} \omega p_2(t)y_{1\star}(t). \tag{3.41}$$

Define $\Phi(t) = p_2(t)y_{1\star}(t)$ so that (up to a modification on a set of measure zero)

$$\gamma_\star(t) = \begin{cases} 0 & \text{if } \Phi(t) > 0, \\ 1 & \text{if } \Phi(t) < 0. \end{cases} \tag{3.42}$$

We remark that Φ is absolutely continuous and

$$\dot{\Phi}(t) = hp_2(t)y_{1\star}(t) - p_1(t)y_{1\star}(t) + p_2(t)y_{2\star}(t).$$

Hence $\dot{\Phi}$ is absolutely continuous as well.

We next show that the the zeros of Φ are isolated. Indeed, consider $t \in [0, \tau_\star]$ such that $\Phi(t) = 0$. Clearly, such a zero is isolated if $\dot{\Phi}(t) \neq 0$. Therefore, one can assume that $\dot{\Phi}(t) = 0$. Since p never vanishes and

$$\Phi(t) = \begin{pmatrix} p_1(t) & p_2(t) \end{pmatrix} \begin{pmatrix} 0 \\ y_{1\star}(t) \end{pmatrix}, \qquad \dot{\Phi}(t) = \begin{pmatrix} p_1(t) & p_2(t) \end{pmatrix} \begin{pmatrix} -y_{1\star}(t) \\ y_{2\star}(t) + hy_{1\star}(t) \end{pmatrix},$$

one immediately concludes that $y_{1\star}(t) = 0$. Therefore, a zero of both Φ and $\dot{\Phi}$ must be a zero of $y_{1\star}$. Since $y_\star$ never vanishes and $\dot{y}_{1\star} = y_{2\star}$, the zeros of $y_{1\star}$ are isolated. Then Φ admits a finite number of zeros on $[0, \tau_\star]$ and $\gamma_\star(t)$ is piecewise constant with values in $\{0, 1\}$.

In order to conclude the proof of the lemma, i.e., to determine the rule of switching for $\gamma_\star$, we adapt the techniques developed in [3] for the analysis of time-optimal two-dimensional control problems. We start by defining the matrices

$$F = \begin{pmatrix} 0 & 1 \\ 0 & -h \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that $\begin{pmatrix} 0 & 1 \\ -3k^{3/2}\gamma & -h \end{pmatrix} y = Fy - 3k^{3/2}\gamma Gy$ and

$$\Phi(t) = p(t)Gy_\star(t), \qquad \dot{\Phi}(t) = p(t)[G, F]y_\star(t),$$

where $[G, F] = GF - FG$ is the commutator of the matrices G and F. We define the functions

$$\Delta_A(y) = \det(Fy, Gy) = y_1 y_2, \qquad \Delta_B(y) = \det(Gy, [G, F]y) = y_1^2.$$

The vectors Fy and Gy are linearly independent outside $\Delta_A^{-1}(0)$. Hence, for every $y \in \mathbb{R}^2 \setminus \Delta_A^{-1}(0)$ there exist $f_S(y), g_S(y) \in \mathbb{R}$ such that $[G,F]y = f_S(y)Fy + g_S(y)Gy$. We have $\Delta_B(y) = f_S(y)\det(Gy, Fy) = -f_S(y)\Delta_A(y)$, which shows that

$$f_S(y) = -\frac{\Delta_B(y)}{\Delta_A(y)} = -\frac{y_1}{y_2}.$$

We now want to characterize the times at which $\gamma_\star$ switches between 0 and 1. We take an open time interval J during which $y_\star$ is outside the axes and we assume that $\gamma_\star$ switches at $t_\star \in J$. Equation (3.42) and the continuity of Φ show that $\Phi(t_\star) = 0$. The discussion above also shows that

$$\dot{\Phi}(t_\star) = p(t_\star)[G,F]y_\star(t_\star) = f_S(y_\star(t_\star))p(t_\star)Fy_\star(t_\star) \neq 0. \tag{3.43}$$

By the PMP, the Hamiltonian

$$t \mapsto p(t)Fy_\star(t) - 3k^{3/2}\gamma_\star(t)p(t)Gy_\star(t) \tag{3.44}$$

is constant almost everywhere and equal to $\frac{\lambda_0}{\tau_\star^2}\ln\|y_\star(\tau_\star)\|$ for some $\lambda_0 \geq 0$. We deduce that $p(t_\star)Fy_\star(t_\star) > 0$ and that the signs of $\dot{\Phi}(t_\star)$ and $f_S(y_\star(t_\star))$ coincide. Hence, at most one switch may happen on J, from 1 to 0 if the trajectory lies in Q_1 and from 0 to 1 if it lies in Q_2.

Let us focus on what happens on the axes. Starting from $y_\star(0) = (-1,0)^{\mathrm{T}}$, the choice of $\gamma_\star(t) = 0$ cannot maximize the cost. Hence the trajectory enters in Q_2 and $\gamma_\star(t) = 1$ in a right-neighborhood of 0. Moreover, it exits Q_2 through the y_2-axis. Since both vector fields corresponding to $\gamma = 0$ and $\gamma = 1$ are transversal to the positive semi-axis y_2 and point towards Q_1, there exists a unique $s_\star$ such that $y_\star(s_\star)$ is in the y_2-axis.

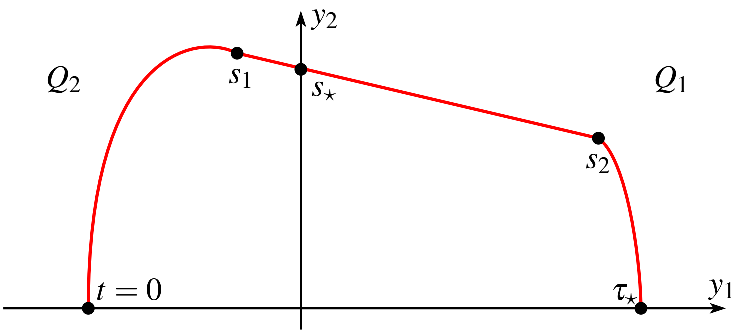

Figure 3.1: Representation of the solution $y_\star$. As stated in Lemma 3.9, $y_\star$ is a solution of (3.39) with $\gamma_\star(t) = 1$ on $[0, s_1)$, $\gamma_\star(t) = 0$ on (s_1, s_2) and $\gamma_\star(t) = 1$ on $(s_2, \tau_\star]$. The solution $y_\star$ lies on Q_2 on $[0, s_1]$ and on Q_1 on $[s_2, \tau_\star]$.

Finally, we remark that the trajectories of the vector field corresponding to $\gamma = 0$ never reach the y_1-axis in finite time unless they start on it. Therefore, either $\gamma_\star$ is identically equal to 1 or it switches twice, once from 1 to 0 in Q_2 and then from 0 to 1 in Q_1 (see Figure 3.1). ■

Lemma 3.9 reduces the optimization problem (3.36) into a maximization over the two scalar parameters s_1 and s_2. A bound on the maximal value of such problem is given by the following lemma.

Lemma 3.10. *Let $K_5(T,\mu,M)$ be as in Lemma 3.6. There exists $K_6(M)$ such that, if $k > K_5(T,\mu,M)$ and $k > K_6(M)$, then*

$$\mathbf{\Lambda}(T,M,k) \le \sqrt{3}k^{3/4}. \tag{3.45}$$

Proof. Assume that $k > K_5(T,\mu,T)$ and take $\tau_\star$, $\gamma_\star$ and $y_\star$ as in Lemma 3.8. Then $\mathbf{\Lambda}(T,M,k) = \frac{1}{\tau_\star}\ln\|y_\star(\tau_\star)\|$. Let s_1 and s_2 be as in Lemma 3.9. Along the interval $[0,s_1]$, then, $\gamma_\star(t)=1$ and $y_\star$ satisfies

$$\dot{y}_\star = \begin{pmatrix} 0 & 1 \\ -3k^{3/2} & -h \end{pmatrix} y_\star, \qquad y_\star(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{3.46}$$

Now take

$$k > \frac{M^2}{9}, \tag{3.47}$$

so that $3k^{3/2} > h^2/4$ and $\omega = \sqrt{3k^{3/2} - h^2/4}$ is well defined and positive. A direct computation shows that the solution of (3.46) is

$$y_{1\star}(t) = -e^{-\frac{h}{2}t}\left(\cos\omega t + \frac{h}{2\omega}\sin\omega t\right), \tag{3.48a}$$

$$y_{2\star}(t) = \left(\omega + \frac{h^2}{4\omega}\right)e^{-\frac{h}{2}t}\sin\omega t. \tag{3.48b}$$

In the interval $[s_1,s_2]$, we have $\gamma_\star(t)=0$ and then $y_\star$ satisfies

$$\dot{y}_\star = \begin{pmatrix} 0 & 1 \\ 0 & -h \end{pmatrix} y_\star,$$

which yields the solution

$$y_{1\star}(t) = \frac{1}{h}\left(1-e^{-h(t-s_1)}\right)y_{2\star}(s_2) + y_{1\star}(s_1), \tag{3.49a}$$

$$y_{2\star}(t) = e^{-h(t-s_1)}y_{2\star}(s_1). \tag{3.49b}$$

Finally, in the interval $[s_2,\tau_\star]$, we have $\gamma_\star(t)=1$ and thus the differential equation satisfied by $y_\star$ is the same as in (3.46), but we now consider the boundary condition $y_\star(\tau_\star) = \begin{pmatrix}\xi & 0\end{pmatrix}^T$ with $\xi > 0$. This yields the solution

$$y_{1\star}(t) = \xi e^{-\frac{h}{2}(t-\tau_\star)}\left(\cos\omega(t-\tau_\star) + \frac{h}{2\omega}\sin\omega(t-\tau_\star)\right), \tag{3.50a}$$

$$y_{2\star}(t) = -\xi\left(\omega + \frac{h^2}{4\omega}\right)e^{-\frac{h}{2}t}\sin\omega(t-\tau_\star). \tag{3.50b}$$

We have

$$\mathbf{\Lambda}(T,M,k) = \frac{\ln\xi}{\tau_\star}. \tag{3.51}$$

To simplify the notation, we write $\sigma = s_2 - s_1$. Using the parametrization of the solution given above and imposing that the curves given in (3.49) and (3.50) coincide at s_2, we get

$$\xi e^{\frac{h}{2}(\tau_\star - s_2)} \sin\omega(\tau_\star - s_2) = \frac{e^{-h\sigma}}{\omega + \frac{h^2}{4\omega}} y_{2\star}(s_1), \tag{3.52a}$$

$$\xi e^{\frac{h}{2}(\tau_\star - s_2)} \cos\omega(\tau_\star - s_2) = y_{1\star}(s_1) + y_{2\star}(s_1)\left[\frac{1}{h}\left(1 - e^{-h\sigma}\right) + \frac{he^{-h\sigma}}{2\omega^2 + h^2/4}\right]. \tag{3.52b}$$

We can thus express ξ in terms of s_1, σ and $\tau_\star$, and rewrite (3.51) as

$$\mathbf{\Lambda}(T,M,k) = \frac{-h(\tau_\star - s_2) + \ln\left[\left(y_{1\star}(s_1) + y_{2\star}(s_1)\left[\frac{1}{h}\left(1 - e^{-h\sigma}\right) + \frac{he^{-h\sigma}}{2\omega^2 + h^2/4}\right]\right)^2 + \left(\frac{e^{-h\sigma} y_{2\star}(s_1)}{\omega + \frac{h^2}{4\omega}}\right)^2\right]}{2[s_1 + \sigma + (\tau_\star - s_2)]}. \tag{3.53}$$

To give a bound on $\mathbf{\Lambda}(T,M,k)$, we first use that $-h(\tau_\star - s_2) \le 0$ and $\tau - s_2 \ge 0$. By the expression (3.48b) of $y_{2\star}$ in $[0, s_1]$, we get

$$\frac{e^{-h\sigma} y_{2\star}(s_1)}{\omega + \frac{h^2}{4\omega}} \le \sin\omega s_1.$$

We also have that $y_{1\star}(s_2) \ge 0$ and $y_{2\star}(s_2) \ge 0$, and then (3.50) implies that $\sin\omega(\tau_\star - s_2) \ge 0$ and $\cos\omega(\tau_\star - s_2) \ge 0$. Equation (3.52b) implies that

$$y_{1\star}(s_1) + y_{2\star}(s_1)\left[\frac{1}{h}\left(1 - e^{-h\sigma}\right) + \frac{he^{-h\sigma}}{2\omega^2 + h^2/4}\right] \ge 0.$$

Recall, moreover, that Lemma 3.9 implies that $y_{1\star}(s_1) \le 0$. We also have $\frac{1}{h}(1 - e^{-h\sigma}) \le \sigma$ and, by (3.48b), we obtain

$$y_{2\star}(s_1)\frac{he^{-h\sigma}}{2\omega^2 + h^2/4} \le \frac{h}{2\omega}\sin\omega s_1.$$

We bound $y_{2\star}(s_1)$ from above by $\left(\omega + \frac{h^2}{4\omega}\right)\sin\omega s_1$ and, combining all the previous estimates, we obtain

$$\mathbf{\Lambda}(T,M,k) \le \frac{\ln(\sin^2\omega s_1) + \ln\left[1 + \left(\sigma\left(\omega + \frac{h^2}{4\omega}\right) + \frac{K}{2\omega}\right)^2\right]}{2(s_1 + \sigma)}.$$

By (3.47), we have $\frac{h}{2\omega} \le 1$ and $\omega + \frac{h^2}{4\omega} \le 2\omega$, which finally yields

$$\mathbf{\Lambda}(T,M,k) \le \frac{\ln(\sin^2\omega s_1) + \ln\left[1 + (2\omega\sigma + 1)^2\right]}{2(s_1 + \sigma)}.$$

We now define $s' = \omega s_1$, $\sigma' = \omega\sigma$, and then we have

$$\mathbf{\Lambda}(T,M,k) \le \omega\frac{\ln(\sin^2 s') + \ln\left[1 + (2\sigma' + 1)^2\right]}{2(s' + \sigma')}.$$

A direct computation shows that the function

$$(s',\sigma') \mapsto \frac{\ln(\sin^2 s') + \ln\left[1+(2\sigma'+1)^2\right]}{2(s'+\sigma')}$$

is upper bounded over $(\mathbb{R}_+^*)^2$ by 1, and, by bounding ω by $\sqrt{3}k^{3/4}$, we obtain the desired estimate (3.45) under the hypothesis $k > \max(K_5(T,\mu,M), K_6(M))$ with $K_6(M) = M^2/9$. ■

By combining this result with Lemma 3.7, we obtain the desired estimate on the growth of y.

Corollary 3.11. *Let $K_5(T,\mu,M)$ be as in Lemma 3.6 and $K_6(M)$ as in Lemma 3.10. If $k >$ $\max(K_5(T,\mu,M), K_6(M))$, then, for every $\gamma \in \mathfrak{D}(T,\mu,M,k)$ and $I = [t_{n-1},t_n] \in \mathfrak{I}_-$, the solution r of* (3.13b) *satisfies*

$$r(t_n) \leq r(t_{n-1}) e^{\sqrt{3}k^{3/4}(t_n - t_{n-1})}.$$

3.3.6 Estimate of y

Now that we estimated the growth of y on intervals of the classes $\mathfrak{I}_+$ and $\mathfrak{I}_-$, we only have to join these results in order to estimate the growth of y over any interval $[0,t]$.

Lemma 3.12. *There exists $K_7(T,\mu,M)$ such that, for $k > K_7(T,\mu,M)$, there exists a constant C depending only on T, M and k such that, for every signal $\alpha \in \mathcal{D}(T,\mu,M)$, every solution y of* (3.9) *and every $t \in \mathbb{R}_+$, we have*

$$\|y(t)\| \leq C \|y(0)\| e^{2k^{3/4}t}. \tag{3.54}$$

Proof. Suppose that k is large enough (that is, larger that the maximal value of the functions $K_1,\dots,K_6$) so that all previous results can be applied. Fix $\alpha \in \mathcal{D}(T,\mu,M)$ and $t \in \mathbb{R}_+$.

Since the sequence $(t_n)_{n\in\mathbb{N}}$ defined in (3.21) tends monotonically to $+\infty$ as $n \to +\infty$, then there exists $N \in \mathbb{N}$ such that $t \in [t_{N-1}, t_N)$ (with the convention that $t_{-1} = 0$). We can use Lemma 3.5 and Corollary 3.11 to estimate the growth of y in each interval I_n, $n = 1,\dots,N-1$. The length of the two intervals $I_0 = [0,t_0]$ and $[t_{N-1},t]$ is bounded by T, since, as proved in Lemma 3.2, $\theta(s+T) - \theta(s) \leq -2\pi$ for every $s \in \mathbb{R}_+$. By Equation (3.13c), we have $\frac{d}{dt}\ln r \leq k^2 + h + 1$, and then

$$r(t_0) \leq r(0) e^{T(k^2+h+1)}, \qquad r(t) \leq r(t_{N-1}) e^{T(k^2+h+1)}.$$

We now combine these two results with (3.25) and (3.45), which yields

$$r(t) \leq e^{2T(k^2+h+1)} r(0) \left(\prod_{\substack{n=1 \\ I_n \in \mathfrak{I}_+}}^{N-1} e^{4Mk^{1/2}(t_n - t_{n-1})} \right) \left(\prod_{\substack{n=1 \\ I_n \in \mathfrak{I}_-}}^{N-1} e^{\sqrt{3}k^{3/4}(t_n - t_{n-1})} \right) \leq C r(0) e^{\sqrt{3}k^{3/4}t + 4Mk^{1/2}t}$$

with $C = e^{2T(k^2+h+1)}$, which depends only on T, k and M (through h). It suffices to take k large enough, and more precisely $k \geq \left(\frac{4M}{2-\sqrt{3}}\right)^4$, in order to obtain (3.54). We then take $K_7(T,\mu,M)$ as the maximum between $\left(\frac{4M}{2-\sqrt{3}}\right)^4$ and the values of the functions $K_1 \dots, K_6$ and the proof is concluded. ■

3.4 Proof of Theorem 3.1

By combining (3.54) and the relation (3.6) between x and y, we can prove Theorem 3.1 .

Proof of Theorem 3.1. Let λ be a real constant. Take $k > K_7(T,\mu,M)$ and consider the feedback gain $K = \begin{pmatrix} k^2 & k \end{pmatrix}$. By (3.6), for every $t \in \mathbb{R}_+$, we have

$$\|x(t)\| \leq e^{-\frac{k}{2}\int_0^t \alpha(s)ds+\frac{h}{2}t}\left(1+\frac{h}{2}+\frac{k}{2}\right)\|y(t)\|, \qquad \|y(t)\| \leq e^{\frac{k}{2}\int_0^t \alpha(s)ds-\frac{h}{2}t}\left(1+\frac{h}{2}+\frac{k}{2}\right)\|x(t)\|,$$

and then, in particular, $\|y(0)\| \leq \left(1+\frac{h}{2}+\frac{k}{2}\right)\|x(0)\|$. Thus, combining these inequalities with (3.54), we obtain that $\|x(t)\| \leq C'\|x(0)\| e^{-\frac{k}{2}\int_0^t \alpha(s)ds+\frac{h}{2}t+2k^{3/4}t}$, where C' is a constant depending only on k, M and T. We now use $\int_0^t \alpha(s)ds \geq \frac{\mu}{T}t-\mu$ to obtain $\|x(t)\| \leq \overline{C}\|x(0)\| e^{(-\frac{k}{2}\frac{\mu}{T}+\frac{h}{2}+2k^{3/4})t}$ for a new constant $\overline{C}$, which depends on k, M, T and μ. There exists $K(T,\mu,M,\lambda)$ such that, for $k > K(T,\mu,M,\lambda)$, we have $-\frac{k}{2}\frac{\mu}{T}+\frac{h}{2}+2k^{3/4} \leq -\lambda$ and then $\|x(t)\| \leq \overline{C}\|x(0)\| e^{-\lambda t}$. This concludes the proof, since $\limsup_{t\to+\infty}\frac{\ln\|x(t)\|}{t} \leq -\lambda$. ■

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