Generic controllability of the bilinear Schrödinger equation on 1-D domains: the case of measurable potentials *

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Abstract

In recent years, several sufficient conditions for the controllability of the Schrödinger equation have been proposed. In this article, we discuss the genericity of these conditions with respect to the variation of the controlled or the uncontrolled potential. In the case where the Schrödinger equation is set on a domain of dimension one, we improve the results in the literature, removing from the previously known genericity results some unnecessary technical assumptions on the regularity of the potentials.

1 Introduction

In this paper we consider controlled Schrödinger equations of the type

$$i\frac{\partial\psi}{\partial t}(t,x) = (-\Delta + V(x) + u(t)W(x))\psi(t,x), \qquad u(t) \in U, \tag{1}$$

where $\psi : [0,+\infty) \times \Omega \to \mathbf{C}$ for some domain Ω of $\mathbf{R}^d$, $d \geq 1$, V, W are real-valued functions and $U = [0,\delta)$ for some $\delta > 0$. We will assume either that Ω, V, W are bounded and that ψ satisfies Dirichlet boundary conditions on $\partial\Omega$ or that $\Omega = \mathbf{R}^d$ and $-\Delta + V + uW$ has discrete spectrum for every $u \in U$. We look at (1) as at a control system evolving in the unit sphere of $L^2(\Omega,\mathbf{C})$, whose state $\psi(t,\cdot)$ is called the *wavefunction* of the Schrödinger equation. When W is in $L^\infty(\Omega,\mathbf{R})$, the multiplication operator $L^2(\Omega,\mathbf{C}) \ni \psi \to W\psi \in L^2(\Omega,\mathbf{C})$ is bounded and then it is known that the Schrödinger equation (1) is not exactly controllable (see [5, 32]). In certain cases, when Ω is of dimension one, a complete description of reachable sets has been provided (see [7, 8]). In the general case, however, such a description seems unattainable and one focuses on the issue of approximate controllability of system (1), that is, on the question

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of whether it is possible to connect any fixed initial state to any neighbourhood of any fixed final state by an admissible trajectory of the control system (1).

Several approaches have been developed to identify conditions on V and W which guarantee approximate controllability of (1). Let us mention in particular the approaches based on: Lyapunov functions [9, 22, 23, 24, 26], adiabatic evolution [1, 14, 20], Lie-bracket conditions in Banach spaces and in partially invariant finite-dimensional subspaces [10, 11, 19].

In this paper we focus on the approach developed in [13, 12, 15, 16], which is based on the idea of dropping the invariance requirement for the finite-dimensional spaces and replacing it with some motion planning strategy which makes some finite-dimensional spaces "almost invariant", in the sense that the norm of the projection of the solution in the finite-dimensional subspace stays as close to one as desired. (An analogous approach for the Navier–Stokes equation has been developed in [3, 30].) The approach allowed to obtain the most general known sufficient conditions for approximate controllability, which can also be applied, for instance, to Schrödinger equations on Riemannian (non-Euclidean) manifolds. Another advantage of this approach is that it also guarantees stronger notions of controllability than approximate controllability among wavefunctions. Indeed, it also implies approximate controllability between density matrices, simultaneous controllability for several initial conditions, tracking up to phases, etc (for details, see [13]). Let us mention that similar notions of controllability have also be obtained in [19].

The aim of this paper is to show that, in the case $d = 1$, the approximate controllability of (1) is generic with respect to V (that is, it holds for all V in a residual—in particular, dense—subset of the set of all possible potentials, endowed with a suitable topology), once some non-constant potential W is fixed and, similarly, that it is also generic with respect to W, once V is fixed. Such results are proved by showing that the sufficient conditions proposed in [12, 13, 16] are generic.

The genericity of the sufficient conditions of [16] has first been studied in [21]. We improve here the results obtained in [21], by removing the assumption that the fixed potential (either V or W in the two cases presented above) is absolutely continuous. We should mention, hover, that the results in [21] concern any dimension $d \in \mathbf{N}$ of the domain Ω, while the technique developed here requires d to be equal to 1.

Let us mention that some other genericity results for the approximate controllability of the Schrödinger equation exist in the literature [6, 24, 25, 27]. Such results are typically obtained by allowing variations of the pair (V, W) or the triple (Ω, V, W), instead of a single element.

Our approach, shared with [21, 27], is based on analytic long-range perturbations. The idea is the following: denote by Γ the class of systems on which the genericity of a certain property P is studied. If we are able to prove the existence of at least one element of Γ satisfying P, then we can *propagate* P if we can express P as the non-annihilation of some functions which are analytic in Γ. In this way we can prove that the property holds in a dense subset of Γ. A key property which allows this propagation to be performed is a result by Teytel in [31], which guarantees that between any two discrete-spectrum operators $-\Delta + V_1$ and $-\Delta + V_2$ (in a suitably defined class) there exists an analytic path $\mu \mapsto -\Delta + V_\mu$ such that all eigenvalues of $-\Delta + V_\mu$ are simple for any $\mu \in (1, 2)$.

The paper is organized as follows. In Section 2 we fix the mathematical framework, introducing the notion of solutions to the control system (1) and the notion of genericity that is investigated in the paper. In Section 3 we recall the sufficient conditions for approximate controllability obtained in [16] and the genericity results proved in [21]. Section 4 contains the main technical argument which allows us to improve the results in [21]. In terms of the discussion above, it contains the proof of the existence of the element whose good properties can be propagated globally by analytic perturbation. Finally, in Section 5 we develop the analytic propagation argument, showing that the approximate controllability is generic separately with respect to V or W.

2 Mathematical framework

2.1 Notation and basic definitions

Let $\mathbf{N}$ be the set of positive integers. For $d \in \mathbf{N}$, denote by Ξ_d the set of nonempty, open, bounded and connected subsets of $\mathbf{R}^d$ and let $\Xi_d^\infty = \Xi_d \cup \{\mathbf{R}^d\}$. Take $U = [0, \delta) \subset \mathbf{R}$ for some $\delta > 0$.

In the following we consider the Schrödinger equation (1) assuming that the potentials V, W are taken in $L^\infty(\Omega, \mathbf{R})$ if Ω belongs to Ξ_d and that $V, W \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R})$ and $\lim_{\|x\| \to \infty} V(x) + uW(x) = +\infty$ for every $u \in U$ if $\Omega = \mathbf{R}^d$. Then, for every $u \in U$, $-\Delta + V + uW$ (with Dirichlet boundary conditions if Ω is bounded) is a skew-adjoint operator on $L^2(\Omega, \mathbf{C})$ with compact resolvent and discrete spectrum (see [17, 28]). We denote by $\sigma(\Omega, V + uW) = (\lambda_j(\Omega, V + uW))_{j \in \mathbf{N}}$ the non-decreasing sequence of eigenvalues of $-\Delta + V + uW$, counted according to their multiplicities, and by $(\phi_j(\Omega, V + uW))_{j \in \mathbf{N}}$ a corresponding sequence of real-valued eigenfunctions, forming an orthonormal basis of $L^2(\Omega, \mathbf{C})$. If $j \in \mathbf{N}$ is such that $\lambda_j(\Omega, V + uW)$ is simple, then $\phi_j(\Omega, V + uW)$ is uniquely defined up to sign.

For every $\Omega \in \Xi_d$ let $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ be equal to $L^\infty(\Omega, \mathbf{R})$. For $\Omega = \mathbf{R}^d$ let

$$\mathcal{V}(\mathbf{R}^d) = \{V \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}) \mid \lim_{\|x\| \to \infty} V(x) = +\infty\},$$

$$\mathcal{W}(\mathbf{R}^d) = \left\{W \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}) \mid \operatorname*{ess\,sup}_{x \in \mathbf{R}^d} \frac{\log(|W(x)| + 1)}{\|x\| + 1} < \infty\right\}.$$

For every $\Omega \in \Xi_d^\infty$ let, moreover,

$$\mathcal{Z}(\Omega, U) = \{(V, W) \mid V \in \mathcal{V}(\Omega),\ W \in \mathcal{W}(\Omega), V + uW \in \mathcal{V}(\Omega) \text{ for every } u \in U\}.$$

If $(V, W) \in \mathcal{Z}(\Omega, U)$, each operator $-\Delta + V + uW$, $u \in U$, generates a group of unitary transformations $e^{it(-\Delta + V + uW)} : L^2(\Omega, \mathbf{C}) \to L^2(\Omega, \mathbf{C})$. In particular, $e^{it(-\Delta + V + uW)}(\mathcal{S}) = \mathcal{S}$ where $\mathcal{S}$ denotes the unit sphere of $L^2(\Omega, \mathbf{C})$. For every piecewise constant control function $u(\cdot)$ with values in U and every initial condition $\psi_0 \in L^2(\Omega, \mathbf{C})$, we can associate a solution

$$\psi(t; \psi_0, u) = e^{-i(t - \sum_{l=1}^{j-1} t_l)(-\Delta + V + u_j W)} \circ e^{-it_{j-1}(-\Delta + V + u_{j-1} W)} \circ \cdots \circ e^{-it_1(-\Delta + V + u_1 W)}(\psi_0), \quad (2)$$

where $0 \le \sum_{l=1}^{j-1} t_l \le t < \sum_{l=1}^{j} t_l$ and

$$u(\tau) = u_k \quad \text{if } \sum_{l=1}^{k-1} t_l \le \tau < \sum_{l=1}^{k} t_l$$

for $k = 1, \ldots, j$.

Definition 2.1. *Given $(V, W) \in \mathcal{Z}(\Omega, U)$ we say that the quadruple (Ω, V, W, U) is approximately controllable if for every $\psi_0, \psi_1 \in \mathcal{S}$ and every $\varepsilon > 0$ there exist $T > 0$ and $u : [0, T] \to U$ piecewise constant such that $\|\psi_1 - \psi(T; \psi_0, u)\| < \varepsilon$.*

2.2 Topologies and genericity

Let us endow $\mathcal{V}(\Omega)$, $\mathcal{W}(\Omega)$ with the topologies induced by the L^∞ distance and $\mathcal{Z}(\Omega, U)$ with the corresponding product topology.

Let us also introduce, for every $V \in \mathcal{V}(\Omega)$ and every $W \in \mathcal{W}(\Omega)$, the topological subspaces of $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ defined, with a slight abuse of notation, by

$$\begin{aligned} \mathcal{V}(\Omega, W, U) &= \{\tilde{V} \in \mathcal{V}(\Omega) \mid (\tilde{V}, W) \in \mathcal{Z}(\Omega, U)\}, \\ \mathcal{W}(\Omega, V, U) &= \{\tilde{W} \in \mathcal{W}(\Omega) \mid (V, \tilde{W}) \in \mathcal{Z}(\Omega, U)\}. \end{aligned}$$

Notice that neither $\mathcal{V}(\Omega, W, U)$ nor $\mathcal{W}(\Omega, V, U)$ is empty. Moreover, both $\mathcal{V}(\Omega, W, U)$ and $\mathcal{W}(\Omega, V, U)$ are invariant by the set addition with $L^\infty(\Omega)$. In particular, they are open in $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ respectively and they coincide with $L^\infty(\Omega)$ when $\Omega \in \Xi_d$.

Let us recall that a topological space X is called a *Baire space* if the intersection of countably many open and dense subsets of X is dense in X. Every complete metric space is a Baire space. (In particular, $\mathcal{V}(\Omega), \mathcal{W}(\Omega), \mathcal{Z}(\Omega, U), \mathcal{V}(\Omega, W, U), \mathcal{W}(\Omega, V, U)$ are Baire spaces.) The intersection of countably many open and dense subsets of a Baire space is called a *residual subset* of X. Given a Baire space X, a boolean function $P : X \to \{0, 1\}$ is said to be a *generic property* if there exists a residual subset Y of X such that every x in Y satisfies property P, that is, $P(x) = 1$.

3 Controllability of the discrete-spectrum Schrödinger equation: sufficient conditions and their genericity

The theorem below recalls the controllability result obtained in [16, Theorems 3.4]. Here and in the following a map $h : \mathbf{N} \to \mathbf{N}$ is called a *reordering of* $\mathbf{N}$ if it is a bijection. Recall that the elements of a sequence $(\nu_j)_{j\in\mathbf{N}}$ are said to be $\mathbf{Q}$-*linearly independent* if for every $K \in \mathbf{N}$ and every $(q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$, it holds $\sum_{j=1}^{K} q_j \nu_j \neq 0$. Recall, moreover, that a $n \times n$ matrix $M = (m_{jk})_{j,k=1}^n$ is said to be *connected* if for every pair of indices $j, k \in \{1, \dots, n\}$ there exists a finite sequence $r_1, \dots, r_l \in \{1, \dots, n\}$ such that $m_{jr_1} m_{r_1 r_2} \cdots m_{r_{l-1} r_l} m_{r_l k} \neq 0$.

Theorem 3.1 ([16])**.** *Let $\Omega \in \Xi_d^\infty$ and $(V, W) \in \mathcal{Z}(\Omega, U)$. Assume that the elements of $\big(\lambda_{k+1}(\Omega, V) - \lambda_k(\Omega, V)\big)_{k\in\mathbf{N}}$ are $\mathbf{Q}$-linearly independent and that there exists a reordering $h : \mathbf{N} \to \mathbf{N}$ such that for infinitely many $n \in \mathbf{N}$ the matrix*

$$B_n^h(\Omega, V, W) := \left(\int_\Omega W(x) \phi_{h(j)}(\Omega, V) \phi_{h(k)}(\Omega, V)\, dx \right)_{j,k=1}^n$$

is connected. Then (Ω, V, W, U) is approximately controllable.

Remark 3.2. *Notice that, even when Ω is unbounded, each integral $\int_\Omega W(x) \phi_j(\Omega, V) \phi_k(\Omega, V)\, dx$ is well defined. Indeed, when $\Omega = \mathbf{R}^d$, the growth of $|W|$ is at most exponential and $e^{a|x|}\phi_j(\mathbf{R}^d, V) \in L^2(\mathbf{R}^d, \mathbf{R})$ for every $a > 0$ and $j \in \mathbf{N}$ (see [2]).*

The papers [12] and [13] present relaxed conditions on V and W which are enough to prove approximate controllability. In particular the $\mathbf{Q}$-linearly independence of $\big(\lambda_{k+1}(\Omega, V) - \lambda_k(\Omega, V)\big)_{k\in\mathbf{N}}$ can be replaced by the assumption that the elements of $\big(|\lambda_k(\Omega, V) - \lambda_j(\Omega, V)|\big)_{k,j\in\mathbf{N}}$

are pairwise distinct, or even less under some additional assumptions (see in particular [12, Theorem 2.6]). However, since our goal is to prove the genericity of the sufficient conditions implying approximate controllability, we prefer to focus on the conditions stated in Theorem 3.1, which contain more informations on the potentials V and W. We therefore introduce the following definition.

Definition 3.3. *Let $V \in \mathcal{V}(\Omega)$ and $W \in \mathcal{W}(\Omega)$. We say that (Ω, V, W) is* fit for control *if $(\lambda_{k+1}(\Omega, V) - \lambda_k(\Omega, V))_{k\in\mathbf{N}}$ is $\mathbf{Q}$-linearly independent and there exists a reordering h such that $B^h_n(\Omega, V, W)$ is connected for infinitely many $n \in \mathbf{N}$. Let (V, W) be an element of $\mathcal{Z}(\Omega, U)$. We say that the quadruple (Ω, V, W, U) is effective if $(\Omega, V + uW, W)$ is fit for control for some $u \in U$.*

Notice that if $(\Omega, V + uW, W)$ is fit for control then Theorem 3.1 implies that the control system $(\Omega, V + uW, W, U')$ is approximate controllable, where $U' = [0, \delta - u)$. In particular, also (Ω, V, W, U) is approximate controllable, since all the admissible dynamics for $(\Omega, V + uW, W, U')$ are also admissible for (Ω, V, W, U). Theorem 3.1 can then be rephrased by saying that being effective is a sufficient condition for approximate controllability.

Let us recall the following result, which can be found in [21, Theorem 3.4].

Theorem 3.4 ([21])**.** *Let Ω belong to Ξ^∞_d. Then, generically with respect to $(V, W) \in \mathcal{Z}(\Omega, U)$ the triple (Ω, V, W) is fit for control.*

In the present paper we give new results on the genericity of approximate controllability when one of the two potentials V and W is fixed. We recall that in [21, Corollaries 4.4, 4.5, Proposition 4.6] the following was proved.

Theorem 3.5 ([21])**.** *Let Ω belong to Ξ^∞_d. Fix an absolutely continuous function $V \in \mathcal{V}(\Omega)$. Then, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple (Ω, V, W, U) is effective. Similarly, if $W \in \mathcal{W}(\Omega)$ is a fixed non-constant and absolutely continuous function on Ω, then, generically with respect to $V \in \mathcal{V}(\Omega, W, U)$, the quadruple (Ω, V, W, U) is effective.*

The goal of this paper is to show that the absolute continuity assumption on the potential that is fixed is purely technical and can be removed. We succeed in our goal at least in the case $d = 1$. Our main result is the following.

Theorem 3.6. *Let Ω belong to Ξ^∞_1. Fix $V \in \mathcal{V}(\Omega)$. Then, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple (Ω, V, W, U) is effective. Similarly, if $W \in \mathcal{V}(\Omega)$ non-constant is fixed then, generically with respect to $V \in \mathcal{V}(\Omega, W, U)$, the quadruple (Ω, V, W, U) is effective.*

Notice that the requirement on W to be non-constant in the second part of the statement is necessary. Indeed, if W is constant, then for every $V \in \mathcal{V}(\Omega, W, U)$ every eigenspace of $-\Delta + V$ is invariant by the flow of the control system (Ω, V, W, U).

4 The basic one-dimensional technical result

The main goal of the section is to obtain a generalisation of the following result obtained in [21] in which we drop the assumption of absolute continuity on the function Z.

Lemma 4.1 ([21]). *Let Ω belong to Ξ_d^∞ and Z be a non-constant absolutely continuous function on Ω. Then there exist $\omega \in \Xi_d$ compactly contained in Ω with Lipschitz continuous boundary and a reordering $h : \mathbf{N} \to \mathbf{N}$ such that $\sigma(\omega, 0)$ is simple and*

$$\int_\omega Z\phi_{h(l)}(\omega,0)\phi_{h(l+1)}(\omega,0) \neq 0 \tag{3}$$

for every $l \in \mathbf{N}$.

We are going to obtain such an extension in the case $d = 1$, assuming that Z is just measurable and non-constant. Let us stress that a function $Z \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R})$ is said to be non-constant if no constant function on Ω coincides with Z almost everywhere.

Proposition 4.2. *Let Ω belong to Ξ_1^∞ and Z be a non-constant function in $L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R})$. Then there exists a nonempty interval ω compactly contained in Ω such that*

$$\int_\omega Z\phi_l(\omega,0)\phi_{l+1}(\omega,0) \neq 0 \tag{4}$$

for every $l \in \mathbf{N}$.

Proof. We look for ω in the form $(a, a+r)$, for some $a \in \Omega$ and $r > 0$ such that $(a, a+r)$ is compactly contained in Ω.

In particular, the simplicity of $\sigma(\omega, 0)$ is guaranteed and

$$\phi_l(\omega,0)(x) = \phi_l^{a,r}(x) = \sqrt{\frac{2}{r}}\sin\left(\frac{l\pi(x-a)}{r}\right), \quad l \in \mathbf{N}.$$

We also define

$$\psi_l^{a,r}(x) = \sqrt{\frac{2}{r}}\cos\left(\frac{l\pi(x-a)}{r}\right), \quad l \in \mathbf{N}.$$

Let us first show that it is enough to prove that there exists (a, r) as above such that

$$\int_a^{a+r} Z\psi_1^{a,r} \neq 0. \tag{5}$$

Indeed, assume that (5) is true and consider a neighbourhood $\mathcal{N}$ of (a, r) is $\Omega \times (0, +\infty)$ such that, for every $(\alpha, \rho) \in \mathcal{N}$, $(\alpha, \alpha+\rho)$ is compactly contained in Ω and

$$\int_\alpha^{\alpha+\rho} Z\psi_1^{\alpha,\rho} \neq 0. \tag{6}$$

Such neighbourhood exists since the map $(\alpha, \rho) \mapsto \int_\alpha^{\alpha+\rho} Z\psi_1^{\alpha,\rho}$ is continuous. Reasoning similarly, we have that for every $l \in \mathbf{N}$ the set $\{(\alpha, \rho) \in \mathcal{N} \mid \int_\alpha^{\alpha+\rho} Z\phi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho} \neq 0\}$ is open in $\mathcal{N}$. If we can guarantee that each of such sets is also dense, then necessarily their intersection is not empty, by the Baire property.

Assume by contradiction that there exists $l \in \mathbf{N}$ such that $\int_\alpha^{\alpha+\rho} Z\phi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho} \equiv 0$ on a nonempty open subset $\mathcal{N}'$ of $\mathcal{N}$. Set

$$F_l(\alpha, \rho) = \int_\alpha^{\alpha+\rho} Z\phi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho}.$$

By differentiating F_l with respect to its first variable, we get that

$$0 \equiv \frac{\partial}{\partial \alpha} F_l(\alpha,\rho) = -\frac{\pi}{\rho}\int_{\alpha}^{\alpha+\rho} Z\left(l\psi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho} + (l+1)\phi_l^{\alpha,\rho}\psi_{l+1}^{\alpha,\rho}\right) \tag{7}$$

on $\mathcal{N}'$. We used in this computation the fact that each function $\phi_j^{\alpha,\rho}$ annihilates at α and $\alpha+\rho$.

Differentiating once more with respect to α, we get that, for every $(\alpha,\rho)\in\mathcal{N}'$,

$$\begin{aligned}0 =& \frac{\partial^2}{\partial \alpha^2} F_l(\alpha,\rho) = \frac{\pi^2}{\rho^2}\int_{\alpha}^{\alpha+\rho} Z\left(l(l+1)\psi_l^{\alpha,\rho}\psi_{l+1}^{\alpha,\rho} - (l^2+(l+1)^2)\phi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho}\right)\\ =& \frac{l(l+1)\pi^2}{\rho^2}\int_{\alpha}^{\alpha+\rho} Z\psi_l^{\alpha,\rho}\psi_{l+1}^{\alpha,\rho},\end{aligned}$$

where the last identity follows from the fact that $F_l(\alpha,\rho)\equiv 0$ on $\mathcal{N}'$.

Differentiating once more $\frac{\partial^2}{\partial\alpha^2}F_l(\alpha,\rho)$ with respect to α, we get that at almost every $(\alpha,\rho)\in\mathcal{N}'$ it holds

$$0 = -\frac{2l(l+1)\pi^2}{\rho^3}(Z(\alpha+\rho)+Z(\alpha)) + \frac{l(l+1)\pi^3}{\rho^3}\int_{\alpha}^{\alpha+\rho} Z\left(l\phi_l^{\alpha,\rho}\psi_{l+1}^{\alpha,\rho} + (l+1)\psi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho}\right).$$

Combining with (7), we deduce that

$$Z(\alpha+\rho)+Z(\alpha) = \frac{\pi}{2}\int_{\alpha}^{\alpha+\rho} Z\left(-\phi_l^{\alpha,\rho}\psi_{l+1}^{\alpha,\rho} + \psi_l^{\alpha,\rho}\phi_{l+1}^{\alpha,\rho}\right) = \frac{\pi\sqrt{2}}{2\sqrt{\rho}}\int_{\alpha}^{\alpha+\rho} Z\phi_1^{\alpha,\rho} \tag{8}$$

almost everywhere on $\mathcal{N}'$, where the last equality follows by standard trigonometric identities.

Let us rewrite (8) as

$$Z(\beta)+Z(\alpha) = \frac{\pi\sqrt{2}}{2\sqrt{\beta-\alpha}}\int_{\alpha}^{\beta} Z\phi_1^{\alpha,\beta-\alpha}. \tag{9}$$

Since the right-hand side of (9) is $\mathcal{C}^1$ with respect to (α,β) on $\{(\alpha,\beta)\in\Omega^2 \mid \alpha<\beta\}$, we deduce that Z is $\mathcal{C}^1$ on the open set $\mathcal{N}_1'\cup\mathcal{N}_2'$, where

$$\mathcal{N}_1' = \{\alpha \mid (\alpha,\rho)\in\mathcal{N}' \text{ for some } \rho>0\},\quad \mathcal{N}_2' = \{\alpha+\rho \mid (\alpha,\rho)\in\mathcal{N}'\}.$$

If there exist $x\in\mathcal{N}_1'\cup\mathcal{N}_2'$ such that $\frac{d}{dx}Z(x)\neq 0$, then the conclusion follows from Lemma 4.1. (The proof of Lemma 4.1 given in [21] shows that in the case $d=1$ one can take $h=\mathrm{id}_{\mathbf{N}}$.) Otherwise, $\frac{d}{dx}Z\equiv 0$ on $\mathcal{N}_1'\cup\mathcal{N}_2'$.

Differentiating (8) with respect to α, we get that

$$\int_{\alpha}^{\alpha+\rho} Z\psi_1^{\alpha,\rho}\equiv 0,$$

for every $(\alpha,\rho)\in\mathcal{N}'$, contradicting (6).

We are left to prove that either Z is absolutely continuous on Ω (and hence Lemma 4.1 applies with $h=\mathrm{id}_{\mathbf{N}}$) or there exists $(a,r)\in\Omega\times(0,\infty)$ such that $(a,a+r)$ is compactly contained in Ω and (5) holds true.

By contradiction, assume that for every $(a,r) \in \Omega \times (0,\infty)$ such that $(a, a+r)$ is compactly contained in Ω we have $\int_a^{a+r} Z\psi_1^{a,r} = 0$. By differentiating with respect to a, we get that for almost every a such that $(a, a+r) \subset \Omega$,

$$Z(a+r) + Z(a) = \frac{\pi}{r} \int_a^{a+r} Z\phi_1^{a,r}.$$

Reasoning as above, we deduce that Z is $\mathcal{C}^1$ on Ω. In particular, Z is absolutely continuous and the proof of the proposition in concluded. $\square$

5 The genericity argument by analytic perturbation

We are going to prove Theorem 3.6 by considering separately the cases where V or W is fixed. Before doing so, let us recall some useful result from the literature.

The first is a technical result allowing to obtain the spectral decomposition of a Laplace–Dirichlet operator on a bounded domain ω as the limit for operators defined on larger spacial domains, whose potential converge uniformly to infinity outside ω.

Lemma 5.1 ([21]). *Let Ω belong to Ξ_d^∞ and ω be a nonempty, open subset compactly contained in Ω and whose boundary is Lipschitz continuous. Let $v \in L^\infty(\omega, \mathbf{R})$ and $(V_k)_{k\in\mathbf{N}}$ be a sequence in $\mathcal{V}(\Omega)$ such that $V_k|_\omega \to v$ in $L^\infty(\omega, \mathbf{R})$ as $k \to \infty$ and $\lim_{k\to\infty} \inf_{\Omega\setminus\omega} V_k = +\infty$. Then, for every $j \in \mathbf{N}$, $\lim_{k\to\infty} \lambda_j(\Omega, V_k) = \lambda_j(\omega, v)$. Moreover, if $\lambda_j(\omega, v)$ is simple then (up to the sign) $\phi_j(\Omega, V_k)$ and $\sqrt{|V_k|}\phi_j(\Omega, V_k)$ converge respectively to $\phi_j(\omega, v)$ and $\sqrt{|v|}\phi_j(\omega, v)$ in $L^2(\Omega, \mathbf{C})$ as k goes to infinity, where $\phi_j(\omega, v)$ is identified with its extension by zero outside ω.*

The second result states that the $\mathbf{Q}$-linear independence of the spectrum of $-\Delta + V$ is a generic property with respect to V. It generalises a classical result on the generic simplicity of the spectrum of $-\Delta + V$ obtained by Albert in [4]. It implies in particular that the spectral gaps $\lambda_{j+1}(\Omega, V) - \lambda_j(\Omega, V)$ of $-\Delta + V$ form generically a $\mathbf{Q}$-linear independent family, as required in the hypotheses of Theorem 3.1.

Proposition 5.2 ([4] and [21]). *Let Ω belong to Ξ_d^∞. For every $K \in \mathbf{N}$ and $q = (q_1, \ldots, q_K) \in \mathbf{Q}^K \setminus \{0\}$, the set*

$$\mathcal{O}_q(\Omega) = \left\{ V \in \mathcal{V}(\Omega) \mid \lambda_1(\Omega, V), \ldots, \lambda_K(\Omega, V) \text{ are simple and } \sum_{j=1}^{K} q_j \lambda_j(\Omega, V) \neq 0 \right\} \quad (10)$$

is open and dense in $\mathcal{V}(\Omega)$.

The third result, based on the conclusions obtained in [31], states the existence of analytic paths of potentials such that the spectrum is simple along them.

Proposition 5.3 ([31] and [21]). *Let Ω belong to Ξ_d^∞ and $V, Z \in \mathcal{V}(\Omega)$ be such that $Z - V \in L^\infty(\Omega, \mathbf{R})$. Then there exists an analytic function $\mu \mapsto W_\mu$ from $[0,1]$ into $L^\infty(\Omega, \mathbf{R})$ such that $W_0 = 0$, $W_1 = Z - V$ and the spectrum of $-\Delta + V + W_\mu$ is simple for every $\mu \in (0,1)$.*

5.1 Proof of Theorem 3.6 in the case where W is fixed

Let $\Omega \in \Xi_1^\infty$ and fix $W \in \mathcal{W}(\Omega)$. Let us consider the following subspace of $\mathcal{V}(\Omega)$

$$\hat{\mathcal{V}}(\Omega, W) = \left\{ V \in \mathcal{V}(\Omega) \mid \operatorname*{ess\,sup}_{x\in\Omega} \frac{|W(x)|}{|V(x)|+1} < +\infty \right\}.$$

Notice that $\hat{\mathcal{V}}(\Omega, W)$ is open in $\mathcal{V}(\Omega, W)$.

Proposition 5.4. *Let Ω belong to Ξ_1^∞ and $W \in \mathcal{W}(\Omega)$ be non-constant. Then, generically with respect to V in $\hat{\mathcal{V}}(\Omega, W)$, the triple (Ω, V, W) is fit for control.*

Proof. By applying Proposition 4.2 to $Z = W$, we deduce that there exists a nonempty interval ω compactly contained in Ω such that

$$\int_\omega W\phi_l(\omega, 0)\phi_{l+1}(\omega, 0) \neq 0 \tag{11}$$

for every $l \in \mathbf{N}$.

Denote by $\mathcal{Q}_n(\Omega, W)$ the set of potentials $V \in \hat{\mathcal{V}}(\Omega, W)$ such that for every $j \in \{1, \ldots, n\}$ the eigenvalue $\lambda_j(\Omega, V)$ is simple and

$$\int_\Omega W\phi_j(\Omega, V)\phi_{j+1}(\Omega, V) \neq 0 \quad \text{for } j = 1, \ldots, n-1.$$

In the case where Ω is bounded the openness of $\mathcal{Q}_n(\Omega, W)$ follows from classical results on the continuity of eigenvalues and eigenfunctions (see, e.g., [18]). For the unbounded case, one should use the fact that each eigenfunction $\phi_r(\Omega, V)$ goes to zero at infinity faster than any exponential. Since W has at most exponential growth, one then deduces that $V \mapsto \sqrt{|W|}\phi_r(\Omega, V)$ is continuous, as a function from the open subset of $\mathcal{V}(\Omega)$ of potentials for which the r-th eigenvalue of $-\Delta + V$ is simple into $L^2(\Omega, \mathbf{C})$ (for details, see [21, Proposition 2.9]).

Let us now prove the density of $\mathcal{Q}_n(\Omega, W)$. Fix $\overline{V} \in \hat{\mathcal{V}}(\Omega, W)$. We should prove that $\overline{V}$ is in the closure of $\mathcal{Q}_n(\Omega, W)$.

Let $(V_k)_{k\in\mathbf{N}}$ be the sequence in $\mathcal{V}(\Omega)$ defined by $V_k = 0$ in ω and $V_k = \overline{V} + k$ in $\Omega \setminus \omega$. Then, for every $j \in \mathbf{N}$, the sequence $\|\sqrt{|V_k|}\phi_j(\Omega, V_k)\|_{L^2(\Omega\setminus\omega,\mathbf{C})}$ converges to 0 as k goes to infinity (Lemma 5.1). By definition of $\hat{\mathcal{V}}(\Omega, W)$, $|W| < C(|\overline{V}| + 1)$ on Ω for some $C > 0$. Hence, for every $j \in \mathbf{N}$, also the sequence $\|\sqrt{|W|}\phi_j(\Omega, V_k)\|_{L^2(\Omega\setminus\omega,\mathbf{C})}$ converges to 0 as k goes to infinity.

Moreover, Lemma 5.1 implies that, for every $j \in \mathbf{N}$, the sequences $\lambda_j(\Omega, V_k)$ and $\phi_j(\Omega, V_k)$ converge, respectively, to $\lambda_j(\omega, 0)$ and $\phi_j(\Omega, 0)$ (up to sign) as k goes to infinity. In particular, $\lambda_1(\Omega, V_k), \ldots, \lambda_n(\Omega, V_k)$ are simple for k large enough and condition (11) allows to conclude that $V_k \in \mathcal{Q}_n(\Omega, W)$ for k large enough.

Fix $\bar{k}$ such that $V_{\bar{k}} \in \mathcal{Q}_n(\Omega, W)$. It follows from Proposition 5.3 that there exists an analytic function $\mu \mapsto W_\mu$ from $[0, 1]$ into $L^\infty(\Omega, \mathbf{R})$ such that $W_0 = 0$, $W_1 = V_{\bar{k}} - \overline{V}$ and the spectrum of $-\Delta + \overline{V} + W_\mu$ is simple for every $\mu \in (0, 1)$.

The conclusion of the proof is based on the analytic dependence of the eigenpairs of $-\Delta + \overline{V} + W_\mu$ with respect to μ. The analytic dependence of a finite set of eigenpairs in a neighborhood of a given μ is a consequence of the classical Kato–Rellich theorem (see [18, Chapter VII]). The global analiticity on the interval $(0,1)$ of the entire infinity family of eigenpairs of $-\Delta+\overline{V}+W_\mu$ can be deduced from the analyticity of $(0,1)\ni\mu\mapsto W_\mu$ in $L^\infty(\Omega)$. In particular, the boundedness of each $\frac{d}{d\mu}W_\mu$ as a function on Ω prevents the analytic branches of the spectrum of $-\Delta+\overline{V}+W_\mu$ to go to infinity as μ tends to some $\mu_0\in[0,1]$. Indeed, since each $\lambda_j(\Omega,\overline{V}+W_\mu)$, $j\in\mathbf{N}$, $\mu\in(0,1)$, is an isolated eigenvalue, one can compute by classical formulas the derivative of $\lambda_j(\Omega,\overline{V}+W_\mu)$ with respect to μ and get

$$\left|\frac{d}{d\mu}\lambda_j(\Omega,\overline{V}+W_\mu)\right|=\left|\int_\Omega\left(\phi_j^2(\Omega,\overline{V}+W_\mu)\frac{d}{d\mu}W_\mu\right|\right)\leq\left\|\frac{d}{d\mu}W_\mu\right\|_\infty$$

(see, for instance, [4]).

We are going to use a stronger analytic dependence property, namely, that the eigenfunctions $\phi_j(\Omega,\overline{V}+W_\mu)$ are analytic from $(0,1)$ to the domain $D(-\Delta+\overline{V})$ endowed with the graph norm (see [29, Theorem 5.6] and also [21, Proposition 2.11]). Recalling that, by definition of $\hat{\mathcal{V}}(\Omega,W)$ and by boundedness of W_μ, $|W|<C(|\overline{V}+W_\mu|+1)$ on Ω for some $C>0$, we deduce that

$$\mu\mapsto\int_\Omega W\phi_j(\Omega,\overline{V}+W_\mu)\phi_{j+1}(\Omega,\overline{V}+W_\mu)$$

is analytic from $(0,1)$ to $\mathbf{R}$, for every $j\in\mathbf{N}$.

Since, moreover, $V_{\bar{k}}=\overline{V}+W_1\in\mathcal{Q}_n(\Omega,W)$, we get that $\overline{V}+W_\mu\in\mathcal{Q}_n(\Omega,W)$ for almost every $\mu\in(0,1)$. Hence $\overline{V}=\lim_{\mu\to0}\overline{V}+W_\mu$ is in the closure $\mathcal{Q}_n(\Omega,W)$. We proved that $\mathcal{Q}_n(\Omega,W)$ is dense in $\hat{\mathcal{V}}(\Omega,W)$.

The set $\cap_{n\in\mathbf{N}}\mathcal{Q}_n(\Omega,W)$ is then residual in $\hat{\mathcal{V}}(\Omega,W)$ and, for every $n\in\mathbf{N}$, the matrix $B_n^{\mathrm{id}_\mathbf{N}}(\Omega,V,W)$ is connected.

The triple (Ω,V,W) is then fit for control if V belongs to

$$\left(\cap_{n\in\mathbf{N}}\mathcal{Q}_n(\Omega,W)\right)\cap\left(\cap_{q\in\cup_{K\in\mathbf{N}}\mathbf{Q}^K\setminus\{0\}}\mathcal{O}_q(\Omega)\right),\tag{12}$$

where the sets $\mathcal{O}_q(\Omega)$ are those introduced in Proposition 5.2. We conclude by noticing that the set in (12) is the intersection of countably many open and dense subsets of $\hat{\mathcal{V}}(\Omega,W)$. □

The next corollary follows immediately from Proposition 5.4.

Corollary 5.5. *Let $\Omega\in\Xi_1$ and $W\in L^\infty(\Omega,\mathbf{R})$ be non-constant. Then, generically with respect to V in $L^\infty(\Omega,\mathbf{R})$, the triple (Ω,V,W) is fit for control.*

In the unbounded case we deduce the following.

Corollary 5.6. *Let $\Omega=\mathbf{R}$ and $W\in\mathcal{W}(\mathbf{R})$ be non-constant. Then, generically with respect to V in $\mathcal{V}(\mathbf{R},W,U)$, the quadruple $(\mathbf{R},V,W,U)$ is effective.*

Proof. Fix $u\in(0,\delta)$. Let $\eta>0$ be such that $[u-\eta,u+\eta]\subset U$. If $V\in\mathcal{V}(\mathbf{R},W,U)$ then both $V+(u-\eta)W$ and $V+(u+\eta)W$ are positive outside some bounded subset Ω_0 of $\mathbf{R}$. In particular $|W|\leq\frac{1}{2\eta}|V+uW|$ outside Ω_0. Since, moreover, W is bounded on Ω_0, then $V+uW$ is in $\hat{\mathcal{V}}(\mathbf{R},W)$.

Since $uW+\mathcal{V}(\mathbf{R},W,U)$ is an open subset of $\hat{\mathcal{V}}(\mathbf{R},W)$, we deduce from Proposition 5.4 that there exist a residual subset $\mathcal{R}$ of $uW+\mathcal{V}(\mathbf{R},W,U)$ such that $(\Omega,\widetilde{V},W)$ is fit for control for every $\widetilde{V}\in\mathcal{R}$. This means that the triple $(\mathbf{R},V+uW,W)$ is fit for control, generically with respect to $V\in\mathcal{V}(\mathbf{R},W,U)$. In particular, the quadruple $(\mathbf{R},V,W,U)$ is effective, generically with respect to V in $\mathcal{V}(\mathbf{R},W,U)$. □

5.2 Proof of Theorem 3.6 in the case where V is fixed

We prove in this section that for a fixed potential V, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple (Ω, V, W, U) is effective. Notice that (Ω, V, W) cannot be fit for control if the spectrum of $-\Delta+V$ is resonant, independently of W. In this regard the result is necessarily weaker than Proposition 5.4 and Corollary 5.5, where the genericity of the fitness for control is proved.

Proposition 5.7. *Let Ω belong to Ξ_1^∞ and $V \in \mathcal{V}(\Omega)$. Then, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple (Ω, V, W, U) is effective.*

Proof. Fix $u \in (0, \delta)$. Notice that $V+u\mathcal{W}(\Omega, V, U)$ is an open subset of $\mathcal{V}(\Omega)$ and that the map $W \mapsto V+uW$ is a homeomoprhism between $\mathcal{W}(\Omega, V, U)$ and $V+u\mathcal{W}(\Omega, V, U)$. In particular, due to Proposition 5.2, for every $K \in \mathbf{N}$ and $q \in \mathbf{Q}^K \setminus \{0\}$, the set $\{W \in \mathcal{W}(\Omega, V, U) \mid V+uW \in \mathcal{O}_q(\Omega)\}$ is open and dense in $\mathcal{W}(\Omega, V, U)$.

For every $W \in \mathcal{W}(\Omega, V, U)$ let, as in the previous section, $\mathcal{Q}_n(\Omega, W)$ be the open and dense subset of $\hat{\mathcal{V}}(\Omega, W)$ made of all potentials $\widetilde{V} \in \hat{\mathcal{V}}(\Omega, W)$ such that for every $j \in \{1, \dots, n\}$ the eigenvalue $\lambda_j(\Omega, \widetilde{V})$ is simple and

$$\int_\Omega W\phi_j(\Omega, \widetilde{V})\phi_{j+1}(\Omega, \widetilde{V}) \neq 0 \quad \text{for } j = 1, \dots, n-1.$$

As proved in Corollary 5.6, for every $W \in \mathcal{W}(\Omega, V, U)$ one has $V+uW \in \hat{\mathcal{V}}(\Omega, W)$. We are going to prove the proposition by showing that for every $n \in \mathbf{N}$, for each W in a open and dense subset of $\mathcal{W}(\Omega, V, U)$, $V+uW$ belongs to $\mathcal{Q}_n(\Omega, W)$.

Define

$$\mathcal{P}_n = \{W \in \mathcal{W}(\Omega, V, U) \mid V+uW \in \mathcal{Q}_n(\Omega, W)\}.$$

Since

$$W \mapsto \int_\Omega W\phi_j(\Omega, V+uW)\phi_k(\Omega, V+uW)$$

is continuous on $\{W \in \mathcal{W}(\Omega, V, U) \mid \lambda_j(\Omega, V+uW), \lambda_k(\Omega, V+uW) \text{ are simple}\}$ for every $j, k \in \mathbf{N}$ (see [21, Proposition 2.9] for details), we deduce that $\mathcal{P}_n$ is open.

Fix $\widetilde{W} \in \mathcal{W}(\Omega, V, U)$. We are left to prove that $\widetilde{W}$ belongs to the closure of $\mathcal{P}_n$.

Consider first the case in which V is constant. In particular, Ω is a bounded interval, $\mathcal{W}(\Omega, V, U) = V+u\mathcal{W}(\Omega, V, U) = L^\infty(\Omega, \mathbf{R})$, and

$$\int_\Omega W\phi_j(\Omega, V+uW)\phi_k(\Omega, V+uW) = \int_\Omega W\phi_j(\Omega, uW)\phi_k(\Omega, uW) \tag{13}$$

for every $j, k \in \mathbf{N}$ and $W \in L^\infty(\Omega, \mathbf{R})$. Fix an interval ω compactly contained in Ω. In particular, the spectrum $\sigma(\omega, 0)$ is simple.

Let $z \in L^\infty(\omega, \mathbf{R})$ be such that

$$\int_\omega z\phi_j(\omega, 0)\phi_k(\omega, 0) \neq 0$$

for every $j, k \in \mathbf{N}$. Then, for every $j, k \in \mathbf{N}$, the derivative of

$$\varepsilon \mapsto \int_\omega \varepsilon z\phi_j(\omega, \varepsilon z)\phi_k(\omega, \varepsilon z)$$

at $\varepsilon = 0$ is equal to

$$\int_\omega z\phi_j(\omega, 0)\phi_k(\omega, 0) \neq 0.$$

By analiticity, there exists $\tilde{\varepsilon} \in \mathbf{R}$ such that the spectrum $\sigma(\omega, \tilde{\varepsilon}z)$ is simple and

$$\int_\omega \tilde{\varepsilon}z\phi_j(\omega, \tilde{\varepsilon}z)\phi_k(\omega, \tilde{\varepsilon}z) \neq 0$$

for every $j, k \in \mathbf{N}$. Set $\tilde{z} = \tilde{\varepsilon}z$.

Let $(W_l)_{l\in\mathbf{N}}$ be a sequence in $L^\infty(\Omega, \mathbf{R})$ such that $\lim_{l\to\infty} W_l|_\omega = \tilde{z}/u$ in $L^2(\omega, \mathbf{R})$ and $\lim_{l\to\infty} \inf_{\Omega\setminus\omega} W_l = +\infty$. By Lemma 5.1 we deduce that there exists $\bar{l}$ large enough such that

$$\int_\Omega W_{\bar{l}}\phi_j(\Omega, uW_{\bar{l}})\phi_k(\Omega, uW_{\bar{l}}) \neq 0 \quad \text{for } j, k = 1, \ldots, n.$$

By Proposition 5.3 we can consider an analytic curve $[0, 1] \ni \mu \mapsto \hat{W}_\mu$ in $L^\infty(\Omega, \mathbf{R})$ such that $\hat{W}_0 = \widetilde{W}$, $\hat{W}_1 = W_{\bar{l}}$ and the spectrum of $-\Delta + u\hat{W}_\mu$ is simple for every $\mu \in (0, 1)$. Since V is constant and by analytic dependence with respect to μ, we have

$$\int_\Omega \hat{W}_\mu\phi_j(\Omega, V + u\hat{W}_\mu)\phi_k(\Omega, V + u\hat{W}_\mu) = \int_\Omega \hat{W}_\mu\phi_j(\Omega, u\hat{W}_\mu)\phi_k(\Omega, u\hat{W}_\mu) \neq 0$$

for $j, k = 1, \ldots, n$ and for almost every $\mu \in (0, 1)$. In particular, taking μ arbitrarily small, we have that $\widetilde{W}$ belongs to the closure of $\mathcal{P}_n$.

Let now V be non-constant. Let $\omega \subset \Omega$ be as in the statement of Proposition 4.2, with V playing the role of Z.

Take a sequence $(W_k)_{k\in\mathbf{N}}$ in $\mathcal{W}(\Omega, V, U)$ such that $W_k - \widetilde{W}$ belongs to $L^\infty(\Omega, \mathbf{R})$ for every k and

$$\lim_{k\to+\infty} \|V + uW_k\|_{L^\infty(\omega,\mathbf{R})} = 0, \qquad \lim_{k\to+\infty} \inf_{\Omega\setminus\omega}(uW_k) = +\infty.$$

According to Lemma 5.1,

$$\lim_{k\to+\infty} \phi_m(\Omega, V + uW_k) = \phi_m(\omega, 0) \quad \text{and} \quad \lim_{k\to+\infty} \sqrt{|V + uW_k|}\phi_m(\Omega, V + uW_k) = 0$$

in $L^2(\Omega, \mathbf{C})$ for every $m \in \mathbf{N}$, where $\phi_m(\omega, 0)$ is identified with its extension by zero on $\Omega \setminus \omega$. In particular, we have that $\sqrt{|V|}\phi_m(\Omega, V + uW_k)$ converges in $L^2(\Omega, \mathbf{C})$ as k tends to infinity to the extension by zero of $\sqrt{|V|}\phi_m(\omega, 0)$ on $\Omega \setminus \omega$. Hence,

$$\lim_{k\to+\infty} \int_\Omega W_k\phi_l(\Omega, V + uW_k)\phi_{l+1}(\Omega, V + uW_k) = -\frac{1}{u}\int_\omega V\phi_l(\omega, 0)\phi_{l+1}(\omega, 0) \neq 0,$$

for every $l \in \mathbf{N}$. For a fixed $n \in \mathbf{N}$, we can choose $\bar{k}$ large enough so that

$$\int_\Omega W_{\bar{k}}\phi_l(\Omega, V + uW_{\bar{k}})\phi_{l+1}(\Omega, V + uW_{\bar{k}}) \neq 0, \quad \text{for } l = 1, \ldots, n-1,$$

in order to guarantee that $W_{\bar{k}} \in \mathcal{P}_n$.

Using again Proposition 5.3, we deduce that there exists an analytic path $\mu \mapsto \hat{W}_\mu$ from $[0,1]$ into $L^\infty(\Omega, \mathbf{R})$ such that $\hat{W}_0 = 0$, $\hat{W}_1 = W_{\bar{k}} - \widetilde{W}$ and the spectrum of $-\Delta + V + u\widetilde{W} + u\hat{W}_\mu$ is simple for every $\mu \in (0,1)$. Therefore, by analyticity, we get that

$$\int_\Omega (\widetilde{W} + \hat{W}_\mu)\phi_l(\Omega, V + u\widetilde{W} + u\hat{W}_\mu)\phi_{l+1}(\Omega, V + u\widetilde{W} + u\hat{W}_\mu) \neq 0$$

for $l = 1, \dots, n-1$ and for almost every $\mu \in (0,1)$. Hence, $\widetilde{W}$ belongs to the closure of $\mathcal{P}_n$. $\square$

6 Conclusion

In this paper we proved that once (Ω, V) or (Ω, W) is fixed (with Ω a one-dimensional domain and W non-constant), the bilinear Schrödinger equation on Ω having V as uncontrolled and W as controlled potential is generically approximately controllable with respect to the other element of the triple (Ω, V, W). This improves the results in [21], where a technical regularity assumption was imposed on the potentials. It remains to prove that the regularity assumption can be dropped also in the case of domain of dimension larger than one.

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