Rolling of Manifolds and Controllability in dimension three*

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Abstract

In this paper, we present the rolling (or development) of one smooth connected complete Riemannian manifold (M,g) onto another one $(\hat{M},\hat{g})$ of equal dimension $n \geq 2$ where there is no relative spin or slip of one manifold with respect to the other one. Relying on geometric control theory, we provide an intrinsic description of the two constraints "without spinning" and "without slipping" in terms of the Levi-Civita connections ∇^g and $\nabla^{\hat{g}}$ by defining corresponding vector fields distributions in the appropriate state space. We then address the issue of complete controllability for that rolling problem. We first establish basic global properties for the reachable set and investigate the associated Lie bracket structure. In particular, we point out the role played by a curvature tensor defined on the state space, that we call the *rolling curvature*. When the two manifolds are three-dimensional, we give a complete local characterization of the reachable sets and, in particular, we identify necessary and sufficient conditions for the existence of a non open orbit. In adition to the trivial case where the manifolds (M,g) and $(\hat{M},\hat{g})$ are (locally) isometric, we show that (local) non controllability occurs if and only if (M,g) and $(\hat{M},\hat{g})$ are either warped products or contact manifolds with additional restrictions that we precisely describe.

Contents

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1 Introduction

In this paper, we study the rolling of a manifold over another one. Unless otherwise precised, manifolds are smooth, connected, oriented, of finite dimension $n \geq 2$, endowed with a complete Riemannian metric. The rolling is assumed to be without spinning nor slipping and we refer to it as the rolling (R) since it is possible to have another rolling problem just assuming a no-slipping condition (cf [24]). When both manifolds are isometrically embedded into an Euclidean space, the rolling problem (R) is classical in differential geometry through the notions of "development of a manifold" and "rolling maps", see [41] and references therein. To get an intuitive grasp of the problem, consider the rolling problem (R) of a 2D convex surface S_1 onto another one S_2 in the euclidean space $\mathbb{R}^3$. The most classical such example is the so-called plate-ball problem, i.e., a sphere rolling onto a plane in $\mathbb{R}^3$, (cf. [22] and [33]). The two surfaces are in contact i.e., they have a common tangent plane at the contact point and, equivalently, their exterior normal vectors are opposite at the contact point. If $\gamma : [0, T] \to S_1$ is a C^1 regular curve on S_1, one says that S_1 rolls onto S_2 along γ without spinning nor slipping if the following holds. The curve traced on S_1 by the contact point is equal to γ and let $\hat{\gamma} : [0, T] \to S_2$ be the curve traced on S_2 by the contact point. At every time $t \in [0, T]$ the relative orientation of S_2 with respect to S_1 is measured by the angle $\theta(t)$ between $\dot{\gamma}(t)$ and $\dot{\hat{\gamma}}(t)$ in the common tangent plane at the contact point and let Q be the state space of the rolling problem (which is therefore five dimensional since a point in Q is defined by fixing a point on S_1, a point on S_2 and an angle in S^1, the unit circle). The no-slipping condition says that $\dot{\hat{\gamma}}(t)$ is equal to $\dot{\gamma}(t)$ rotated by the angle $\theta(t)$ and the no-spinning condition characterizes $\dot{\theta}(t)$ in term of the surface elements at $\gamma(t)$ and $\hat{\gamma}(t)$ respectively. Then, once a point on S_2 and an angle are chosen at time $t = 0$, the curves $\hat{\gamma}$ and θ are uniquely determined. The most basic issue in geometric control theory linked to the rolling problem (R) is that of *controllability* i.e., to determine, for two given points q_{init} and q_{final} in the state space Q, if there exists a curve γ so that the rolling of S_1 onto S_2 along γ steers the system from q_{init} to q_{final}. If this is the case for every points q_{init} and q_{final} in Q, then the rolling of S_1 onto S_2 is said to be *completely controllable.*

If the manifolds rolling on each other are two-dimensional, the controllability issue is well-understood thanks to the work of [3], [6] and [27] especially. For instance, in the simply connected case, the rolling (R) is completely controllable if and only if the manifolds are not isometric. In the case where the manifolds are isometric, [3] also provides a description of the reachable sets in terms of isometries between the manifolds. In particular, these reachable sets are immersed submanifolds of Q of dimension either 2 or 5. In case the manifolds rolling on each other are isometric convex surfaces, [27] provides a beautiful description of a two dimensional reachable set: consider the initial configuration given by two (isometric) surfaces in contact so that one is the image of the other one by the symmetry with respect to the (common) tangent plane at the contact point. Then, this symmetry property (chirality) is preserved along the rolling (R). Note that if the (isometric) convex surfaces are not spheres nor planes, the reachable set starting at a contact point where the Gaussian curvatures are distinct, is open (and thus of dimension 5).

From a robotics point of view, once the controllability is well-understood, the next issue to address is that of *motion planning*, i.e., defining an effective procedure that produces, for every pair of points (q_{init}, q_{final}) in the state space Q, a curve $\gamma_{q_{\text{init}},q_{\text{final}}}$ so that the rolling of S_1 onto S_2 along $\gamma_{q_{\text{init}},q_{\text{final}}}$ steers the system from q_{init} to q_{final}. In [9], an algorithm based on the continuation method was proposed to tackle the rolling problem (R) of a strictly convex compact surface onto an Euclidean plane. That algorithm was also proved in [9] to be convergent and it was numerically implemented in [1] (see also [28] for another algorithm).

The rolling problem (R) is traditionally presented by isometrically embedding the rolling manifolds M and $\hat{M}$ in an Euclidean space (cf. [29], [41], [20]) since it is the most intuitive way to provide a rigorous meaning to the notions of relative spin (or twist) and relative slip of one manifold with respect to the other one. However, the rolling model will depend in general on the embedding. For instance, rolling two 2D spheres of different radii on each other can be isometrically embedded in (at least) two ways in $\mathbb{R}^3$: the smaller sphere can roll onto the bigger one either inside of it or outside. Then one should be able to define rolling without having to resort to any isometric embedding into an Euclidean space. To be satisfactory, that *intrinsic* formulation of the rolling should also allow one to address at least the controllability issue.

Let us first provide an intrinsic definition of the state space Q. For $n \geq 3$, the relative orientation betwen two manifolds is defined (in coordinates) by an element of $SO(n)$. Therefore the state space Q is locally diffeomorphic to neighborhoods of $M \times \hat{M} \times \mathrm{SO}(n)$ and thus of dimension $2n + n(n-1)/2$. There are two main approaches for an intrinsic formulation of the rolling problem (R), first considered by [3] and [6] respectively. Note that the two references only deal with the two dimensional case but it is immediate to generalize them to higher dimensions. In [3], the state space Q is given by

$$Q = \{A : T|_x M \to T|_{\hat{x}} \hat{M} \mid A \text{ o-isometry},\ x \in M,\ \hat{x} \in \hat{M}\},$$

where "o-isometry" means positively oriented isometry, (see Definition 3.1 below) while in [6], one has equivalently

$$Q = (F_{\mathrm{OON}}(M) \times F_{\mathrm{OON}}(\hat{M}))/\Delta,$$

where $F_{\mathrm{OON}}(M)$, $F_{\mathrm{OON}}(\hat{M})$ be the oriented orthonormal frame bundles of (M, g), $(\hat{M}, \hat{g})$ respectively, and Δ is the diagonal right $\mathrm{SO}(n)$-action.

The next step consists of using either the parallel transports with respect to ∇^g and $\hat{\nabla}^{\hat{g}}$ (Agrachev-Sachkov's approach) or alternatively, orthonormal moving frames and the structure equations (Bryant-Hsu's approach) to translate the constraints of no-spinning and no-slipping and derive the admissible curves, i.e., the curves of Q describing the rolling (R), (cf. Eq.(12) below). Finally, one defines either a distribution or a codistribution depending which approach is chosen. In the present paper, we adopt the Agrachev-Sachkov's approach and we construct an n-dimensional distribution $\mathcal{D}_{\mathrm{R}}$ on Q so that the locally absolutely continuous curves tangent to $\mathcal{D}_{\mathrm{R}}$ are exactly the admissible curves for the rolling problem, (cf. Definition 3.17 below). The construction of $\mathcal{D}_{\mathrm{R}}$ comes along with the construction of (local) basis of vector fields, which allow one to compute the Lie algebraic structure associated to $\mathcal{D}_{\mathrm{R}}$. (See also [30, 16] for alternative constructions of of the rolling problem (R).) Note also that the precise definition of the rolling (R) has been extended to the case of manifolds with different dimensions in [17].

We now describe precisely the results of the present paper. In Section 2, are gathered the notations used throughout the paper. The control system associated to the rolling problem (R) is presented in Section 3 by giving a precise definition of the state space Q and of the set of admissible controls, which is equal the set of locally absolutely continuous (l.a.c.) curves on M only. We thus obtain a driftless control systems affine in the control $(\Sigma)_R$ and also provide, in Appendix A, expressions in local coordinates for these control systems.

In Section 3, we construct the rolling distribution $\mathcal{D}_{\mathrm{R}}$ and we provide (local) basis of vector fields for $\mathcal{D}_{\mathrm{R}}$. We show that the rolling (R) of M over $\hat{M}$ is symmetric to that of $\hat{M}$ over M i.e., the reachable sets are diffeomorphic. The controllability issue turns out to be a delicate one since, in general, there is no "natural" principal bundle structure on $\pi_{Q,M} : Q \to M$ which leaves invariant the rolling distribution $\mathcal{D}_{\mathrm{R}}$. Despite this fact, we prove that reachable sets are smooth bundle over M (cf. Proposition 4.2) and have an equivariance property of the reachable sets of $\mathcal{D}_{\mathrm{R}}$ with respect to isometries from M and $\hat{M}$. We deduce from that complete controllability for the rolling problem (R) associated to a pair of manifolds M and $\hat{M}$ is equivalent to that of the rolling problem (R) associated to their universal Riemannian coverings. Therefore, as far as complete controllability is concerned, one can assume without loss of generality that M and $\hat{M}$ are simply connected.

We then compute the first order Lie brackets of the vector fields generating $\mathcal{D}_{\mathrm{R}}$ and find that they are (essentially) equal to vector fields given by the vertical lifts of

$$\mathsf{Rol}(X,Y)(A) := AR(X,Y) - \hat{R}(AX,AY)A, \tag{1}$$

where X,Y are smooth vector fields of M, $q = (x,\hat{x};A) \in Q$ and $R(\cdot,\cdot)$, and $\hat{R}(\cdot,\cdot)$ are the curvature tensors of g and $\hat{g}$ respectively. We call the vertical vector field defined in Eq. (1) the *Rolling Curvature*, (cf Definition 4.9 below). Higher order Lie brackets can now be expressed as linear combinations of covariant derivatives of the Rolling Curvature for the vertical part and evaluations on $\hat{M}$ of the images of the Rolling Curvature and its covariant derivatives.

In dimension two, the Rolling Curvature is (essentially) equal to $K^M(x) - K^{\hat{M}}(\hat{x})$, where $K^M(\cdot)$, $K^{\hat{M}}(\cdot)$ are the Gaussian curvatures of M and $\hat{M}$ respectively. At some point $q \in Q$ where $K^M(x) - K^{\hat{M}}(\hat{x}) \neq 0$, one immediately deduces that the dimension of the evaluation at q of the Lie algebra of the vector fields spanning $\mathcal{D}_{\mathrm{R}}$ is equal to five, (the dimension of Q) and thus the reachable set from q is open in Q. From that fact, one has the following alternative: (a) there exists $q_0 \in Q$ so that $K^M - K^{\hat{M}} \equiv 0$ over the reachable set from q_0, yielding easily that M and $\hat{M}$ have the same Riemannian covering space (cf. [3] and [6]); (b) all the reachable sets are open and then the rolling problem (R) is completely controllable. In dimension $n \geq 3$, the rolling curvature cannot be reduced to a scalar and it seems difficult compute in general the rank of the evaluations of the Lie algebra of the vector fields spanning $\mathcal{D}_{\mathrm{R}}$. We however can derive an easy sufficent condition for complete controllability, reminiscent of the 2D case: if, for every point $q \in Q$, the vertical part of T_qQ belongs to the tangent space at q of the reachable set from q, then $(\Sigma)_R$ is completely controllable, cf. Proposition 4.18 (see also [16] for a similar result). Moreover, in the case where one of the manifolds (let say $(\hat{M},\hat{g})$) is of constant (Gaussian) curvature, a key simplification occurs namely the state space Q carries the structure of a principal bundle compatible with $\mathcal{D}_{\mathrm{R}}$ (cf. [10]). One can further show that the constant curvature assumption to get a principal bundle structure for rolling is (essentially) a necessary condition (cf. [12]). In that situation, the orbits obtained by rolling along loops of (M, g) become Lie subgroups of the structure group of $\pi_{Q,M} : Q \to M$ which can be realized as holonomy groups of either certain vector bundle connections (that we call rolling connections) when the curvature of the space form is non-zero, or of an affine connection (in the sense of [23]) in the zero curvature case. Note that in the latter case, the rolling (R) is nothing else but the development of a manifold onto its tangent space which was used first by Cartan in [8] in order to define the (affine) holonomy group of the affine connection. By studying the rolling connections, one is able to prove precise controllability results (cf. [10, 11, 13, 14]).

Section 5 collects our results for the rolling (R) of three-dimensional Riemannian manifolds. We are able to provide a complete classification of the possible local structures of a non open orbit, and to each of them, to characterize precisely the manifolds (M, g) and $(\hat{M}, \hat{g})$ giving rise to such orbits.

Roughly speaking, we show that the rolling problem (R) *is not* completely controllable i.e. $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ if and only if the Riemannian manifolds (M, g) and $(\hat{M}, \hat{g})$ are *locally* of the following types (i.e., in open dense sets):

(i) isometric,

(ii) both are warped products with similar warping functions or

(iii) both are of class $\mathcal{M}_\beta$ with the same $\beta > 0$.

Here, the manifolds of class $\mathcal{M}_\beta$ are defined as three-dimensional Riemannian manifolds carrying a contact structure of particular type, as described in [2] and that we recall in Appendix C.1. The possible values of the orbit dimension d of a non open orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (i.e. $d = \dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$) are correspondingly in (i) $d = 3$, (ii) $d = 6$ or $d = 8$ and finally (iii) $d = 7$ or $d = 8$, where the alternatives in (ii) and (iii) depend on the initial orientation A_0. Consequently, it follows that the possible orbit dimensions for the rolling of 3D manifolds are

$$\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \in \{3, 6, 7, 8, 9\},$$

where dimension $d = 9$ corresponds to an open orbit (in Q). Note that we do not answer here to the question of global structure of (M, g), $(\hat{M}, \hat{g})$ when the rolling problem (R) is not completely controllable. We finally gather in a series of appendices several results either used in the text or directly related to it.

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2 Notations

For any sets A, B, C and $U \subset A \times B$ and any map $F : U \to C$, we write U_a and U^b for the sets defined by $\{b \in B \mid (a, b) \in U\}$ and $\{a \in A \mid (a, b) \in U\}$ respectively. Similarly, let $F_a : U_a \to C$ and $F^b : U^b \to C$ be defined by $F_a(b) := F(a, b)$ and

$F^b(a) := F(a,b)$ respectively. For any sets $V_1, \dots, V_n$ the map $\mathrm{pr}_i : V_1 \times \cdots \times V_n \to V_i$ denotes the projection onto the i-th factor. For a real matrix A, we use A^i_j to denote the real number on the i-th row and j-th column and the matrix A can then be denoted by $[A^i_j]$. If, for example, one has $A^i_j = a_{ij}$ for all i,j, then one uses the notation $A^i_j = (a_{ij})^i_j$ and thus $A = [(a_{ij})^i_j]$. The matrix multiplication of $A = [A^i_j]$ and $B = [B^i_j]$ is therefore given by $AB = \big[\big(\sum_k A^i_k B^k_j\big)^i_j\big]$. Suppose V, W are finite dimensional $\mathbb{R}$-linear spaces, $L : V \to W$ is an $\mathbb{R}$-linear map and $F = (v_i)_{i=1}^{\dim V}$, $G = (w_i)_{i=1}^{\dim W}$ are bases of V, W respectively. The $\dim W \times \dim V$-real matrix corresponding to L w.r.t. the bases F and G is denoted by $\mathcal{M}_{F,G}(L)$. In other words, $L(v_i) = \sum_j \mathcal{M}_{F,G}(L)^j_i w_j$ (corresponding to the right multiplication by a matrix of a row vector). Notice that, if $K : W \to U$ is yet another $\mathbb{R}$-linear map to a finite dimensional linear space U with basis $H = (u_i)_{i=1}^{\dim U}$, then

$$\mathcal{M}_{F,H}(K \circ L) = \mathcal{M}_{G,H}(K)\mathcal{M}_{F,G}(L).$$

If (V,g), (W,h) are inner product spaces with inner products g and h, one defines $L^{T_{g,h}} : W \to V$ as the transpose (adjoint) of A w.r.t g and h i.e., $g(L^{T_{g,h}}w, v) = h(w, Lv)$. With bases F and G as above, one has $\mathcal{M}_{F,G}(L)^T = \mathcal{M}_{G,F}(L^{T_{g,h}})$, where T on the left is the usual transpose of a real matrix i.e., the transpose w.r.t standard Euclidean inner products in $\mathbb{R}^N$, $N \in \mathbb{N}$.

In this paper, by a smooth manifold, one means a smooth finite-dimensional, second countable, Hausdorff manifold (see e.g. [26]). By a smooth submanifold of M, we always mean a smooth embedded submanifold. For any smooth map $\pi : E \to M$ between smooth manifolds E and M, the set $\pi^{-1}(\{x\}) =: \pi^{-1}(x)$ is called the π-fiber over x and it is sometimes denoted by $E|_x$, when π is clear from the context. The set of smooth sections of π is denoted by $\Gamma(\pi)$. The value $s(x)$ of a section s at x is usually denoted by $s|_x$. A smooth manifold M is oriented if there exists a smooth (or continuous) section, defined on all of M, of the bundle of n-forms $\pi_{\bigwedge^n(M)} : \bigwedge^n(M) \to M$ where $n = \dim M$. If not otherwise mentioned, the smooth manifolds considered in this paper are connected and oriented. For a smooth map $\pi : E \to M$ and $y \in E$, let $V|_y(\pi)$ be the set of all $Y \in T|_yE$ such that $\pi_*(Y) = 0$. If π is a smooth bundle, the collection of spaces $V|_y(\pi)$, $y \in E$, defines a smooth submanifold $V(\pi)$ of $T(E)$ and the restriction $\pi_{T(E)} : T(E) \to E$ to $V(\pi)$ is denoted by $\pi_{V(\pi)}$. In this case $\pi_{V(\pi)}$ is a vector subbundle of $\pi_{T(E)}$ over E. For a smooth manifold M, one uses $\mathrm{VF}(M)$ to denote the set of smooth vector fields on M i.e., the set of smooth sections of the tangent bundle $\pi_{T(M)} : T(M) \to M$. The flow of a vector field $Y \in \mathrm{VF}(M)$ is a smooth onto map $\Phi_Y : D \to M$ defined on an open subset D of $\mathbb{R} \times M$ containing $\{0\} \times M$ such that $\frac{\partial}{\partial t}\Phi_Y(t,y) = Y|_{\Phi_Y(t,y)}$ for $(t,y) \in D$ and $\Phi_Y(0,y) = y$ for all $y \in M$. As a default, we will take D to be the maximal flow domain of Y.

For any distribution $\mathcal{D}$ on a manifold M, we use $\mathrm{VF}_{\mathcal{D}}$ to denote the set of vector fields $X \in \mathrm{VF}(M)$ tangent to $\mathcal{D}$ (i.e., $X|_x \in \mathcal{D}|_x$ for all $x \in M$) and we define inductively for $k \geq 2$, $\mathrm{VF}^k_{\mathcal{D}} = \mathrm{VF}^{k-1}_{\mathcal{D}} + [\mathrm{VF}_{\mathcal{D}}, \mathrm{VF}^{k-1}_{\mathcal{D}}]$, where $\mathrm{VF}^1_{\mathcal{D}} := \mathrm{VF}_{\mathcal{D}}$. The Lie algebra generated by $\mathrm{VF}_{\mathcal{D}}$ is denoted by $\mathrm{Lie}(\mathcal{D})$ and it equals $\bigcup_k \mathrm{VF}^k_{\mathcal{D}}$. For any maps $\gamma : [a,b] \to X$, $\omega : [c,d] \to X$ into a set X such that $\gamma(b) = \omega(c)$ we define

$$\omega \sqcup \gamma : [a, b+d-c] \to X; \quad (\omega \sqcup \gamma)(t) = \begin{cases} \gamma(t), & t \in [a,b], \\ \omega(t-b+c), & t \in [b, b+d-c]. \end{cases}$$

A map $\gamma : [a,b] \to X$ is a loop in X based at $x_0 \in X$ if $\gamma(a) = \gamma(b) = x_0$. In the space of loops $[0,1] \to X$ based at some given point x_0, one defines an operation "." of concatenation, by

$$\omega.\gamma := (t \mapsto \omega(\tfrac{t}{2})) \sqcup (t \mapsto \gamma(\tfrac{t}{2})).$$

If N is a smooth manifold and $y \in N$, we use $\Omega_y(N)$ to denote the set of all piecewise C^1-loops $[0,1] \to N$ of N based at y.

Given a smooth distribution $\mathcal{D}$ on a smooth manifold M, we call an absolutely continuous curve $c : I \to M$, $I \subset \mathbb{R}$, *$\mathcal{D}$-admissible* if c it is tangent to $\mathcal{D}$ almost everywhere (a.e.) i.e., if for almost all $t \in I$ it holds that $\dot{c}(t) \in \mathcal{D}|_{c(t)}$. For $x_0 \in M$, the endpoints of all the $\mathcal{D}$-admissible curves of M starting at x_0 form the set called *$\mathcal{D}$-orbit through* x_0 and denoted $\mathcal{O}_{\mathcal{D}}(x_0)$. More precisely,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{c(1) \mid c : [0,1] \to M,\ \mathcal{D}-\text{admissible},\ c(0) = x_0\}. \tag{2}$$

By the Orbit Theorem (see [4]), it follows that $\mathcal{O}_{\mathcal{D}}(x_0)$ is an immersed smooth submanifold of M containing x_0. It is also known that one may restrict to piecewise smooth curves in the description of the orbit i.e.,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{c(1) \mid c : [0,1] \to M \text{ piecewise smooth and } \mathcal{D}-\text{admissible},\ c(0) = x_0\}.$$

We call a smooth distribution $\mathcal{D}'$ on M a subdistribution of $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$. An immediate consequence of the definition of the orbit shows that in this case, for all $x_0 \in M$, $\mathcal{O}_{\mathcal{D}'}(x_0) \subset \mathcal{O}_{\mathcal{D}}(x_0)$.

If $\pi : E \to M$, $\eta : F \to M$ are two smooth maps (e.g. bundles), let $C^\infty(\pi, \eta)$ be the set of all bundle maps $\pi \to \eta$ i.e., smooth maps $g : E \to F$ such that $\eta \circ g = \pi$. For a manifold M, let $\pi_{M_{\mathbb{R}}} : M \times \mathbb{R} \to M$ be the projection onto the first factor i.e., $(x,t) \mapsto x$ (i.e., $\pi_{M_{\mathbb{R}}} = \mathrm{pr}_1$). If $\pi : E \to M$, $\eta : F \to M$ are any smooth vector bundles over a smooth manifold M, $f \in C^\infty(\pi, \eta)$ and $u, w \in \pi^{-1}(x)$, one defines the vertical derivative f at u in the direction w by

$$\nu(w)|_u(f) := (D_\nu f)(u)(w) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(u + tw). \tag{3}$$

Here $w \mapsto (D_\nu f)(u)(w) = \nu(w)|_u(f)$ is an $\mathbb{R}$-linear map between fibers $\pi^{-1}(x) \to \eta^{-1}(x)$. In a similar way, in the case of $f \in C^\infty(E)$ and $u, w \in \pi^{-1}(x)$, one defines the π-vertical derivative $\nu(w)|_u(f) := D_\nu f(u)(w) := \frac{\mathrm{d}}{\mathrm{d}t}|_0 f(u+tw)$ at u in the direction w. This definition agrees with the above one modulo the canonical bijection $C^\infty(E) \cong C^\infty(\mathrm{id}_E, \pi_{E_{\mathbb{R}}})$. This latter definition means that $\nu(w)|_u$ can be viewed as an element of $V|_u(\pi)$ and the mapping $w \mapsto \nu(w)|_u$ gives a (natural) $\mathbb{R}$-linear isomorphism between $\pi^{-1}(x)$ and $V|_u(\pi)$ where $\pi(u) = x$. If $\tilde{w} \in \Gamma(\pi)$ is a smooth π-section, let $\nu(\tilde{w})$ be the π-vertical vector field on E defined by $\nu(\tilde{w})|_u(f) = \nu(\tilde{w}|_x)|_u(f)$, where $\pi(u) = x$ and $f \in C^\infty(E)$. The same remark holds also locally.

In the case of smooth manifolds M and $\hat{M}$, $x \in M$, $\hat{x} \in \hat{M}$, we will use freely and without mention the natural inclusions ($\subset$) and isomorphisms ($\cong$): $T|_x M, T|_{\hat{x}} \hat{M} \subset T|_{(x,\hat{x})}(M \times \hat{M}) \cong T|_x M \oplus T|_{\hat{x}} \hat{M}$, $T^*|_x M, T^*|_{\hat{x}} \hat{M} \subset T^*|_{(x,\hat{x})}(M \times \hat{M}) \cong T^*|_x M \oplus T^*|_{\hat{x}} \hat{M}$. An element of $T|_{(x,\hat{x})}(M \times \hat{M}) \cong T|_x(M) \oplus T|_{\hat{x}}(\hat{M})$ with respect to the direct sum splitting is denoted usually by $(X, \hat{X})$, where $X \in T|_x M$, $\hat{X} \in T|_{\hat{x}} \hat{M}$. Sometimes it is even more convenient to write $X + \hat{X} := (X, \hat{X})$ when we make the identifications $(X,0)=X$, $(0,\hat{X})=\hat{X}$. Let (M,g), $(\hat{M},\hat{g})$ be smooth Riemannian manifolds. A map $f:M\to\hat{M}$ is a *local isometry* if it is smooth, surjective and for all $x\in M$, $f_*|_x : T|_xM\to T|_{f(x)}\hat{M}$ is an isometric linear map. A bijective local isometry $f:M\to\hat{M}$ is called an *isometry* and then (M,g), $(\hat{M},\hat{g})$ are said to be *isometric*. In this text we say that two Riemannian manifolds (M,g), $(\hat{M},\hat{g})$ are *locally isometric*, if there is a Riemannian manifold (N,h) and local isometries $F:N\to M$ and $G:N\to\hat{M}$ which are also covering maps i.e. if they are *Riemannian covering maps*. One calls (N,h) a common Riemannian covering space of (M,g) and $(\hat{M},\hat{g})$. Notice that being locally isometric is an equivalence relation in the class of smooth Riemannian manifolds (the fact that we assume F,G to be Riemannian covering maps, and not only local isometries, implies the transitivity of this relation).

The space $\overline{M}=M\times\hat{M}$ is a Riemannian manifold, called the Riemannian product manifold of (M,g), $(\hat{M},\hat{g})$, when endowed with the product metric $\overline{g}:=g\oplus\hat{g}$. One often writes this as $(M,g)\times(\hat{M},\hat{g})$. Let ∇, $\hat{\nabla},\overline{\nabla}$ (resp. $R,\hat{R},\overline{R}$) denote the Levi-Civita connections (resp. the Riemannian curvature tensors) of (M,g), $(\hat{M},\hat{g}),(\overline{M}=M\times\hat{M},\overline{g}=g\oplus\hat{g})$ respectively. From Koszul's formula (cf. [26]), one has

$$\overline{\nabla}_{(X,\hat{X})}(Y,\hat{Y})=(\nabla_XY,\hat{\nabla}_{\hat{X}}\hat{Y}), \tag{4}$$

when $X,Y\in\mathrm{VF}(M)$, $\hat{X},\hat{Y}\in\mathrm{VF}(\hat{M})$ and hence from the definition of the Riemannian curvature tensor

$$\overline{R}((X,\hat{X}),(Y,\hat{Y}))(Z,\hat{Z})=(R(X,Y)Z,\hat{R}(\hat{X},\hat{Y})\hat{Z}), \tag{5}$$

where $X,Y,Z\in T|_xM$, $\hat{X},\hat{Y},\hat{Z}\in T|_{\hat{x}}\hat{M}$. For any (k,m)-tensor field T on M we define ∇T to be the $(k,m+1)$-tensor field such that (see [39], p. 30)

$$(\nabla T)(X_1,\dots,X_m,X)=(\nabla_XT)(X_1,\dots,X_m), \tag{6}$$

$X_1,\dots,X_m,X\in T|_xM$.

The parallel transport of a tensor $T_0\in T^k_m|_{\gamma(0)}(M)$ from $\gamma(0)$ to $\gamma(t)$ along an absolutely continuous curve $\gamma:I\to M$ (with $0\in I$) and with respect to the Levi-Civita connection of (M,g) is denoted by $(P^{\nabla^g})^t_0(\gamma)T_0$. In the notation of the Levi-Civita connection ∇^g (resp. parallel transport P^{∇^g}), the upper index g (resp. ∇^g) referring to the Riemannian metric g (resp. the connection ∇^g) is omitted if it is clear from the context. Let $(\gamma,\hat{\gamma}):I\to M\times\hat{M}$ be a smooth curve on $M\times\hat{M}$ defined on an open real interval I containing 0. If $(X(t),\hat{X}(t)):I\to T(M\times\hat{M})$ is a smooth vector field on $M\times\hat{M}$ along $(\gamma,\hat{\gamma})$ i.e., $(X(t),\hat{X}(t))\in T|_{(\gamma(t),\hat{\gamma}(t))}(M\times\hat{M})$ then one has

$$\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(X,\hat{X})=(\nabla_{\dot{\gamma}(t)}X,\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X}) \tag{7}$$

if the covariant derivatives on the right-hand side are well defined.

If (N,h) is a Riemannian manifold we define $\mathrm{Iso}(N,h)$ to be the (smooth Lie) group of isometries of (N,h) (cf. [39], Lemma III.6.4, p. 118). It is clear that the isometries respect parallel transport in the sense that for any absolutely continuous $\gamma:[a,b]\to N$ and $F\in\mathrm{Iso}(N,g)$ one has (cf. [39], p. 41, Eq. (3.5))

$$F_*|_{\gamma(t)}\circ(P^{\nabla^h})^t_a(\gamma)=(P^{\nabla^h})^t_a(F\circ\gamma)\circ F_*|_{\gamma(a)}. \tag{8}$$

The following result is standard.

Theorem 2.1 Let (N, h) be a Riemannian manifold and for any absolutely continuous $\gamma : [0, 1] \to M$, $\gamma(0) = y_0$, define

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = \int_0^t (P^{\nabla^h})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s \in T|_{y_0}N, \quad t \in [0, 1].$$

Then the map $\Lambda^{\nabla^h}_{y_0} : \gamma \mapsto \Lambda^{\nabla^h}_{y_0}(\gamma)(\cdot)$ is an injection from the set of absolutely continuous curves $[0, 1] \to N$ starting at y_0 onto an open subset of the Banach space of absolutely continuous curves $[0, 1] \to T|_{y_0}N$ starting at 0.

Moreover, the map $\Lambda^{\nabla^h}_{y_0}$ is a bijection onto the latter Banach space if (and only if) (N, h) is a complete Riemannian manifold.

Remark 2.2 (i) For example, in the case where γ is the geodesic $t \mapsto \exp_{y_0}(tY)$ for $Y \in T|_{y_0}N$, one has

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = tY.$$

(ii) It is directly seen from the definition of $\Lambda^{\nabla^h}_{y_0}$ that it maps injectively (piecewise) C^k-curves, $k = 1, \ldots, \infty$, starting at y_0 to (piecewise) C^k-curves starting at 0. Moreover, these correspondences are bijective if (N, h) is complete.

3 State Space, Distributions and Computational Tools

3.1 State Space

3.1.1 Definition of the state space

After [3], [4] we make the following definition.

Definition 3.1 The *state space* $Q = Q(M, \hat{M})$ for the rolling of two n-dimensional *connected, oriented* smooth Riemannian manifolds $(M, g), (\hat{M}, \hat{g})$ is defined as

$$Q = \{A : T|_x M \to T|_{\hat{x}}\hat{M} \mid A \text{ o-isometry}, \ x \in M, \ \hat{x} \in \hat{M}\},$$

where "o-isometry" stands for "orientation preserving isometry" i.e., if $(X_i)_{i=1}^n$ is a positively oriented g-orthonormal frame of M at x then $(AX_i)_{i=1}^n$ is a positively oriented $\hat{g}$-orthonormal frame of $\hat{M}$ at $\hat{x}$.

The linear space of $\mathbb{R}$-linear map $A : T|_x M \to T|_{\hat{x}}\hat{M}$ is canonically isomorphic to the tensor product $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$. On the other hand, by using the canonical inclusions $T^*|_x M \subset T^*|_{(x,\hat{x})}(M \times \hat{M})$, $T|_{\hat{x}}\hat{M} \subset T|_{(x,\hat{x})}(M \times \hat{M})$, the space $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ is canonically included in the space $T^1_1(M \times \hat{M})|_{(x,\hat{x})}$ of $(1, 1)$-tensors of $M \times \hat{M}$ at $(x, \hat{x})$. These inclusions make $T^*M \otimes T\hat{M} := \bigcup_{(x,\hat{x}) \in M \times \hat{M}} T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ a subset of $T^1_1(M \times \hat{M})$ such that $\pi_{T^*M \otimes T\hat{M}} := \pi_{T^1_1(M \times \hat{M})}|_{T^*M \otimes T\hat{M}} : T^*M \otimes T\hat{M} \to M \times \hat{M}$ is a smooth vector subbundle of the bundle of $(1, 1)$-tensors $\pi_{T^1_1(M \times \hat{M})}$ on $M \times \hat{M}$.

The state space $Q = Q(M, \hat{M})$ can be described as a subset of $T^*M \otimes T\hat{M}$ as

$$Q = \{A \in T^*M \otimes T\hat{M}|_{(x,\hat{x})} \mid (x, \hat{x}) \in M \times \hat{M}, \\ \|AX\|_{\hat{g}} = \|X\|_g, \ \forall X \in T|_x M, \ \det(A) = 1\}.$$

In the next subsection, we will show that $\pi_Q := \pi_{T^*M\otimes T\hat{M}}|_Q$ is moreover a smooth subbundle of $\pi_{T^*M\otimes T\hat{M}}$. It is also sometimes convenient to consider the manifold $T^*M \otimes T\hat{M}$ and we will refer to it as the *extended state space* for the rolling. This concept of extended state space naturally makes sense also in the case where M and $\hat{M}$ are not assumed to be oriented (or connected). A point $A \in T^*M \otimes T\hat{M}$ with $\pi_{T^*M\otimes T\hat{M}}(A) = (x, \hat{x})$ (or $A \in Q$ with $\pi_Q(A) = (x, \hat{x})$) will be usually denoted by $(x, \hat{x}; A)$ to emphasize the fact that $A : T|_x M \to T|_{\hat{x}}\hat{M}$. Thus the notation $q = (x, \hat{x}; A)$ simply means that $q = A$.

3.1.2 The Bundle Structure of Q

In this subsection, it is shown that π_Q is a bundle with typical fiber $\mathrm{SO}(n)$.

Definition 3.2 Suppose the vector fields $X_i \in \mathrm{VF}(M)$ (resp. $\hat{X}_i \in \mathrm{VF}(\hat{M})$), $i = 1, \dots, n$ form a g-orthonormal (resp. $\hat{g}$-orthonormal) frame of vector fields on an open subset U of M (resp. $\hat{U}$ of $\hat{M}$). We denote $F = (X_i)_{i=1}^n$, $\hat{F} = (\hat{X}_i)_{i=1}^n$ and for $x \in U$, $\hat{x} \in \hat{U}$ we let $F|_x = (X_i|_x)_{i=1}^n$, $\hat{F}|_{\hat{x}} = (\hat{X}_i|_{\hat{x}})_{i=1}^n$ Then a local trivialization $\tau = \tau_{F,\hat{F}}$ of Q over $U \times \hat{U}$ *induced by* $F, \hat{F}$ is given by

$$\begin{aligned}\tau : \pi_Q^{-1}(U \times \hat{U}) &\to (U \times \hat{U}) \times \mathrm{SO}(n)\\ (x, \hat{x}; A) &\mapsto \big((x, \hat{x}), \mathcal{M}_{F|_x,\hat{F}|_{\hat{x}}}(A)\big),\end{aligned}$$

where $\mathcal{M}_{F|_x,\hat{F}|_{\hat{x}}}(A)_i^j = \hat{g}(AX_i, \hat{X}_j)$ since $AX_i|_x = \sum_j \hat{g}(AX_i|_x, \hat{X}_j|_{\hat{x}})\hat{X}_j|_{\hat{x}}$.

For the sake of clarity, we shall write $\mathcal{M}_{F|_x,\hat{F}|_{\hat{x}}}(A)$ as $\mathcal{M}_{F,\hat{F}}(A)$. Obviously $\|AX\|_{\hat{g}} = \|X\|_g$ for all $X \in T|_x M$ is equivalent to $A^{T_{g,\hat{g}}}A = \mathrm{id}_{T|_x M}$ and we get

$$\mathcal{M}_{F,\hat{F}}(A)^T \mathcal{M}_{F,\hat{F}}(A) = \mathcal{M}_{\hat{F},F}(A^{T_{g,\hat{g}}})\mathcal{M}_{F,\hat{F}}(A) = \mathcal{M}_{F,F}(\mathrm{id}_{T|_x M}) = \mathrm{id}_{\mathbb{R}^n},$$

where T denotes the usual transpose in $\mathfrak{gl}(n)$, the set of Lie algebra of $n \times n$-real matrices. Since $\det \mathcal{M}_{F,\hat{F}}(A) = \det(A) = +1$, one has $\mathcal{M}_{F,\hat{F}}(A) \in \mathrm{SO}(n)$.

Remark 3.3 Notice that the above local trivializations $\tau_{F,\hat{F}}$ of π_Q are just the restrictions of the vector bundle local trivializations

$$(\pi_{T^*(M)\otimes T(\hat{M})})^{-1}(U \times \hat{U}) \to (U \times \hat{U}) \times \mathfrak{gl}(n)$$

of the bundle $\pi_{T^*(M)\otimes T(\hat{M})}$ induced by $F, \hat{F}$ and defined by the same formula as $\tau_{F,\hat{F}}$. In this setting, one does not even have to assume that the local frames F, $\hat{F}$ are g- or $\hat{g}$-orthonormal. Hence π_Q is a smooth subbundle of $\pi_{T^*M\otimes T\hat{M}}$ with Q a smooth submanifold of $T^*M \otimes T\hat{M}$.

One has the following simple proposition.

Proposition 3.4 Let $q = (x, \hat{x}; A) \in Q$ and $B \in T^*(M) \otimes T(\hat{M})|_{(x,\hat{x})}$. Every π_Q-vertical tangent vector (i.e., an element of $V|_q(\pi_Q)$) is of the form $\nu(B)|_q$ for a unique $B \in T^*M \otimes T\hat{M}|_{(x,\hat{x})}$ and $\nu(B)|_q$ is tangent to Q if and only if

$$\hat{g}(AX, BY) + \hat{g}(BX, AY) = 0,$$

for all $X, Y \in T|_x M$ or simply $B \in A(\mathfrak{so}(T|_x M))$.

We use $\overline{T}$ to denote the $(g, \hat{g})$-transpose operation $T_{g,\hat{g}}$ in the sequel. The proposition says that $V|_q(\pi_Q)$ is naturally $\mathbb{R}$-linearly isomorphic to $A(\mathfrak{so}(T|_x M))$.

3.2 Distribution and the Control Problem

3.2.1 From Rolling to Distributions

Each point $(x, \hat{x}; A)$ of the state space $Q = Q(M, \hat{M})$ can be viewed as describing a contact point of the two manifolds which is given by the points x and $\hat{x}$ of M and $\hat{M}$, respectively, and an isometry A of the tangent spaces $T|_x M$, $T|_{\hat{x}}\hat{M}$ at this contact point. The isometry A can be viewed as measuring the relative orientation of these tangent spaces relative to each other in the sense that rotation of, say, $T|_{\hat{x}}\hat{M}$ corresponds to a unique change of the isometry A from $T|_x M$ to $T|_{\hat{x}}\hat{M}$. A curve $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ in Q can then be seen as a motion of M against $\hat{M}$ such that at an instant t, $\gamma(t)$ and $\hat{\gamma}(t)$ represent the common point of contact in M and $\hat{M}$, respectively, and $A(t)$ measures the relative orientation of coinciding tangent spaces $T|_{\gamma(t)}M$, $T|_{\hat{\gamma}(t)}\hat{M}$ at this point of contact.

In order to call this motion *rolling*, there are two kinematic constraints that will be demanded (see e.g. [3], [4] Chapter 24, [9]) namely

(i) the *no-spinning* condition;

(ii) the *no-slipping* condition.

In this section, these conditions will be defined explicitly and it will turn out that they are modeled by certain smooth distributions on the state space Q. The subsequent sections are then devoted to the detailed definitions and analysis of the distribution $\mathcal{D}_{\rm NS}$ and $\mathcal{D}_{\rm R}$ on the state space Q, the former capturing the no-spinning condition (i) while the latter capturing both of the conditions (i) and (ii).

The first restriction (i) for the motion is that the relative orientation of the two manifolds should not change along motion. This *no-spinning condition* (also known as the no-twisting condition) can be formulated as follows.

Definition 3.5 An absolutely continuous (*a.c.*) curve

$$q: \ I \to Q, \qquad t \mapsto (\gamma(t), \hat{\gamma}(t); A(t)),$$

defined on some real interval $I = [a, b]$, is said to describe a *motion without spinning* of M against $\hat{M}$ if, for every a.c. curve $[a, b] \to TM$; $t \mapsto X(t)$ of vectors along $t \mapsto \gamma(t)$, we have

$$\nabla_{\dot{\gamma}(t)} X(t) = 0 \implies \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t)) = 0 \quad \text{for a.e. } t \in [a, b]. \tag{9}$$

(See also [30] for a similar definition.) Note that Condition (9) is equivalent to the following: for a. e. t and all parallel vector fields $X(\cdot)$ along $x(\cdot)$, one has

$$(\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(t)) X(t) = 0.$$

Since the parallel translation $P_0^t(\gamma) : T|_{\gamma(0)}M \to T|_{\gamma(t)}M$ along $\gamma(\cdot)$ is an (isometric) isomorphism (here $X(t) = P_0^t(\gamma)X(0)$), then (9) is equivalent to

$$\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(t) = 0 \quad \text{for a.e. } t \in [a, b]. \tag{10}$$

The second restriction (ii) is that the manifolds should not slip along each other as they move i.e., the velocity of the contact point should be the same w.r.t both manifolds. This *no-slipping condition* can be formulated as follows.

Definition 3.6 An a.c. curve $q : I \to Q;\ t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$, defined on some real interval $I = [a, b]$, is said to describe a *motion without slipping* of M against $\hat{M}$ if

$$A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t) \quad \text{for a.e. } t \in [a, b]. \tag{11}$$

Definition 3.7 An a.c. curve $q : I \to Q;\ t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$, defined on some real interval $I = [a, b]$, is said to describe a *rolling motion* i.e., a *motion without slipping or spinning* of M against $\hat{M}$ if it satisfied both of the conditions (9),(11) (or equivalently (10),(11)). The corresponding curve $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ that satisfies these conditions is called a *rolling curve*.

It is easily seen that $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, is a rolling curve if and only if it satisfies the following driftless control affine system

$$(\Sigma)_R \quad \begin{cases} \dot{\gamma}(t) = u(t), & \\ \dot{\hat{\gamma}}(t) = A(t)u(t), & \text{for a.e. } t \in [a, b], \\ \overline{\nabla}_{(u(t), A(t)u(t))} A(t) = 0, & \end{cases} \tag{12}$$

where the control u belongs to $\mathcal{U}(M)$, the set of measurable TM-valued functions u defined on some interval $I = [a, b]$ such that there exists a.c. $y : [a, b] \to M$ verifying $u = \dot{y}$ a.e. on $[a, b]$. Conversely, given any control $u \in \mathcal{U}(M)$ and $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, a solution $q(\cdot)$ to this control system exists on a subinterval $[a, b']$, $a < b' \leq b$ satisfying the initial condition $q(a) = q_0$. The fact that System (12) is driftless and control affine can be seen from its representation in local coordinates (see Eqs. (53)-(55) in Appendix A).

We begin by recalling some basic observations on parallel transport. As is clear, if one starts with a $(1, 1)$-tensor $A_0 \in T^1_1|_{(x_0, \hat{x}_0)}(M \times \hat{M})$ and has an a.c. curve $t \mapsto (\gamma(t), \hat{\gamma}(t))$ on $M \times \hat{M}$ with $\gamma(0) = x_0$, $\hat{\gamma}(0) = \hat{x}_0$, defined on an open interval $I \ni 0$, then the parallel transport $A(t) = P^t_0(\gamma, \hat{\gamma})A_0$ exists on I and determines an a.c. curve in $T^1_1(M \times \hat{M})$. But now, if A_0 rather belongs to the subspace $T^*M \otimes T\hat{M}$ or Q of $T^1_1(M \times \hat{M})$, it will actually happen that the parallel translate $A(t)$ belongs to this subspace as well for all $t \in I$. This is the content of the next proposition, whose proof is straightforward.

Proposition 3.8 Let $t \mapsto (\gamma(t), \hat{\gamma}(t))$ be an absolutely continuous curve in $M \times \hat{M}$ defined on some real interval $I \ni 0$. Then we have

$$A_0 \in T^*M \otimes TM \Longrightarrow A(t) = P^t_0(\gamma, \hat{\gamma})A_0 \in T^*M \otimes T\hat{M} \quad \forall t \in I,$$
$$A_0 \in Q \quad \Longrightarrow \quad A(t) = P^t_0(\gamma, \hat{\gamma})A_0 \in Q \quad \forall t \in I,$$

and

$$P^t_0(\gamma, \hat{\gamma})A_0 = P^t_0(\hat{\gamma}) \circ A_0 \circ P^0_t(\gamma) \quad \forall t \in I. \tag{13}$$

Let $T(M \times \hat{M}) \times_{M \times \hat{M}} (T^*(M) \otimes T(\hat{M}))$ be the total space of the product vector bundle $\pi_{T(M \times \hat{M})} \times_{M \times \hat{M}} \pi_{T^*(M) \otimes T(\hat{M})}$ over $M \times \hat{M}$. We will define certain *lift* operations corresponding to parallel translation of elements of $T^*M \otimes T\hat{M}$.

Definition 3.9 The *No-Spinning lift* is defined to be the map

$$\mathscr{L}_{\rm NS} : T(M\times\hat{M})\times_{M\times\hat{M}}\big(T^*(M)\otimes T(\hat{M})\big)\to T\big(T^*(M)\otimes T(\hat{M})\big),$$

such that, if $q=(x,\hat{x};A)\in T^*(M)\otimes T(\hat{M})$, $X\in T|_xM$, $\hat{X}\in T|_{\hat{x}}\hat{M}$ and $t\mapsto(\gamma(t),\hat{\gamma}(t))$ is a smooth curve on in $M\times\hat{M}$ defined on an open interval $I\ni 0$ s.t. $\dot{\gamma}(0)=X$, $\dot{\hat{\gamma}}(0)=\hat{X}$, then one has

$$\mathscr{L}_{\rm NS}((X,\hat{X}),q)=\frac{\rm d}{{\rm d}t}\big|_0 P_0^t(\gamma,\hat{\gamma})A\in T|_q\big(T^*(M)\otimes T(\hat{M})\big). \tag{14}$$

The smoothness of the map $\mathscr{L}_{\rm NS}$ can be easily seen by using local trivializations. We will usually use a notation $\mathscr{L}_{\rm NS}(\overline{X})|_q$ for $\mathscr{L}_{\rm NS}(\overline{X},q)$ when $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$ and $q=(x,\hat{x};A)\in T^*(M)\otimes T(\hat{M})$. In particular, when $\overline{X}\in{\rm VF}(M\times\hat{M})$, we get a *lifted vector field* on $T^*(M)\otimes T(\hat{M})$ given by $q\mapsto\mathscr{L}_{\rm NS}(\overline{X})|_q$. The smoothness of $\mathscr{L}_{\rm NS}(\overline{X})$ for $\overline{X}\in{\rm VF}(M\times\hat{M})$ follows immediately from the smoothness of the map $\mathscr{L}_{\rm NS}$. Notice that, by Proposition 3.8, the No-Spinning lift map $\mathscr{L}_{\rm NS}$ restricts to

$$\mathscr{L}_{\rm NS} : T(M\times\hat{M})\times_{M\times\hat{M}} Q\to TQ,$$

where $T(M\times\hat{M})\times_{M\times\hat{M}}Q$ is the total space of the fiber product $\pi_{T(M\times\hat{M})}\times_{M\times\hat{M}}\pi_Q$.

Definition 3.10 The *No-Spinning (NS) distribution* $\mathcal{D}_{\rm NS}$ on $T^*M\otimes T\hat{M}$ is a $2n$-dimensional smooth distribution defined pointwise by

$$\mathcal{D}_{\rm NS}|_q=\mathscr{L}_{\rm NS}(T|_{(x,\hat{x})}(M\times\hat{M}))|_q, \tag{15}$$

with $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$. Since $\mathcal{D}_{\rm NS}|_Q\subset TQ$ (by Proposition 3.8) this distribution restricts to a $2n$-dimensional smooth distribution on Q which we also denote by $\mathcal{D}_{\rm NS}$ (instead of $\mathcal{D}_{\rm NS}|_Q$).

The No-Spinning lift $\mathscr{L}_{\rm NS}$ will also be called $\mathcal{D}_{\rm NS}$-lift since it maps vectors of $M\times\hat{M}$ to vectors in $\mathcal{D}_{\rm NS}$. The distribution $\mathcal{D}_{\rm NS}$ is smooth since $\mathscr{L}_{\rm NS}(\overline{X})$ is smooth for any smooth vector field $\overline{X}\in{\rm VF}(M\times\hat{M})$. Also, the fact that the rank of $\mathcal{D}_{\rm NS}$ exactly is $2n$ follows from the next proposition, which itself follows immediately from Eq. (14).

Proposition 3.11 For every $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ and $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$, one has

$$(\pi_{T^*M\otimes T\hat{M}})_*(\mathscr{L}_{\rm NS}(\overline{X})|_q)=\overline{X},$$

and in particular $(\pi_Q)_*(\mathscr{L}_{\rm NS}(\overline{X})|_q)=\overline{X}$ if $q\in Q$.

Thus $(\pi_{T^*M\otimes T\hat{M}})_*$ (resp. $(\pi_Q)_*$) maps $\mathcal{D}_{\rm NS}|_{(x,\hat{x};A)}$ isomorphically onto $T|_{(x,\hat{x})}(M\times\hat{M})$ for every $(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ (resp. $(x,\hat{x};A)\in Q$) and the inverse map of $(\pi_{T^*M\otimes T\hat{M}})_*|_{\mathcal{D}_{\rm NS}|_q}$ (resp. $(\pi_Q)_*|_{\mathcal{D}_{\rm NS}|_q}$) is $\overline{X}\mapsto\mathscr{L}_{\rm NS}(\overline{X})|_q$.
The following basic formula for the lift $\mathscr{L}_{\rm NS}$ will be useful.

Theorem 3.12 For $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$ and $A\in\Gamma(\pi_{T^*M\otimes T\hat{M}})$, we have

$$\mathscr{L}_{\rm NS}(\overline{X})|_{A|_{(x,\hat{x})}}=A_*(\overline{X})-\nu\big(\overline{\nabla}_{\overline{X}}A\big)|_{A|_{(x,\hat{x})}}, \tag{16}$$

where ν denotes the vertical derivative in the vector bundle $\pi_{T^*M\otimes T\hat{M}}$ and A_* is the map $T(M\times\hat{M})\to T(T^*M\otimes T\hat{M})$.

Proof. Choose smooth paths $\gamma:[-1,1]\to M$, $\hat{\gamma}:[-1,1]\to\hat{M}$ such that $(\dot{\gamma}(0),\dot{\hat{\gamma}}(0))=\overline{X}$ and take an arbitrary $f\in C^\infty(T^*M\otimes T\hat{M})$. Define $\tilde{A}(t)=P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}$. Then

$$\mathscr{L}_{\rm NS}(\overline{X})|_{A|_{(x,\hat{x})}}=\dot{\tilde{A}}(0)=\tilde{A}_*(\frac{\partial}{\partial t}).$$

Also, it is known that (see e.g. [39], p.29)

$$P_t^0(\gamma,\hat{\gamma})(A|_{(\gamma(t),\hat{\gamma}(t))})=A|_{(x,\hat{x})}+t\overline{\nabla}_{\overline{X}}A+t^2F(t), \tag{17}$$

with $t\mapsto F(t)$ a C^∞-function $]-1,1[\to T^*|_xM\otimes T|_{\hat{x}}\hat{M}$. Moreover, one has

$$\begin{aligned}
&\big(A_*(\overline{X})-\tilde{A}_*(\frac{\partial}{\partial t})\big)f=\lim_{t\to 0}\frac{f(A|_{(\gamma(t),\hat{\gamma}(t))})-f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})})}{t}\\
=&\lim_{t\to 0}\frac{f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}+tP_0^t(\gamma,\hat{\gamma})\overline{\nabla}_{\overline{X}}A+t^2P_0^t(\gamma,\hat{\gamma})F(t))-f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})})}{t}\\
=&\lim_{t\to 0}\frac{1}{t}\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}f\big(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}+sP_0^t(\gamma,\hat{\gamma})\overline{\nabla}_{\overline{X}}A+s^2P_0^t(\gamma,\hat{\gamma})F(t)\big)\mathrm{d}s\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}f\big(A|_{(x,\hat{x})}+s\overline{\nabla}_{\overline{X}}A+s^2F(0)\big)=\nu(\overline{\nabla}_{\overline{X}}A)|_{A|_{(x,\hat{x})}}f.
\end{aligned}$$

□

We shall write Eq. (16) from now on with a compressed notation

$$\mathscr{L}_{\rm NS}(\overline{X})|_A=A_*(\overline{X})-\nu(\overline{\nabla}_{\overline{X}}A)|_A.$$

Remark 3.13 If $A\in\Gamma(\pi_{T^*(M)\otimes T(\hat{M})})$ and $q:=A|_{(x,\hat{x})}\in Q$ (e.g. if $A\in\Gamma(\pi_Q)$), then on the right hand side of (16), both terms are elements of $T|_q(T^*M\otimes T\hat{M})$ but their difference is actually an element of $T|_qQ$.

As a trivial corollary of the theorem, one gets the following.

Corollary 3.14 Suppose $t\mapsto q(t)=(\gamma(t),\hat{\gamma}(t);A(t))$ is an a.c. curve on $T^*M\otimes T\hat{M}$ or Q defined on an open real interval I. Then, for a.e. $t\in I$,

$$\mathscr{L}_{\rm NS}\big(\dot{\gamma}(t),\dot{\hat{\gamma}}(t)\big)\big|_{q(t)}=\dot{A}(t)-\nu\big(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A\big)\big|_{q(t)}.$$

Remark 3.15 The controllability of the control system associated to the distribution $\mathcal{D}_{\rm NS}$ is studied more thoroughly in [24]. In particular, it is shown there that the orbits of $\mathcal{D}_{\rm NS}$ can be completely characterized in terms of the holonomy groups of (M,g) and $(\hat{M},\hat{g})$.

3.2.2 The Rolling Distribution $\mathcal{D}_{\rm R}$

We next define a distribution which will correspond to the rolling with neither slipping nor spinning. As regards the rolling of one manifold onto another one, the admissible curve $q(\cdot)$ must verify the no-spinning condition (9) and no-slipping condition (11) that we recall next. Since $q(\cdot)$ is tangent to $\mathcal{D}_{\rm NS}$, we have $A(t)=$

$P_0^t(x,\hat{x})A(0)$, and the no-slipping condition (11) writes $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$. It forces one to have, for a.e. t,

$$\dot{q}(t) = \mathscr{L}_{\mathrm{NS}}\big(\dot{\gamma}(t), A(t)\dot{\gamma}(t)\big)\big|_{q(t)}.$$

Evaluating at $t = 0$ and noticing that if $q_0 := q(0)$, with $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\dot{\gamma}(0) =: X \in T|_{x_0}M$ are arbitrary, we get

$$\dot{q}(0) = \mathscr{L}_{\mathrm{NS}}(X, A_0X)|_{q_0}.$$

This motivates the following definition.

Definition 3.16 For $q = (x, \hat{x}; A) \in Q$, we define the *Rolling lift* or $\mathcal{D}_{\mathrm{R}}$-lift as a bijective linear map

$$\mathscr{L}_{\mathrm{R}} : T|_x M \times Q|_{(x,\hat{x})} \to T|_q Q,$$

given by

$$\mathscr{L}_{\mathrm{R}}(X, q) = \mathscr{L}_{\mathrm{NS}}(X, AX)|_q. \tag{18}$$

This map naturally induces $\mathscr{L}_{\mathrm{R}} : \mathrm{VF}(M) \to \mathrm{VF}(Q)$ as follows. For $X \in \mathrm{VF}(M)$ we define $\mathscr{L}_{\mathrm{R}}(X)$, the *Rolling lifted* vector field associated to X, by

$$\begin{aligned} \mathscr{L}_{\mathrm{R}}(X) : Q &\to T(Q), \\ q &\mapsto \mathscr{L}_{\mathrm{R}}(X)|_q, \end{aligned}$$

where $\mathscr{L}_{\mathrm{R}}(X)|_q := \mathscr{L}_{\mathrm{R}}(X, q)$.

The Rolling lift map $\mathscr{L}_{\mathrm{R}}$ allows one to construct a distribution on Q (see [7]) reflecting both of the rolling restrictions of motion defined by the no-spinning condition, Eq. (9), and the no-slipping condition, Eq. (11).

Definition 3.17 The *rolling distribution* $\mathcal{D}_{\mathrm{R}}$ on Q is the n-dimensional smooth distribution defined pointwise by

$$\mathcal{D}_{\mathrm{R}}|_q = \mathscr{L}_{\mathrm{R}}(T|_x M)|_q, \tag{19}$$

for $q = (x, \hat{x}; A) \in Q$.

The Rolling lift $\mathscr{L}_{\mathrm{R}}$ will also be called $\mathcal{D}_{\mathrm{R}}$-lift since it maps vectors of M to vectors in $\mathcal{D}_{\mathrm{R}}$. Thus an absolutely continuous curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q is a rolling curve if and only if it is a.e. tangent to $\mathcal{D}_{\mathrm{R}}$ i.e., $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}|_{q(t)}$ for a.e. t or, equivalently, if $\dot{q}(t) = \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))|_{q(t)}$ for a.e. t.

Define $\pi_{Q,M} = \mathrm{pr}_1 \circ \pi_Q : Q \to M$ and notice that its differential $(\pi_{Q,M})_*$ maps each $\mathcal{D}_{\mathrm{R}}|_{(x,\hat{x};A)}$, $(x, \hat{x}; A) \in Q$, isomorphically onto $T|_x M$. Similarly one defines $\pi_{Q,\hat{M}} = \mathrm{pr}_2 \circ \pi_Q : Q \to \hat{M}$.

Proposition 3.18 For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and absolutely continuous $\gamma : [0, a] \to M$, $a > 0$, such that $\gamma(0) = x_0$, there exists a unique absolutely continuous $q : [0, a'] \to Q$, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, with $0 < a' \leq a$ (and a' maximal with the latter property), which is tangent to $\mathcal{D}_{\mathrm{R}}$ a.e. and $q(0) = q_0$. We denote this unique curve q by

$$t \mapsto q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)),$$

and refer to it as the *rolling curve along γ with initial position* q_0. In the case that $\hat{M}$ is a complete manifold one has $a' = a$.

Conversely, any absolutely continuous curve $q : [0, a] \to Q$, which is a.e. tangent to $\mathcal{D}_{\mathrm{R}}$, is a rolling curve along $\gamma = \pi_{Q,M} \circ q$ i.e., has the form $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q(0))$.

Proof. We need to show only that completeness of $(\hat{M}, \hat{g})$ implies that $a' = a$. In fact, $\hat{X}(t) := A_0 \int_0^t P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s$ defines an a.c. curve $t \mapsto \hat{X}(t)$ in $T|_{\hat{x}_0}\hat{M}$ defined on $[0, a]$ and the completeness of $\hat{M}$ implies that there is a unique a.c. curve $\hat{\gamma}$ on $\hat{M}$ defined on $[0, a]$ such that $\hat{X}(t) = \int_0^t P_s^0(\hat{\gamma})\dot{\hat{\gamma}}(s)\mathrm{d}s$ for all $t \in [0, a]$. Defining $A(t) = P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma)$, $t \in [0, a]$ we notice that $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ is the rolling curve along γ starting at q_0 that is defined on the interval $[0, a]$. Hence $a' = a$. □

Remark 3.19 It follows immediately from the uniqueness statement of the previous theorem that, if $\gamma : [a, b] \to M$ and $\omega : [c, d] \to M$ are two a.c. curves with $\gamma(b) = \omega(c)$ and $q_0 \in Q$, then

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega \sqcup \gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(b)) \sqcup q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0). \tag{20}$$

On the space $\Omega_{x_0}(M)$ of piecewise differentiable loops of M based at x_0 one has

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega.\gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)).q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0),$$

where $\gamma, \omega \in \Omega_{x_0}(M)$.

In the case where the curve γ on M is a geodesic, we can give a more precise form of the rolling curve along γ with a given initial position.

Proposition 3.20 Consider $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $X \in T|_{x_0}M$ and $\gamma : [0, a] \to M$; $\gamma(t) = \exp_{x_0}(tX)$, a geodesic of (M, g) with $\gamma(0) = x_0$, $\dot{\gamma}(0) = X$. Then the rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) = (\gamma, \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) : [0, a'] \to Q$, $0 < a' \leq a$, along γ with initial position q_0 is given by

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = \widehat{\exp}_{\hat{x}_0}(tA_0X), \quad A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = P_0^t(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) \circ A_0 \circ P_t^0(\gamma).$$

Of course, $a' = a$ if $\hat{M}$ is complete.

Proof. Let $0 < a' \leq a$ such that $\hat{\gamma}(t) := \widehat{\exp}_{\hat{x}_0}(tA_0X)$ is defined on $[0, a']$. Then, by proposition 3.8, $q(t) := (\gamma(t), \hat{\gamma}(t); A(t))$ with $A(t) := P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma)$, $t \in [0, a']$, is a curve on Q and $A(t)$ is parallel to $(\gamma, \hat{\gamma})$ in $M \times \hat{M}$. Therefore $t \mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ on $[0, a']$ and thus $\dot{q}(t) = \mathscr{L}_{\mathrm{NS}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{q(t)}$. Moreover, since γ and $\hat{\gamma}$ are geodesics,

$$A(t)\dot{\gamma}(t) = (P_0^t(\hat{\gamma}) \circ A_0)(P_t^0(\gamma)\dot{\gamma}(t)) = P_0^t(\hat{\gamma})(A_0X) = \dot{\hat{\gamma}}(t),$$

which shows that for $t\in[0,a']$,

$$\begin{aligned}\dot{q}(t) &= \mathscr{L}_{\mathrm{NS}}(\dot{\gamma}(t),A(t)\dot{\gamma}(t))|_{q(t)} \\ &= \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))\big|_{q(t)}.\end{aligned}$$

Hence $t\mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{R}}$ i.e., it is a rolling curve along γ with initial position $q(0)=(\gamma(0),\hat{\gamma}(0);A(0))=(x_0,\hat{x}_0;A_0)=q_0$.

□

Remark 3.21 If $\gamma(t)=\exp_{x_0}(tA_0X)$ and $q_0=(x_0,\hat{x}_0;A_0)$, the statement of the proposition can be written in a compact form as

$$A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(t)=P_0^t\big(s\mapsto\overline{\exp}_{(x_0,\hat{x}_0)}(s(X,A_0X))\big)A_0,$$

for all t where defined.

The next proposition describes the symmetry of the study of the rolling problem of (M,g) rolling against $(\hat{M},\hat{g})$ to the problem of $(\hat{M},\hat{g})$ rolling against (M,g).

Proposition 3.22 Let $\widehat{\mathcal{D}_{\mathrm{R}}}$ be the rolling distribution in $\hat{Q}:=Q(\hat{M},M)$. Then the map

$$\iota:Q\to\hat{Q};\quad \iota(x,\hat{x};A)=(\hat{x},x;A^{-1})$$

is a diffeomorphism of Q onto $\hat{Q}$ and

$$\iota_*\mathcal{D}_{\mathrm{R}}=\widehat{\mathcal{D}_{\mathrm{R}}}.$$

In particular, $\iota(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q))=\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q))$.

Proof. It is obvious that ι is a diffeomorphism (with the obvious inverse map) and for an a.c. path $q(t)=(\gamma(t),\hat{\gamma}(t);A(t))$ in Q, $(\iota\circ q)(t)=(\hat{\gamma}(t),\gamma(t);A(t)^{-1})$ is a.c. in $\hat{Q}$ and for a.e. t,

$$\begin{cases}\dot{\hat{\gamma}}(t)=A(t)\dot{\gamma}(t)\\ A(t)=P_0^t(\hat{\gamma})\circ A(0)\circ P_t^0(\gamma)\end{cases}\iff\begin{cases}\dot{\gamma}(t)=A(t)^{-1}\dot{\hat{\gamma}}(t)\\ A(t)^{-1}=P_0^t(\gamma)\circ A(0)^{-1}\circ P_t^0(\hat{\gamma})\end{cases}.$$

These simple remarks prove the claims.

□

Remark 3.23 Notice that Definitions 3.16 and 3.17 make sense not only in Q but also in the space $T^*M\otimes T\hat{M}$. It is easily seen that $\mathcal{D}_{\mathrm{R}}$ defined on $T^*M\otimes T\hat{M}$ by Eq. (19) is actually tangent to Q so its restriction to Q gives exactly $\mathcal{D}_{\mathrm{R}}$ on Q as defined above. Similarly, Propositions 3.18, 3.20 and 3.22 still hold if we replace Q by $T^*M\otimes T\hat{M}$ and $\hat{Q}$ by $T^*\hat{M}\otimes TM$ everywhere in their statements.

3.3 Lie brackets of vector fields on Q

In this section, we compute commutators of the vectors fields of $T^*M\otimes T\hat{M}$ and Q with respect to the splitting of $T(T^*M\otimes T\hat{M})$ (resp. TQ) as a direct sum $\mathcal{D}_{\mathrm{NS}}\oplus V(\pi_{T^*M\otimes T\hat{M}})$ (resp. $\mathcal{D}_{\mathrm{NS}}\oplus V(\pi_Q)$). The main results are Propositions 3.35, 3.35 and 3.37. These computations will serve as preliminaries for the Lie bracket computations relative to the rolling distribution $\mathcal{D}_{\mathrm{R}}$ studied in the next section. It is convenient to make computations in $T^*M\otimes T\hat{M}$ and then to restrict the results to Q.

3.3.1 Computational tools

The next lemmas will be useful in the subsequent calculations.

Lemma 3.24 Let $(x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ (resp. $(x, \hat{x}; A) \in Q$). Then there exists a local $\pi_{T^*M\otimes T\hat{M}}$-section (resp. π_Q-section) $\tilde{A}$ around $(x, \hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$ and $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$.

Proof. Let U be an open neighborhood of the origin of $T|_{(x,\hat{x})}(M \times \hat{M})$, where the $\overline{g}$-exponential map $\overline{\exp} : U \to M \times \hat{M}$ is a diffeomorphism onto its image. Parallel translate A along geodesics $t \mapsto \overline{\exp}(t\overline{X})$, $\overline{X} \in U$, to get a local section $\tilde{A}$ of $T^*(M) \otimes T(\hat{M})$ in a neighborhood of $\overline{x} = (x, \hat{x})$. More explicitly, one has

$$\tilde{A}|_{\overline{y}} = P_0^1\big(t \mapsto \overline{\exp}\big(t(\overline{\exp}_{\overline{x}})^{-1}(\overline{y})\big)\big)A,$$

for $\overline{y} \in U$. If $(x, \hat{x}; A) \in Q$, this actually provides a local π_Q-section. Moreover, we clearly have $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$.

□

Notice that the choice of $\tilde{A}$ corresponding to $(x, \hat{x}; A)$ is, of course, not unique. The proof of the following lemma is obvious and hence omitted.

Lemma 3.25 Let $\tilde{A}$ be a smooth local $\pi_{T^*M\otimes T\hat{M}}$-section and $\tilde{A}|_{(x,\hat{x})} = A$. Then, for any vector fields $\overline{X}, \overline{Y} \in \mathrm{VF}(M \times \hat{M})$ such that $\overline{X}|_{(x,\hat{x})} = (X, \hat{X})$, $\overline{Y}|_{(x,\hat{x})} = (Y, \hat{Y})$, one has

$$([\overline{\nabla}_{\overline{X}}, \overline{\nabla}_{\overline{Y}}]\tilde{A})|_{(x,\hat{x})} = -AR(X, Y) + \hat{R}(\hat{X}, \hat{Y})A + (\overline{\nabla}_{[\overline{X},\overline{Y}]}\tilde{A})|_{(x,\hat{x})}. \tag{21}$$

Here $[\overline{\nabla}_{\overline{X}}, \overline{\nabla}_{\overline{Y}}]$ is given by $\overline{\nabla}_{\overline{X}} \circ \overline{\nabla}_{\overline{Y}} - \overline{\nabla}_{\overline{Y}} \circ \overline{\nabla}_{\overline{X}}$ and is an $\mathbb{R}$-linear map on the set of local sections of $\pi_{T^*M\otimes T\hat{M}}$ around $(x, \hat{x})$.

We next define the actions of vectors $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q \in T|_q(T^*M \otimes T\hat{M})$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$, and $\nu(B)|_q \in V|_q(\pi_{T^*M\otimes T\hat{M}})$, $B \in T|^*_x M \otimes T|_{\hat{x}}\hat{M}$, on certain bundle maps instead of just functions (e.g. from $C^\infty(T^*M \otimes T\hat{M})$). Recall that if $\eta : E \to N$ is a vector bundle and $y \in N$, $u \in E|_y = \eta^{-1}(y)$, we have defined the isomorphism

$$\nu_\eta|_u : E|_y \to V|_u(\eta); \quad \nu_\eta|_u(v)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(u + tv), \quad \forall f \in C^\infty(E).$$

We normally omit the index η in ν_η, when it is clear from the context, and simply write ν instead of ν_η and it is sometimes more convenient to write $\nu(v)|_u$ for $\nu|_u(v)$. By using this we make the following definition.

Definition 3.26 Suppose B is a smooth manifold, $\eta : E \to N$ a vector bundle, $\tau : B \to N$ and $F : B \to E$ smooth maps such that $\eta \circ F = \tau$. Then, for $b \in B$ and $\mathcal{V} \in V|_b(\tau)$, we define the vertical derivative of F as

$$\mathcal{V}F := \nu|^{-1}_{F(b)}(F_*\mathcal{V}) \in E|_{\tau(b)}.$$

This is well defined since $F_*\mathcal{V} \in V|_{F(b)}(\eta)$. In this matter, we will show the following simple lemma that will be used later on.

Lemma 3.27 Let N be a smooth manifold, $\eta : E \to N$ a vector bundle, $\tau : B \to N$ a smooth map, $\mathcal{O} \subset B$ an immersed submanifold and $F : \mathcal{O} \to E$ a smooth map such that $\eta \circ F = \tau|_{\mathcal{O}}$.

(i) For every $b_0 \in \mathcal{O}$, there exists an open neighbourhood V of b_0 in $\mathcal{O}$, an open neighbourhood $\tilde{V}$ of b_0 in B s. t. $V \subset \tilde{V}$ and a smooth map $\tilde{F} : \tilde{V} \to E$ such that $\eta \circ \tilde{F} = \tau|_{\tilde{V}}$ and $\tilde{F}|_V = F|_V$. We call $\tilde{F}$ a local extension of F around b_0.

(ii) Suppose $\tau : B \to N$ is also a vector bundle and $\tilde{F}$ is any local extension of F around b_0 as in case (i). Then if $v \in B|_{\tau(b_0)}$ is such that $\nu|_{b_0}(v) \in T|_{b_0}\mathcal{O}$, one has

$$\nu|_{b_0}(v)(F) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \tilde{F}(b_0 + tv) \in E|_{\tau(b_0)},$$

where on the right hand side one views $t \mapsto \tilde{F}(b_0 + tv)$ as a map into a fixed (i.e. independent of t) vector space $E|_{F(b_0)}$ and the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is just the classical derivative of a vector valued map (and not a tangent vector).

Proof. (i) For a given $b_0 \in \mathcal{O}$, take a neighbourhood W of $y_0 := \tau(b_0)$ in N such that there exists a local frame $v_1, \dots, v_k$ of η defined on W (here $k = \dim E - \dim N$). Since $\eta \circ F = \tau|_{\mathcal{O}}$, it follows that

$$F(b) = \sum_{i=1}^{k} f_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tau^{-1}(W) \cap \mathcal{O},$$

for some smooth functions $f_i : \tau^{-1}(W) \cap \mathcal{O} \to \mathbb{R}$, $i = 1, \dots, k$. Now one can choose a small open neighbourhood V of b_0 in $\mathcal{O}$ and an open neigbourhood $\tilde{V}$ of b_0 in B such that $V \subset \tilde{V} \subset \tau^{-1}(W)$ and there exist smooth $\tilde{f}_1, \dots, \tilde{f}_k : \tilde{V} \to \mathbb{R}$ extending the functions $f_i|_V$ i.e. $\tilde{f}_i|_V = f_i|_V$, $i = 1, \dots, k$. To finish the proof of case (i), it suffices to define $\tilde{F} : \tilde{V} \to E$ by

$$\tilde{F}(b) = \sum_{i=1}^{k} \tilde{f}_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tilde{V}.$$

(ii) The fact that $t \mapsto \tilde{F}(b_0 + tv)$ is a map into a fixed vector space $E|_{F(b_0)}$ is clear since $\tilde{F}(b_0 + tv) \in E|_{\eta(\tilde{F}(b_0+tv))} = E|_{\tau(b_0+tv)} = E|_{\tau(b_0)}$. Since $F|_V = \tilde{F}|_V$ and $\nu|_{b_0}(v) \in T|_{b_0}V$, we have $F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v)$. Also, $t \mapsto b_0 + tv$ is a curve in $E|_{\tau(b_0)}$, and hence in E, whose tangent vector at $t = 0$ is exactly $\nu|_{b_0}(v)$. Hence

$$\nu|_{F(b_0)}(\nu|_{b_0}(v)F) = F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \tilde{F}(b_0 + tv).$$

Here on the rightmost side, the derivative $=: T$ is still viewed as a tangent vector of E at $\tilde{F}(b_0)$ i.e. $t \mapsto \tilde{F}(b_0 + tv)$ is thought of as a map into E. On the other hand, if one views $t \mapsto \tilde{F}(b_0 + tv)$ as a map into a fixed linear space $E|_{\tau(b_0)}$, its derivative $=: D$ at $t = 0$, as the usual derivative of vector valued maps, is just $D = \nu|_{F(b_0)}^{-1}(T)$. In the statement, it is exactly D whose expression we wrote as $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{F}(b_0 + tv)$. This completes the proof. $\square$

Remark 3.28 The advantage of the formula in case (ii) of the above lemma is that it simplifies in many cases the computations of τ-vertical derivatives because $t\mapsto\tilde{F}(b_0+tv)$ is a map from a real interval into a *fixed* vector space $E|_{F(b_0)}$ and hence we may use certain computational tools (e.g. Leibniz rule) coming from the ordinary vector calculus.

Let $\mathcal{O}$ be an immersed submanifold of $T^*M\otimes T\hat{M}$ and write $\pi_{\mathcal{O}}:=\pi_{T^*M\otimes T\hat{M}}|_{\mathcal{O}}$. Then if $\overline{T}:\mathcal{O}\to T^k_m(M\times\hat{M})$ with $\pi_{T^k_m(M\times\hat{M})}\circ\overline{T}=\pi_{\mathcal{O}}$ (i.e. $\overline{T}\in C^\infty(\pi_{\mathcal{O}},\pi_{T^k_m(M\times\hat{M})})$) and if $q=(x,\hat{x};A)\in\mathcal{O}$ and $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$ are such that $\mathscr{L}_{\rm NS}(\overline{X})|_q\in T|_q\mathcal{O}$, we next want to define what it means to take the derivative $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$. Our main interest will be the case where $k=0$, $m=1$ i.e. $T^k_m(M\times\hat{M})=T(M\times\hat{M})$, but some arguments below require this slightly more general setting.

First, for a moment, we take $\mathcal{O}=T^*M\otimes T\hat{M}$. Choose some local $\pi_{T^*M\otimes T\hat{M}}$-section $\tilde{A}$ defined on a neighbourhood of $(x,\hat{x})$ such that $\tilde{A}|_{(x,\hat{x})}=A$ and define

$$\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}:=\overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A}))-\nu(\overline{\nabla}_{\overline{X}}\tilde{A})|_q\overline{T}\in T^k_m|_{(x,\hat{x})}(M\times\hat{M}),\tag{22}$$

which is inspired by Eq. (16). Here as usual, $\tilde{T}(\tilde{A})=\tilde{T}\circ\tilde{A}$ is a locally defined (k,m)-tensor field on $M\times\hat{M}$. Note that this does not depend on the choice of $\tilde{A}$ since if $\overline{\omega}\in\Gamma(\pi_{T^m_k(M\times\hat{M})})$ and if we write $(\overline{T}\overline{\omega})(q):=\overline{T}(q)\overline{\omega}|_{(x,\hat{x})}$ as a full contraction for $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$, whence $\overline{T}\overline{\omega}\in C^\infty(T^*M\otimes T\hat{M})$, we may compute (where all the contractions are full)

$$\begin{aligned}(\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T})\overline{\omega}=&\big(\overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A}))\big)\overline{\omega}-\big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+t\overline{\nabla}_{\overline{X}}\tilde{A})\big)\overline{\omega}\\=&\overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A})\overline{\omega})-\overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big(\overline{T}(A+t\overline{\nabla}_{\overline{X}}\tilde{A})\overline{\omega}\big)\\=&\overline{\nabla}_{\overline{X}}((\overline{T}\overline{\omega}))(\tilde{A}))-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(\overline{T}\overline{\omega})(A+t\overline{\nabla}_{\overline{X}}\tilde{A})-\overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}\end{aligned}$$

i.e.

$$(\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T})\overline{\omega}=\mathscr{L}_{\rm NS}(\overline{X})|_q(\overline{T}\overline{\omega})-\overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega},\tag{23}$$

for all $\overline{\omega}\in\Gamma(\pi_{T^m_k(M\times\hat{M})})$ and where $\mathscr{L}_{\rm NS}(\overline{X})|_q$ on the right hand side acts as a tangent vector to a function $\overline{T}\overline{\omega}\in C^\infty(T^*M\otimes T\hat{M})$ as defined previously.

The right hand side is independent of any choice of local extension $\tilde{A}$ of A (i.e. $\tilde{A}|_{(x,\hat{x})}=A$), it follows that the definition of $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$ is independent of this choice as well. Now if $\mathcal{O}\subset T^*M\otimes T\hat{M}$ is just an immersed submanifold, we take the formula (23) as the definition of $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$.

Definition 3.29 Let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold and $q=(x,\hat{x};A)\in\mathcal{O}$, $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$ be such that $\mathscr{L}_{\rm NS}(\overline{X})|_q\in T|_q\mathcal{O}$. Then for $\overline{T}:\mathcal{O}\to T^k_m(M\times\hat{M})$ such that $\pi_{T^k_m(M\times\hat{M})}\circ\overline{T}=\pi_{\mathcal{O}}$, we define $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$ to be the unique element in $T^k_m|_{(x,\hat{x})}(M\times\hat{M})$ such that Eq. (23) holds for every $\overline{\omega}\in\Gamma(\pi_{T^m_k(M\times\hat{M})})$, and call it the derivative of $\overline{T}$ with respect to $\mathscr{L}_{\rm NS}(\overline{X})|_q$.

We now to provide the (unique) decomposition of any vector field of $T^*M\otimes T\hat{M}$ defined over $\mathcal{O}$ (not necessarily tangent to it) according to the decomposition $T(T^*M\otimes T\hat{M})=\mathcal{D}_{\rm NS}\oplus V(\pi_{T^*M\otimes T\hat{M}})$.

Proposition 3.30 Let $\mathcal{X} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(T^*M\otimes T\hat{M})})$ be a smooth bundle map (i.e. a vector field of $T^*M \otimes T\hat{M}$ along $\mathcal{O}$) where $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is a smooth immersed submanifold. Then there are unique smooth bundle maps $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, $U \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ such that

$$\mathcal{X}|_q = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q + \nu(U(q))|_q, \quad q \in \mathcal{O}. \tag{24}$$

Proof. First of all, there are unique smooth vector fields

$$\mathcal{X}^h, \mathcal{X}^v \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(T^*M\otimes T\hat{M})}),$$

of $T^*M \otimes T\hat{M}$ along $\mathcal{O}$ such that

$$\mathcal{X}^h|_q \in \mathcal{D}_{\mathrm{NS}}|_q,\ \mathcal{X}^v|_q \in V|_q(\pi_{T^*M\otimes T\hat{M}}),$$

for all $q \in \mathcal{O}$ and $\mathcal{X} = \mathcal{X}^h + \mathcal{X}^v$. Then, we define

$$\overline{T}(q) = (\pi_{T^*M\otimes T\hat{M}})_*\mathcal{X}^h|_q, \quad U(q) = \nu|_q^{-1}(\mathcal{X}^v|_q),$$

where $q = (x, \hat{x}; A) \in \mathcal{O}$ and $\nu|_q$ is the isomorphism

$$T^*|_x M \otimes T|_{\hat{x}}\hat{M} \to V|_q(\pi_{T^*M\otimes T\hat{M}}); \quad B \mapsto \nu(B)|_q.$$

This clearly proves the claims. □

Remark 3.31 The previous results shows that to know how to compute the Lie brackets of two vector fields $\mathcal{X}, \mathcal{Y} \in \mathrm{VF}(\mathcal{O})$ where $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is an immersed submanifold (e.g. $\mathcal{O} = Q$), one needs, in practice, just to know how to compute the Lie brackets between vectors fields of the form $q \mapsto \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q, \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))$ and $q \mapsto \nu(U(q))|_q, \nu(V(q))|_q$ where $\mathcal{X}|_q = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q + \nu(U(q))|_q$ and $\mathcal{Y}|_q = \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q + \nu(V(q))|_q$ as above.

Remark 3.32 Notice that if $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$, $\mathcal{X} \in T|_q\mathcal{O}$ and $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$, then we may define the derivative $\mathcal{X}\overline{T} \in T^k_m(M \times \hat{M})$ by decomposing $\mathcal{X} = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q + \nu(U)|_q$ for the unique $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $U \in (T^*M \otimes T\hat{M})|_{(x,\hat{x})}$.

We finish this subsection with some obvious but useful rules of calculation, that will be useful in the computations of Lie brackets on $\mathcal{O} \subset T^*M \otimes T\hat{M}$ and we will make use of them especially in section 5.

Lemma 3.33 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$, $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$, $F \in C^\infty(\mathcal{O})$, $h \in C^\infty(\mathbb{R})$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ such that $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q \in T|_q\mathcal{O}$ and finally $U \in (T^*M \times T\hat{M})|_{(x,\hat{x})}$ such that $\nu(U)|_q \in T|_q\mathcal{O}$ Then

(i) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(F\overline{T}) = (\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q F)\overline{T}(q) + F(q)\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T}$,

(ii) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(h \circ F) = h'(F(q))\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q F$,

(iii) $\nu(U)|_q(F\overline{T}) = (\nu(U)|_q F)\overline{T}(q) + F(q)\nu(U)|_q\overline{T}$,

(iv) $\nu(U)|_q(h\circ F) = h'(F(q))\nu(U)|_q F$.

If $T:\mathcal{O}\to TM\subset T(M\times\hat{M})$ such that $T(q)\in T|_x M$ for all $q=(x,\hat{x};A)\in\mathcal{O}$ and one writes (see Remark 3.34 below)

$$(\cdot)T(\cdot):\mathcal{O}\to T\hat{M}\subset T(M\times\hat{M});\quad q=(x,\hat{x};A)\mapsto AT(q),$$

then

(v) $\mathscr{L}_{\rm NS}(\overline{X})|_q\big((\cdot)T(\cdot)\big) = A\mathscr{L}_{\rm NS}(\overline{X})|_q T\in T|_{\hat{x}}\hat{M}$,

(vi) $\nu(U)|_q\big((\cdot)T(\cdot)\big) = UT(q)+A\nu(U)|_q T\in T|_{\hat{x}}\hat{M}$,

where $\mathscr{L}_{\rm NS}(\overline{X})|_q T, \nu(U)|_q T\in T|_x M$. Finally, if $Y\in \mathrm{VF}(M)$ is considered as a map $\mathcal{O}\to TM$; $(x',\hat{x}';A')\mapsto Y|_{x'}$ and if we write $\overline{X}=(X,\hat{X})\in T|_x M\oplus T|_{\hat{x}}\hat{M}$, then

(vii) $\mathscr{L}_{\rm NS}(\overline{X})|_q Y = \nabla_X Y$.

Remark 3.34 In the cases (v) and (vii) we think of $T:\mathcal{O}\to TM$, to adapt to our previous notations, as a map $T:\mathcal{O}\to \mathrm{pr}_1^*(TM)$ where $\mathrm{pr}_1: M\times\hat{M}\to M$ is the projection onto the first factor. Here $\mathrm{pr}_1^*(\pi_{TM})$ is a vector subbundle of $\pi_{T(M\times\hat{M})}$ which we wrote, slightly imprecisely, as $TM\subset T(M\times\hat{M})$ in the statement of the proposition. Thus $T(q')\in T|_{x'}M$ for all $q'=(x',\hat{x}';A')\in\mathcal{O}$ just means that $\mathrm{pr}_1^*(\pi_{TM})\circ T=\pi_{\mathcal{O}}$.

Proof. Items (i)-(iv) are immediate to derive. We next turn to an argument for the others. We take a small open neighbourhood V of q in $\mathcal{O}$, a small open neighbourhood $\tilde{V}$ of q in $T^*M\otimes T\hat{M}$ such that $V\subset\tilde{V}$ a smooth $\tilde{T}:\tilde{V}\to TM$ such that $\tilde{T}|_V=T|_V$ and $\tilde{T}(q')\in T|_{x'}M$ for all $q'=(x',\hat{x}';A')\in\tilde{V}$. Such an extension $\tilde{T}$ of T is provided by Lemma 3.27 by taking $b_0=q$, $\tau=\pi_{T^*M\otimes T\hat{M}}$, $\eta=\mathrm{pr}_1^*(\pi_{TM})$ with $\mathrm{pr}_1: M\times\hat{M}=M$ the projection onto the first factor (see also Remark 3.34 above). Then taking $t\mapsto\Gamma(t)=(\gamma(t),\hat{\gamma}(t);A(t))$ to be any curve in $\mathcal{O}$ with $\Gamma(0)=q$, $\dot{\Gamma}(0)=\mathscr{L}_{\rm NS}(\overline{X})|_q$, we have

$$\begin{aligned}
&\mathscr{L}_{\rm NS}(\overline{X})|_q((\cdot)T(\cdot)) = \mathscr{L}_{\rm NS}(\overline{X})|_q((\cdot)\tilde{T}(\cdot))\\
=&\overline{\nabla}_{\overline{X}}(A(\cdot)\tilde{T}(A(\cdot))) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0(A+t\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(A+t\overline{\nabla}_{\overline{X}}A(\cdot))\\
=&(\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(q)+A\overline{\nabla}_{\overline{X}}(\tilde{T}(A(\cdot))) - (\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(q) - A\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\tilde{T}(A+t\overline{\nabla}_{\overline{X}}A(\cdot))\\
=&A\mathscr{L}_{\rm NS}(\overline{X})|_q\tilde{T} = A\mathscr{L}_{\rm NS}(\overline{X})|_q T,
\end{aligned}$$

where the first and the last steps follow from the facts that $((\cdot)\tilde{T}(\cdot))|_V=((\cdot)T(\cdot))|_V$ and $\tilde{T}|_V=T|_V$. This gives (v).

To prove (vi) we compute

$$\begin{aligned}
&\nu(U)|_q\big((\cdot)T(\cdot)\big)=\nu(U)|_q\big((\cdot)\tilde{T}(\cdot)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(A+tU)\tilde{T}(A+tU)\\
=&\big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(A+tU)\big)\tilde{T}(q)+A\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\tilde{T}(A+tU)=UT(q)+A\nu(U)|_q\tilde{T}\\
=&UT(q)+A\nu(U)|_q T.
\end{aligned}$$

Finally, we prove (vii). Suppose that $Y \in \mathrm{VF}(M)$. Then the map $\mathcal{O} \to TM;\ (x', \hat{x}'; A') \mapsto Y|_{x'}$ is nothing more than $Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}$ where $\mathrm{pr}_1 : M \times \hat{M} \to M$ is the projection onto the first factor. Take a local $\pi_{T^*M\otimes T\hat{M}}$-section $\tilde{A}$ with $\tilde{A}|_{(x,\hat{x})} = A$. Then since $Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}} = Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}}|_{\mathcal{O}}$, we have

$$\begin{aligned}&\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}) = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}})\\ =&\overline{\nabla}_{(X,\hat{X})}(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}} \circ \tilde{A}) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}})(A + t\overline{\nabla}_{\overline{X}}\tilde{A}).\end{aligned}$$

But $(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}} \circ \tilde{A})|_{(x',\hat{x}')} = Y|_{x'} = (Y, 0)|_{(x,\hat{x})}$ for all $(x', \hat{x}')$ and $(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}})(A + t\overline{\nabla}_{\overline{X}}\tilde{A}) = Y|_x$ for all t and hence

$$\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}) = \overline{\nabla}_{(X,\hat{X})}(Y, 0) - 0 = \nabla_X Y.$$

□

3.3.2 Computation of Lie brackets

We now embark into the computation of Lie brackets.

Proposition 3.35 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{T} = (T, \hat{T}), \overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$ with $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q, \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q \in T|_q\mathcal{O}$ for all $q = (x, \hat{x}; A) \in \mathcal{O}$. Then, for every $q \in \mathcal{O}$, one has

$$\begin{aligned}[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)), \mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q =&\mathscr{L}_{\mathrm{NS}}\big(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\overline{T}\big)\big|_q\\ &+ \nu\big(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A\big)\big|_q,\end{aligned} \tag{25}$$

with both sides tangent to $\mathcal{O}$.

Proof. We will deal first with the case where $\mathcal{O}$ is an open subset of $T^*M \otimes T\hat{M}$. Take a local $\pi_{T^*M\otimes T\hat{M}}]$ section $\tilde{A}$ around $(x, \hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$; see Lemma 3.24.

Let $f \in C^\infty(T^*M \otimes T\hat{M})$. By using the definition of $\mathscr{L}_{\mathrm{NS}}$ and ν, one obtains

$$\begin{aligned}&\mathscr{L}_{\mathrm{NS}}(\overline{T}(A))|_q(\mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot)(f)))\\ =&\ \overline{T}(A)(\mathscr{L}_{\mathrm{NS}}(\overline{S}(\tilde{A}))|_{\tilde{A}}(f)) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \mathscr{L}_{\mathrm{NS}}(\overline{S}(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f)\\ =&\ \overline{T}(A)\big(\overline{S}(\tilde{A})(f(\tilde{A})) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\tilde{A} + t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\\ &- \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A})(f(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\ &+ \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A} + s\overline{\nabla}_{\overline{S}(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A})}(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big).\end{aligned}$$

We use the fact that $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M\times\hat{M})$ and $\frac{\partial}{\partial t}$ and $\overline{T}(\tilde{A})$ commute (as the obvious vector fields on $M \times \hat{M} \times \mathbb{R}$ with points $(x, \hat{x}, t)$) to write the last expression in the form

$$\begin{aligned}&\overline{T}(A)(\overline{S}(\tilde{A})(f(\tilde{A}))) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{T}(A)(f(\tilde{A} + t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A)(f(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\ &+ \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big).\end{aligned}$$

By interchanging the roles of $\overline{T}$ and $\overline{S}$ and using the definition of commutator of vector fields, we get from this

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)),\mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q(f)\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_q(f(\tilde{A}))+\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big)\\
&-\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+st\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\big)\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_q(f(\tilde{A}))+\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(t\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q(f)\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(t\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q(f)\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_q(f(\tilde{A}))+\nu(\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q(f)-\nu(\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q(f)\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_q(f(\tilde{A}))-\nu([\overline{\nabla}_{\overline{T}(\tilde{A})},\overline{\nabla}_{\overline{S}(\tilde{A})}]\tilde{A}))|_q(f).
\end{aligned}$$

Using Lemma 3.25, we get that the last line is equal to

$$\begin{aligned}
&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}(f(\tilde{A}))\\
&-\nu(\overline{\nabla}_{[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}}\tilde{A}-AR(T(A),S(A))+\hat{R}(\hat{T}(A),\hat{S}(A))A)|_q(f),
\end{aligned}$$

from which, by using the definition of $\mathscr{L}_{\mathrm{NS}}$, linearity of $\nu(\cdot)|_q$ and arbitrariness of $f\in C^\infty(T^*M\otimes T\hat{M})$, we get

$$\begin{aligned}
[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)),\mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q=&\mathscr{L}_{\mathrm{NS}}([\overline{T}(\tilde{A}),\overline{S}(\tilde{A})])|_q\\
&+\nu(AR(T(A),S(A))-\hat{R}(\hat{T}(A),\hat{S}(A))A)|_q.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{S}(A+t\underbrace{\overline{\nabla}_{\overline{T}(q)}\tilde{A}}_{=0})=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{S}(A)=0,\\
\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+t\underbrace{\overline{\nabla}_{\overline{S}(q)}\tilde{A}}_{=0})=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A)=0,
\end{aligned}$$

since $\overline{T}(q),\overline{S}(q)\in T|_{(x,\hat{x})}(M\times\hat{M})$ and hence by Eq. (22),

$$[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]=\overline{\nabla}_{\overline{T}(q)}(\overline{S}(\tilde{A}))-\overline{\nabla}_{\overline{S}(q)}(\overline{T}(\tilde{A}))=\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S}-\mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\overline{T}.$$

The claim thus holds in this case (i.e. when $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$). We let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ to be an immersed submanifold and $\overline{T},\overline{S}:\mathcal{O}\to T(M\times\hat{M})$ are such that, for all $q=(x,\hat{x};A)\in\mathcal{O}$, $\overline{T}(x,\hat{x};A)$, $\overline{S}(x,\hat{x};A)$ belong to $T|_{(x,\hat{x})}(M\times\hat{M})$ and $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q$, $\mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q$ belong to $T|_q\mathcal{O}$. For a fixed $q=(x,\hat{x};A)\in\mathcal{O}$, we may, thanks to Lemma 3.27 (taking $\tau=\pi_{T^*M\otimes T\hat{M}}$, $\eta=\pi_{T(M\times\hat{M})}$, $b_0=q$ and $F=\overline{T}$ or $F=\overline{S}$ there) take a small open neighbourhood V of q in $\mathcal{O}$, a neighbourhood $\tilde{V}$ of q in Q such that $V\subset\tilde{V}$ and some extensions $\tilde{\overline{T}},\tilde{\overline{S}}:\tilde{V}\to T(M\times\hat{M})$ of $\overline{T}|_V,\overline{S}|_V$ with $\tilde{\overline{T}}(x',\hat{x}';A'),\tilde{\overline{S}}(x',\hat{x}';A')\in T|_{(x',\hat{x}')}(M\times\hat{M})$ for all $(x',\hat{x}';A')\in\tilde{V}$. Then since

$\mathscr{L}_{\rm NS}(\overline{T}(\cdot))|_V = \mathscr{L}_{\rm NS}(\tilde{\overline{T}}(\cdot))|_V$, $\mathscr{L}_{\rm NS}(\overline{S}(\cdot))|_V = \mathscr{L}_{\rm NS}(\tilde{\overline{S}}(\cdot))|_V$, we compute, because of what has been shown already,

$$\begin{aligned}
&[\mathscr{L}_{\rm NS}(\overline{T}), \mathscr{L}_{\rm NS}(\overline{S})]|_q = [\mathscr{L}_{\rm NS}(\tilde{\overline{T}})|_V, \mathscr{L}_{\rm NS}(\tilde{\overline{S}})|_V]|_q = ([\mathscr{L}_{\rm NS}(\tilde{\overline{T}}), \mathscr{L}_{\rm NS}(\tilde{\overline{S}})]|_V)|_q \\
=&\mathscr{L}_{\rm NS}\big(\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\tilde{\overline{S}} - \mathscr{L}_{\rm NS}(\overline{S}(q))|_q\tilde{\overline{T}}\big)\big|_q + \nu\big(AR(T(q),S(q)) - \hat{R}(\hat{T}(q),\hat{S}(q))A\big)\big|_q,
\end{aligned}$$

where in the last line we used that $\tilde{\overline{T}}(q) = \overline{T}(q) = (T(q),\hat{T}(q))$, $\tilde{\overline{S}}(q) = \overline{S}(q) = (S(q),\hat{S}(q))$. Take any $\overline{\omega} \in \Gamma(\pi_{T^m_k(M\times\hat{M})})$. Since $\mathscr{L}_{\rm NS}(\overline{T}(q))|_q \in T|_q\mathcal{O} = T|_qV$ by assumption and since $(\overline{S}\overline{\omega})|_V = (\tilde{\overline{S}}\overline{\omega})|_V$, we have $\mathscr{L}_{\rm NS}(\overline{T}(q))|_q(\overline{S}\overline{\omega}) = \mathscr{L}_{\rm NS}(\overline{T}(q))|_q(\tilde{\overline{S}}\overline{\omega})|_V$. But then Eq. (23) i.e. the definition of $\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\overline{S}$ implies that

$$\begin{aligned}
&(\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\overline{S})\overline{\omega} = \mathscr{L}_{\rm NS}(\overline{T}(q))|_q(\overline{S}\overline{\omega}) - \overline{S}(q)\overline{\nabla}_{\overline{T}(q)}\overline{\omega} \\
=&\mathscr{L}_{\rm NS}(\overline{T}(q))|_q(\tilde{\overline{S}}\overline{\omega}) - \tilde{\overline{S}}(q)\overline{\nabla}_{\overline{T}(q)}\overline{\omega} = (\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\tilde{\overline{S}})\overline{\omega}
\end{aligned}$$

i.e. $\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\overline{S} = \mathscr{L}_{\rm NS}(\overline{T}(q))|_q\tilde{\overline{S}}$ and similarly $\mathscr{L}_{\rm NS}(\overline{S}(q))|_q\overline{T} = \mathscr{L}_{\rm NS}(\overline{S}(q))|_q\tilde{\overline{T}}$. This shows that on $\mathcal{O}$ we have the formula

$$\begin{aligned}
[\mathscr{L}_{\rm NS}(\overline{T}), \mathscr{L}_{\rm NS}(\overline{S})]|_q =&\mathscr{L}_{\rm NS}\big(\mathscr{L}_{\rm NS}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{\rm NS}(\overline{S}(q))|_q\overline{T}\big)\big|_q \\
&+ \nu\big(AR(T(q),S(q)) - \hat{R}(\hat{T}(q),\hat{S}(q))A\big)\big|_q,
\end{aligned}$$

where both sides belong to $T|_q\mathcal{O}$ (since the left hand side obviously belongs to $T|_q\mathcal{O}$). □

Proposition 3.36 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{T} = (T,\hat{T}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, $U \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ be such that, for all $q = (x,\hat{x};A) \in \mathcal{O}$,

$$\mathscr{L}_{\rm NS}(\overline{T}(q))|_q \in T|_q\mathcal{O}, \quad \nu(U(q))|_q \in T|_q\mathcal{O}.$$

Then

$$[\mathscr{L}_{\rm NS}(\overline{T}(\cdot)), \nu(U(\cdot))]|_q = -\mathscr{L}_{\rm NS}(\nu(U(q))|_q\overline{T})|_q + \nu(\mathscr{L}_{\rm NS}(\overline{T}(q))|_qU)|_q,$$

with both sides tangent to $\mathcal{O}$.

Proof. As in the proof of Proposition 3.35, we will deal first with the case where $\mathcal{O}$ is an open subset of $T^*M \otimes T\hat{M}$. Take a local $\pi_{T^*M\otimes T\hat{M}]}$ section $\tilde{A}$ around $(x,\hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$; see Lemma 3.24. In some expressions we will write $q = A$ for clarity.

Let $f \in C^\infty(T^*M \otimes T\hat{M})$. Then $\mathscr{L}_{\rm NS}(\overline{T}(A))|_q\big(\nu(U(\cdot))(f)\big)$ is equal to

$$\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(U(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))\big|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f),$$

which is equal to $\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big)$ once we recall that $\overline{\nabla}_{\overline{T}(A)}\tilde{A} = 0$. In addition, one has

$$\begin{aligned}
&\nu(U(A))|_q\big(\mathscr{L}_{\rm NS}(\overline{T}(\cdot))(f)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\mathscr{L}_{\rm NS}(\overline{T}(A + tU(A))\big|_{A+tU(A)}(f) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A + tU(A))\big(f(\tilde{A} + tU(\tilde{A}))\big) \\
&- \frac{\partial^2}{\partial s\partial t}\big|_0 f\big(A + tU(A) + s\overline{\nabla}_{\overline{T}(A+tU(A))}(\tilde{A} + tU(\tilde{A}))\big) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A + tU(A))\big(f(\tilde{A} + tU(\tilde{A}))\big) - \frac{\partial^2}{\partial s\partial t}\big|_0 f\big(A + tU(A) + st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big),
\end{aligned}$$

since $\overline{\nabla}_{\overline{T}(A+tU(A))}\tilde{A}=0$. We next simplify the first term on the last line to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+tU(A))\big(f(\tilde{A}+tU(\tilde{A}))\big)\\ =&(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big)+\overline{T}(A)\big(\nu(U(\tilde{A}))|_{\tilde{A}}(f)\big)\end{aligned}$$

and then, for the second term, one obtains

$$\begin{aligned}&\frac{\partial^2}{\partial s\partial t}\big|_0 f\big(A+tU(A)+st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big)\\ =&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0 f_*|_q\nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big(tU(A)+st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big)\Big)\Big|_q\\ =&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0 f_*|_q\nu\big(U(A)+s\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q\\ =&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0\Big(f_*|_q\nu(U(A))|_q+sf_*|_q\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q\Big)\\ =&f_*\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q=\nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_q f.\end{aligned}$$

Therefore one deduces

$$\begin{aligned}&[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)),\nu(U(\cdot))]|_q(f)=-(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big)+\nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_q f\\ =&-\tilde{A}_*(\nu(U(A))|_q\overline{T})(f)+\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f)\\ =&-\mathscr{L}_{\mathrm{NS}}(\nu(U(A))|_q\overline{T})|_q(f)+\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f),\end{aligned}$$

where the last line follows from the definition of $\mathscr{L}_{\mathrm{NS}}$ and the fact that $\overline{\nabla}_{\nu(U(A))|_q\overline{T}}\tilde{A}=0$. Finally, Eq. (22) implies

$$\overline{\nabla}_{T(q)}(U(\tilde{A}))=\overline{\nabla}_{T(q)}(U(\tilde{A}))-\nu(\underbrace{\overline{\nabla}_{T(q)}\tilde{A}}_{=0})|_qU=\mathscr{L}_{\mathrm{NS}}(\overline{T}(A))|_qU.$$

Thus the claimed formula holds in the special case where $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$. More generally, let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold, and $\overline{T}=(T,\hat{T}):\mathcal{O}\to T(M\times\hat{M})=TM\times T\hat{M}$, $U:\mathcal{O}\to T^*M\times T\hat{M}$ as in the statement of this proposition.

For a fixed $q=(x,\hat{x};A)\in\mathcal{O}$, Lemma 3.27 implies the existence of a neighbourhood V of q in $\mathcal{O}$, a neighbourhood $\tilde{V}$ of q in $T^*M\otimes T\hat{M}$ and smooth $\tilde{\overline{T}}:\tilde{V}\to T(M\times\hat{M})$, $\tilde{U}:\tilde{V}\to T^*M\otimes T\hat{M}$ such that $\tilde{\overline{T}}(x,\hat{x};A)\in T|_{(x,\hat{x})}(M\times\hat{M})$, $\tilde{U}(x,\hat{x};A)\in(T^*M\otimes T\hat{M})|_{(x,\hat{x})}$ and $\tilde{\overline{T}}|_V=\overline{T}|_V$, $\tilde{U}|_V=U|_V$ (for the case of an extension $\tilde{U}$ of U, take in Lemma 3.27, $\tau=\pi_{T^*M\otimes T\hat{M}}$, $\eta=\pi_{T_1^1(M\times\hat{M})}$, $F=U$, $b_0=q$). In the same way as in the proof of Proposition 3.35, we have $[\mathscr{L}_{\mathrm{NS}}(\overline{T}),\nu(U)]|_q=[\mathscr{L}_{\mathrm{NS}}(\tilde{\overline{T}}),\nu(\tilde{U})]_q$ and $\mathscr{L}_{\mathrm{NS}}(\tilde{\overline{T}}(q))|_q\tilde{U}=\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_qU$. Hence by what was already shown above,

$$[\mathscr{L}_{\mathrm{NS}}(\overline{T}),\nu(U)]|_q=-\mathscr{L}_{\mathrm{NS}}(\nu(U(q))|_q\tilde{\overline{T}})|_q+\nu(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_qU)|_q.$$

We are left to show that $\nu(U(q))|_q\tilde{\overline{T}}=\nu(U(q))|_q\overline{T}$ and for that, it suffices to show that $\nu(\nu(U(q))|_q\tilde{\overline{T}})|_{\overline{T}(q)}=\nu(\nu(U(q))|_q\overline{T})|_{\overline{T}(q)}$: if $f\in C^\infty(T(M\times\hat{M}))$, then

$$\begin{aligned}\nu(\nu(U(q))|_q\tilde{\overline{T}})|_{\overline{T}(q)}f=&\big(\tilde{\overline{T}}_*\nu(U(q))|_q\big)f=\nu(U(q))|_q(f\circ\tilde{\overline{T}})=\nu(U(q))|_q(f\circ\overline{T})\\ =&\big(\overline{T}_*\nu(U(q))|_q\big)f=\nu(\nu(U(q))|_q\overline{T})|_{\overline{T}(q)}f,\end{aligned}$$

where at the 3rd equality we used the fact that $(f\circ\tilde{\overline{T}})|_V=(f\circ\overline{T})|_V$ and $\nu(U(q))|_q\in T|_q\mathcal{O}=T|_qV$. This completes the proof.

□

Finally, we derive a formula for the commutators of two vertical vector fields.

Proposition 3.37 Let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold and $U,V\in C^\infty(\pi_{\mathcal{O}},\pi_{T^*M\otimes T\hat{M}})$ be such that $\nu(U(q))|_q,\nu(V(q))|_q\in T|_q\mathcal{O}$ for all $q\in\mathcal{O}$. Then

$$[\nu(U(\cdot)),\nu(V(\cdot))]|_q=\nu\big(\nu(U(q))|_qV-\nu(V(q))|_qU\big)|_q. \tag{26}$$

Proof. Again we begin with the case where $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$ and write $q=(x,\hat{x};A)\in\mathcal{O}$ simply as A. Let $f\in C^\infty(T^*M\otimes T\hat{M})$. Then,

$$\begin{aligned}
&\nu(U(A))|_q\big(\nu(V(\cdot))(f)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0\nu(V(A+tU(A))|_{A+tU(A)}(f)\\
=&\frac{\partial^2}{\partial t\partial s}\Big|_0 f(A+tU(A)+sV(A+tU(A)))\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 f_*|_q\nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0\big(tU(A)+sV(A+tU(A))\big)\Big)\Big|_q\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 f_*\nu\big(U(A)+s\nu(U(A))|_qV\big)|_q\\
=&f_*\nu\big(\nu(U(A))|_qV\big)|_q=\nu(\nu(U(A))|_qV)|_qf.
\end{aligned}$$

from which the result follows in the case that $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$. The case where $\mathcal{O}$ is only an immersed submanifold of $T^*M\otimes T\hat{M}$ can be treated by using Lemma 3.27 in the same way as in the proofs of Propositions 3.35, 3.36.

□

As a corollary to the previous three propositions, we have the following, whose proof is immediate.

Corollary 3.38 Let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold and $\mathcal{X},\mathcal{Y}\in\mathrm{VF}(\mathcal{O})$. Letting for $q\in\mathcal{O}$,

$$\mathcal{X}|_q=\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q+\nu(U(q))|_q,\quad \mathcal{Y}|_q=\mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q+\nu(V(q))|_q,$$

to be the unique decompositions given by Proposition 3.30. Writing $\overline{T}=(T,\hat{T})$, $\overline{S}=(S,\hat{S})$ corresponding to $T(M\times\hat{M})=TM\times T\hat{M}$, we get

$$\begin{aligned}
[\mathcal{X},\mathcal{Y}]|_q=&\big(\mathscr{L}_{\mathrm{NS}}(\mathcal{X}|_q\overline{S})|_q+\nu(\mathcal{X}|_qV)|_q\big)-\big(\mathscr{L}_{\mathrm{NS}}(\mathcal{Y}|_q\overline{T})|_q+\nu(\mathcal{Y}|_qU)|_q\big)\\
&+\nu\big(AR(T(q),S(q))-\hat{R}(\hat{T}(q),\hat{S}(q))A\big)|_q
\end{aligned}$$

(for the notation, see the second remark after Proposition 3.30).

4 Study of the Rolling problem (R)

In this section, we investigate the rolling problem as a control system $(\Sigma)_R$ associated to the distribution $\mathcal{D}_{\mathrm{R}}$.

4.1 Global properties of a $\mathcal{D}_{\mathrm{R}}$-orbit

We begin with the following remark.

Remark 4.1 Notice that the map $\pi_{Q,M} : Q \to M$ is in fact a bundle. Indeed, let $F = (X_i)_{i=1}^n$ be a local oriented orthonormal frame of M defined on an open set U. Then the local trivialization of $\pi_{Q,M}$ induced by F is

$$\tau_F : \pi_{Q,M}^{-1}(U) \to U \times F_{\mathrm{OON}}(\hat{M}); \quad \tau_F(x, \hat{x}; A) = (x, (AX_i|_x)_{i=1}^n),$$

is a diffeomorphism. Note also that since $\pi_{Q,M}$-fibers are diffeomorphic to $F_{\mathrm{OON}}(\hat{M})$, in order that there would be a principal G-bundle structure for $\pi_{Q,M}$, it is necessary that $F_{\mathrm{OON}}(\hat{M})$ is diffeomorphic to the Lie-group G.

From Proposition 3.20, we deduce that each $\mathcal{D}_{\mathrm{R}}$-orbit is a smooth bundle over M. This is given in the next proposition.

Proposition 4.2 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and suppose that $\hat{M}$ is complete. Then

$$\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} := \pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M,$$

is a smooth subbundle of $\pi_{Q,M}$.

One defines similarly $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}} := \pi_{Q,\hat{M}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to \hat{M}$.

Proof. First, surjectivity of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ follows from completeness of $\hat{M}$ by using Proposition 3.18. Since $\mathcal{D}_{\mathrm{R}}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for every $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and $(\pi_{Q,M})_*$ maps $\mathcal{D}_{\mathrm{R}}|_q$ isomorphically onto $T|_{\pi_{Q,M}(q)}M$, one immediately deduces that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is also a submersion. This implies that each fiber $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x)$, $x \in M$, is a smooth closed submanifold of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Choose next, for each $x \in M$, an open convex $U_x \subset T|_xM$ such that $\exp_x|_{U_x}$ is a diffeomorphism onto its image and $0 \in U$. Define

$$\begin{aligned} &\tau_x : \pi_{Q,M}^{-1}(U_x) \to U_x \times \pi_{Q,M}^{-1}(x), \\ &q = (y, \hat{y}; A) \mapsto \big(y, \big(x, \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma_{y,x}, q)(1); A_{\mathcal{D}_{\mathrm{R}}}(\gamma_{y,x}, q)(1)\big)\big), \end{aligned}$$

where $\gamma_{y,x} : [0, 1] \to M$; $\gamma_{y,x}(t) = \exp_x((1-t)\exp_x^{-1}(y))$ is a geodesic from y to x. It is obvious that τ_x is a smooth bijection. Moreover, restricting τ_x to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ clearly gives a smooth bijection

$$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(U_x) \to U_x \times (\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x)).$$

The inverse of τ_x, $\tau_x^{-1} : U_x \times \pi_{Q,M}^{-1}(x) \to \pi_{Q,M}^{-1}(U_x)$ is constructed with a formula similar to that of τ_x and is seen, in the same way, to be smooth. This inverse restricted to $U_x \times (\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x))$ maps bijectively onto $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(U_x)$ and thus τ_x is a smooth local trivialization of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This completes the proof.

□

Remark 4.3 In the case where $\hat{M}$ is not complete, the result of Proposition 4.2 remains valid if we just claim that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a bundle over its image $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, which is an open connected subset of M.

Write $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$. Then using the diffeomorphism $\iota : Q := Q(M,\hat{M}) \to \hat{Q} := Q(\hat{M}, M)$; $(x,\hat{x};A) \mapsto (\hat{x}, x; A^{-1})$ (Proposition 3.22) one gets

$$\begin{aligned}\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),\hat{M}} =&\pi_{Q,\hat{M}}|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)} = \pi_{Q,\hat{M}} \circ \iota^{-1}|_{\mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q_0))} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)} \\ =&\pi_{\hat{Q},\hat{M}}|_{\mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q_0))} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)} = \pi_{\mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q_0)),\hat{M}} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)},\end{aligned}$$

from which we see that $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),\hat{M}}$ is also a bundle over its image $\hat{M}^\circ$ since $\iota|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\rm R}}(q_0) \to \mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q_0))$ is a diffeomorphism and since by the previous proposition and the above remark $\pi_{\mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q_0)),\hat{M}}$ is a bundle over its image, which necessarily is $\hat{M}^\circ$. Notice also that if M is complete, then $\hat{M}^\circ = \hat{M}$.

The next proposition can be useful in case one of the manifolds has a large group of isometries. We do not provide an argument for this proposition since it is immediate.

Proposition 4.4 For any Riemannian isometries $F \in \mathrm{Iso}(M,g)$ and $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$ of (M,g), $(\hat{M},\hat{g})$ respectively, one defines smooth free right and left actions of $\mathrm{Iso}(M,g)$, $\mathrm{Iso}(\hat{M},\hat{g})$ on Q by

$$q_0 \cdot F := (F^{-1}(x_0), \hat{x}_0; A_0 \circ F_*|_{F^{-1}(x_0)}), \quad \hat{F} \cdot q_0 := (x_0, \hat{F}(\hat{x}_0); \hat{F}_*|_{\hat{x}_0} \circ A_0),$$

where $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. We also set

$$\hat{F} \cdot q_0 \cdot F := (\hat{F} \cdot q_0) \cdot F = \hat{F} \cdot (q_0 \cdot F).$$

Then for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, a.c. $\gamma : [0,1] \to M$, $\gamma(0) = x_0$, and $F \in \mathrm{Iso}(M,g)$, $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$, one has

$$\hat{F} \cdot q_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t) \cdot F = q_{\mathcal{D}_{\rm R}}(F^{-1} \circ \gamma, \hat{F} \cdot q_0 \cdot F)(t), \tag{27}$$

for all $t \in [0,1]$. In particular,

$$\hat{F} \cdot \mathcal{O}_{\mathcal{D}_{\rm R}}(q_0) \cdot F = \mathcal{O}_{\mathcal{D}_{\rm R}}(\hat{F} \cdot q_0 \cdot F).$$

We derive the following consequence.

Corollary 4.5 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\gamma, \omega : [0,1] \to M$ be absolutely continuous such that $\gamma(0) = \omega(0) = x_0$, $\gamma(1) = \omega(1)$. Then assuming that $q_{\mathcal{D}_{\rm R}}(\gamma, q_0)$, $q_{\mathcal{D}_{\rm R}}(\omega, q_0)$, $q_{\mathcal{D}_{\rm R}}(\omega^{-1}.\gamma, q_0)$ exist and if there exists $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$ such that

$$\hat{F} \cdot q_0 = q_{\mathcal{D}_{\rm R}}(\omega^{-1}.\gamma, q_0)(1),$$

then

$$\hat{F} \cdot q_{\mathcal{D}_{\rm R}}(\omega, q_0)(1) = q_{\mathcal{D}_{\rm R}}(\gamma, q_0)(1).$$

Proof.

$$\begin{aligned}&q_{\mathcal{D}_{\rm R}}(\gamma, q_0)(1) = q_{\mathcal{D}_{\rm R}}(\omega.\omega^{-1}.\gamma, q_0)(1) = \big(q_{\mathcal{D}_{\rm R}}(\omega, q_{\mathcal{D}_{\rm R}}(\omega^{-1}.\gamma, q_0)(1)).q_{\mathcal{D}_{\rm R}}(\omega^{-1}.\gamma, q_0)\big)(1) \\ =&\big(q_{\mathcal{D}_{\rm R}}(\omega, \hat{F} \cdot q_0).q_{\mathcal{D}_{\rm R}}(\omega^{-1}.\gamma, q_0)\big)(1) = q_{\mathcal{D}_{\rm R}}(\omega, \hat{F} \cdot q_0)(1) = \hat{F} \cdot q_{\mathcal{D}_{\rm R}}(\omega, q_0)(1).\end{aligned}$$

□

Proposition 4.6 Let $\pi : (M_1, g_1) \to (M, g)$ and $\hat{\pi} : (\hat{M}_1, \hat{g}_1) \to (\hat{M}, \hat{g})$ be Riemannian coverings. Write $Q_1 = Q(M_1, \hat{M}_1)$ and $(\mathcal{D}_{\mathrm{R}})_1$ for the rolling distribution in Q_1. Then the map

$$\Pi : Q_1 \to Q; \quad \Pi(x_1, \hat{x}_1; A_1) = \big(\pi(x_1), \hat{\pi}(\hat{x}_1); \hat{\pi}_*|_{\hat{x}_1} \circ A_1 \circ (\pi_*|_{x_1})^{-1}\big)$$

is a covering map of Q_1 over Q and

$$\Pi_*(\mathcal{D}_{\mathrm{R}})_1 = \mathcal{D}_{\mathrm{R}}.$$

Moreover, for every $q_1 \in Q_1$ the restriction onto $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)$ of Π is a covering map $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. Then, for every $q_1 \in Q_1$, $\Pi(\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$ and one has $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1$ if and only if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q$.

As an immediate corollary of the above proposition, we obtain the following result regarding the complete controllability of $(\mathcal{D}_{\mathrm{R}})$.

Corollary 4.7 Let $\pi : (M_1, g_1) \to (M, g)$ and $\hat{\pi} : (\hat{M}_1, \hat{g}_1) \to (\hat{M}, \hat{g})$ be Riemannian coverings. Write $Q = Q(M, \hat{M})$, $\mathcal{D}_{\mathrm{R}}$ and $Q_1 = Q(M_1, \hat{M}_1)$, $(\mathcal{D}_{\mathrm{R}})_1$ respectively for the state space and for the rolling distribution in the respective state space. Then the control system associated to $\mathcal{D}_{\mathrm{R}}$ is completely controllable if and only if the control system associated to $(\mathcal{D}_{\mathrm{R}})_1$ is completely controllable. As a consequence, when one addresses the complete controllability issue for the rolling distribution $\mathcal{D}_{\mathrm{R}}$, one can assume with no loss of generality that both manifolds M and $\hat{M}$ are simply connected.

We now proceed with the proof of Proposition 4.6.

Proof. It is clear that Π is a local diffeomorphism onto Q. To show that it is a covering map, let $q_1 = (x_1, \hat{x}_1; A_1)$ and choose evenly covered w.r.t π, $\hat{\pi}$ open sets U and $\hat{U}$ of M, $\hat{M}$ containing $\pi(x_1)$, $\hat{\pi}(\hat{x}_1)$, respectively. Thus $\pi^{-1}(U) = \bigcup_{i \in I} U_i$ and $\hat{\pi}^{-1}(\hat{U}) = \bigcup_{i \in \hat{I}} \hat{U}_i$ where U_i, $i \in I$ (resp. $\hat{U}_i$, $i \in \hat{I}$) are mutually disjoint connected open subsets of M_1 (resp. $\hat{M}_1$) such that π (resp. $\hat{\pi}$) maps each U_i (resp. $\hat{U}_i$) diffeomorphically onto U (resp. $\hat{U}$). Then

$$\Pi^{-1}(\pi_Q^{-1}(U \times \hat{U})) = \pi_{Q_1}^{-1}((\pi \times \hat{\pi})^{-1}(U \times \hat{U})) = \bigcup_{i \in I, j \in \hat{I}} \pi_{Q_1}^{-1}(U_i \times \hat{U}_j),$$

where $\pi_{Q_1}^{-1}(U_i \times \hat{U}_j)$ for $(i,j) \in I \times \hat{I}$ are clearly mutually disjoint and connected. Now if for a given $(i,j) \in I \times \hat{I}$ we have $(y_1, \hat{y}_1, B_1), (z_1, \hat{z}_1; C_1) \in \pi_{Q_1}^{-1}(U_i \times \hat{U}_j)$ such that $\Pi(y_1, \hat{y}_1; B_1) = \Pi(z_1, \hat{z}_1, C_1)$, then $y_1 = z_1$, $\hat{y}_1 = \hat{z}_1$ and hence $B_1 = C_1$, which shows that Π restricted to $\pi_{Q_1}^{-1}(U_i \times \hat{U}_j)$ is injective. It is also a local diffeomorphism, as mentioned above, and clearly surjective onto $\pi_Q^{-1}(U \times \hat{U})$, which proves that $\pi_Q^{-1}(U \times \hat{U})$ is evenly covered with respect to Π. This finishes the proof that Π is a covering map. Suppose next that $q_1(t) = (\gamma_1(t), \hat{\gamma}_1(t); A_1(t))$ is a smooth path on Q_1 tangent to $(\mathcal{D}_{\mathrm{R}})_1$ and defined on an interval containing $0 \in \mathbb{R}$. Define $q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) := (\Pi \circ q_1)(t)$. Then

$$\begin{aligned}
\dot{\hat{\gamma}}(t) =& \hat{\pi}_* \dot{\hat{\gamma}}_1(t) = \hat{\pi}_* A_1(t) \dot{\gamma}_1(t) = A(t) \pi_* \dot{\gamma}_1(t) = A(t) \dot{\gamma}(t) \\
A(t) =& \hat{\pi}_*|_{\hat{\gamma}_1(t)} \circ P_0^t(\hat{\gamma}_1(t)) \circ A_1(0) \circ P_t^0(\gamma_1) \circ (\pi_*|_{\gamma_1(t)})^{-1} \\
=& P_0^t(\hat{\gamma}(t)) \circ \hat{\pi}_*|_{\hat{\gamma}_1(t)} \circ A_1(0) \circ (\pi_*|_{\gamma_1(t)})^{-1} \circ P_t^0(\gamma) \\
=& P_0^t(\hat{\gamma}(t)) \circ A(0) \circ P_t^0(\gamma),
\end{aligned}$$

which shows that $q(t)$ is tangent to $\mathcal{D}_{\mathrm{R}}$. This shows that $\Pi_*(\mathcal{D}_{\mathrm{R}})_1 \subset \mathcal{D}_{\mathrm{R}}$ and the equality follows from the fact that Π is a local diffeomorphism and the ranks of $(\mathcal{D}_{\mathrm{R}})_1$ and $\mathcal{D}_{\mathrm{R}}$ are the same i.e., $= n$.

Let $q_1 = (x_1, \hat{x}_1; A_1)$. We proceed to show that the restriction of Π gives a covering $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. First, since $\Pi_*(\mathcal{D}_{\mathrm{R}})_1 = \mathcal{D}_{\mathrm{R}}$ and $\Pi : Q_1 \to Q$ is a covering map, it follows that $\Pi(\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. Let $q := \Pi(q_1)$ and let $U \subset Q$ be an evenly covered neighbourhood of q w.r.t. Π. By the Orbit Theorem, there exists vector fields $Y_1, \dots, Y_d \in \mathrm{VF}(Q)$ tangent to $\mathcal{D}_{\mathrm{R}}$ and $(u_1, \dots, u_d) \in (L^1([0,1]))^d$ and a connected open neighbourhood W of $(u_1, \dots, u_d)$ in $(L^1([0,1]))^d$ such that the image of the end point map $\mathrm{end}_{(Y_1,\dots,Y_d)}(q, W)$ is an open subset of the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ containing q and included in the Π-evenly covered set U. Let $(Y_i)_1$, $i = 1, \dots, d$, be the unique vector fields on Q_1 defined by $\Pi_*(Y_i)_1 = Y_i$, $i = 1, \dots, d$. Since $\Pi_*(\mathcal{D}_{\mathrm{R}})_1 = \mathcal{D}_{\mathrm{R}}$, it follows that $(Y_i)_1$ are tangent to $(\mathcal{D}_{\mathrm{R}})_1$ and also, $\Pi \circ \mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)} = \mathrm{end}_{(Y_1,\dots,Y_d)} \circ (\Pi \times \mathrm{id})$. It follows that $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1', W)$ is an open subset of $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)$ contained in $\Pi^{-1}(U)$ for every $q_1' \in (\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)$. Since $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}$ is continuous and W is connected, it thus follows that for each $q_1' \in (\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)$, the connected set $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1', W)$ is contained in a single component of $\Pi^{-1}(U)$ which, since U was evenly covered, is mapped diffeomorphically by Π onto U. But then Π maps $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1', W)$ diffeomorphically onto $\mathrm{end}_{(Y_1,\dots,Y_d)}(q, W)$. Since it is also obvious that

$$(\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}\big(\mathrm{end}_{(Y_1,\dots,Y_d)}(q, W)\big) = \bigcup_{q_1' \in (\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)} \mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1', W),$$

we have proved that $\mathrm{end}_{(Y_1,\dots,Y_d)}(q, W)$ is an evenly covered neighbourhood of q in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ w.r.t $\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)}$.

Finally, let us prove that for every $q_1 \in Q_1$, the following implication holds true,

$$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q \implies \mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1,$$

(the converse statement being trivial). Indeed, if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q$, then, for every $q \in Q$, $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) = Q$ and, on the other hand, the fact that Π restricts to a covering map $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1') \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1')) = Q$ for any $q_1' \in Q_1$ implies that all the orbits $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1')$, $q_1' \in Q_1$, are open on Q_1. But Q_1 is connected (and orbits are non-empty) and hence there cannot be but one orbit. In particular, $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1$. □

4.2 Rolling Curvature and Lie Algebraic Structure of $\mathcal{D}_{\mathrm{R}}$

4.2.1 Rolling Curvature

We compute some commutators of the vector fields of the form $\mathscr{L}_{\mathrm{R}}(X)$ with $X \in \mathrm{VF}(M)$. The formulas obtained hold both in Q and $T^*M \otimes T\hat{M}$ and thus we do them in the latter space.

The first commutators of the $\mathcal{D}_{\mathrm{R}}$-lifted fields are given in the following theorem.

Proposition 4.8 *If $X, Y \in \mathrm{VF}(M)$, $q = (x_0, \hat{x}_0; A) \in T^*(M) \otimes T(\hat{M})$, then the commutator of the lifts $\mathscr{L}_{\mathrm{R}}(X)$ and $\mathscr{L}_{\mathrm{R}}(Y)$ at q is given by*

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{R}}([X, Y])|_q + \nu(AR(X, Y) - \hat{R}(AX, AY)A)|_q. \qquad (28)$$

Proof. Choosing $\overline{T}(B) = (X, BX)$, $\overline{S}(B) = (Y, BY)$ for $B \in T^*(M) \otimes T(\hat{M})$ in proposition 3.35 we have

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{NS}}(\mathscr{L}_{\mathrm{NS}}(X, AX)|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(Y, AY)|_q\overline{T})|_q \\ + \nu(AR(X,Y) - \hat{R}(AX, AY)A)|_q.$$

By Lemma 3.33 one has

$$\mathscr{L}_{\mathrm{NS}}(X, AX)|_q\overline{S} = \mathscr{L}_{\mathrm{NS}}(X, AX)|_q(Y + (\cdot)Y) = \mathscr{L}_{\mathrm{NS}}(X, AX)|_qY + A\mathscr{L}_{\mathrm{NS}}(X, AX)|_qY \\ = \nabla_X Y + A\nabla_X Y,$$

so

$$\mathscr{L}_{\mathrm{NS}}(\mathscr{L}_{\mathrm{NS}}(X, AX)|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(Y, AY)|_q\overline{T})|_q = \mathscr{L}_{\mathrm{NS}}(\nabla_X Y + A\nabla_X Y - \nabla_Y X - A\nabla_Y X)|_q \\ = \mathscr{L}_{\mathrm{R}}(\nabla_X Y - \nabla_Y X)|_q,$$

which proves the claim after noticing that, by torsion freeness of ∇, one has $\nabla_X Y - \nabla_Y X = [X, Y]$. ☐

Proposition 4.8 justifies the next definition.

Definition 4.9 Given vector fields $X, Y, Z_1, \ldots, Z_k \in \mathrm{VF}(M)$, we define the *Rolling Curvature* of the rolling of M against $\hat{M}$ as the smooth mapping

$$\mathsf{Rol}(X, Y) : \pi_{T^*M\otimes T\hat{M}} \to \pi_{T^*M\otimes T\hat{M}},$$

by

$$\mathsf{Rol}(X, Y)(A) := AR(X, Y) - \hat{R}(AX, AY)A, . \tag{29}$$

Moreover, for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, we use Rol_q to denote the linear map $\wedge^2 T|_x M \to T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ defined on pure elements of $\wedge^2 T|_x M$ by

$$\mathsf{Rol}_q(X \wedge Y) = \mathsf{Rol}(X, Y)(A). \tag{30}$$

Similarly, for $k \geq 0$, we define the smooth mapping

$$\overline{\nabla}^k\mathsf{Rol}(X, Y, Z_1, \ldots, Z_k) : \pi_{T^*M\otimes T\hat{M}} \to \pi_{T^*M\otimes T\hat{M}},$$

by

$$\overline{\nabla}^k\mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) := A\nabla^k R(X, Y, (\cdot), Z_1, \ldots, Z_k) \\ - \hat{\nabla}^k\hat{R}(AX, AY, A(\cdot), AZ_1, \ldots, AZ_k). \tag{31}$$

Restricting to Q, we have

$$\mathsf{Rol}(X, Y), \overline{\nabla}^k\mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) \in C^\infty(\pi_Q, \pi_{T^*M\otimes T\hat{M}}),$$

such that, for all $(x, \hat{x}; A) \in Q$,

$$\mathsf{Rol}(X, Y)(A), \overline{\nabla}^k\mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) \in A(\mathfrak{so}(T|_x M)).$$

Remark 4.10 With this notation, Eq. (28) of Proposition 4.8 can be written as

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{R}}([X,Y])|_q + \nu(\mathsf{Rol}(X,Y)(A))|_q.$$

Recall that using the metric g, one may identify $T^*|_xM \wedge T|_xM = \mathfrak{so}(T|_xM)$ with $\wedge^2 T|_xM$ as we usually do without mention. In order to take advantage of the spectral properties of a (real) symmetric endomorphism, we introduce the following operator associated to the rolling curvature.

Definition 4.11 If $q = (x, \hat{x}; A) \in Q$, let $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_xM \to \wedge^2 T|_xM$ be the (real) symmetric endomorphism defined by

$$\widetilde{\mathsf{Rol}}_q := A^{\overline{T}} \mathsf{Rol}_q. \tag{32}$$

4.2.2 Computation of more Lie brackets

Proposition 4.12 Let $X, Y, Z \in \mathrm{VF}(M)$. Then, for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, one has

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(Z), \nu(\mathsf{Rol}(X,Y)(\cdot))]|_q =& -\mathscr{L}_{\mathrm{NS}}(\mathsf{Rol}(X,Y)(A)Z)|_q + \nu\big(\overline{\nabla}^1 \mathsf{Rol}(X,Y,Z)(A)\big)\big|_q \\ &+ \nu\big(\mathsf{Rol}(\nabla_Z X, Y)(A)\big)\big|_q + \nu\big(\mathsf{Rol}(X, \nabla_Z Y)(A)\big)\big|_q.\end{aligned}$$

Proof. Taking $\overline{T}(B) = (Z, BZ)$ and $U = \mathsf{Rol}(X,Y)$ for $B \in T^*M \otimes T\hat{M}$ in Proposition 3.36, we get

$$\begin{aligned}&[\mathscr{L}_{\mathrm{R}}(Z), \nu(\mathsf{Rol}(X,Y)(\cdot))]|_q \\ =& -\mathscr{L}_{\mathrm{NS}}(\nu(\mathsf{Rol}(X,Y)(A))|_q(Z + (\cdot)Z))|_q + \nu(\mathscr{L}_{\mathrm{NS}}(Z, AZ)|_q \mathsf{Rol}(X,Y)(\cdot))|_q.\end{aligned}$$

By Lemma 3.33 one has

$$\nu(\mathsf{Rol}(X,Y)(A))|_q(Z + (\cdot)Z) = \mathsf{Rol}(X,Y)(A)Z,$$

while by taking a local $\pi_{T^*M\otimes T\hat{M}}$-section $\tilde{A}$ s. t. $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$, one gets

$$\begin{aligned}\mathscr{L}_{\mathrm{NS}}(Z, AZ)|_q \mathsf{Rol}(X,Y)(\cdot) =& \overline{\nabla}_{Z+AZ}(\mathsf{Rol}(X,Y)(\tilde{A})) \\ =& \overline{\nabla}^1 \mathsf{Rol}(X,Y,Z)(A) + \mathsf{Rol}(\nabla_Z X, Y)(A) + \mathsf{Rol}(X, \nabla_Z Y)(A).\end{aligned}$$

□

By Proposition 4.8, the last two terms (when considered as vector fields on $T^*M \otimes T\hat{M}$) on the right hand side belong to $\mathrm{VF}^2_{\mathcal{D}_{\mathrm{R}}}$.

Since for $X, Y \in \mathrm{VF}(M)$ and $q = (x, \hat{x}; A) \in Q$ we have $\nu(\mathsf{Rol}(X,Y)(A))|_q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ by Proposition 4.8, it is reasonable to compute the Lie-bracket of two elements of this type. This is given in the following proposition.

Proposition 4.13 For any $q = (x, \hat{x}; A) \in Q$ and $X, Y, Z, W \in \mathrm{VF}(M)$ we have

$$\begin{aligned}&\big[\nu(\mathsf{Rol}(X,Y)(\cdot)), \nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\ =&\nu\big(\mathsf{Rol}(X,Y)(A)R(Z,W) - \hat{R}(\mathsf{Rol}(X,Y)(A)Z, AW)A - \hat{R}(AZ, \mathsf{Rol}(X,Y)(A)W)A \\ &- \hat{R}(AZ, AW)\mathsf{Rol}(X,Y)(A) - \mathsf{Rol}(Z,W)(A)R(X,Y) + \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A \\ &+ \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A + \hat{R}(AX, AY)\mathsf{Rol}(Z,W)(A)\big)\big|_q.\end{aligned}$$

Proof. We use Proposition 3.37 where for U, V we take $U(A) = \mathsf{Rol}(X,Y)(A)$ and $V(A) = \mathsf{Rol}(Z,W)(A)$. First compute for B such that $\nu(B)|_q \in V|_q(Q)$ that

$$\begin{aligned}\nu(B)|_q U =&\nu(B)|_q\big(\tilde{A} \mapsto \tilde{A}R(X,Y) - \hat{R}(\tilde{A}X, \tilde{A}Y)\tilde{A}\big)\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big((A+tB)R(X,Y) - \hat{R}((A+tB)X,(A+tB)Y)(A+tB)\big)\\ =&BR(X,Y) - \hat{R}(BX,AY)A - \hat{R}(AX,BY)A - \hat{R}(AX,AY)B.\end{aligned}$$

So by taking $B = V(A)$ we get

$$\begin{aligned}\nu(V(A))|_q U =&\mathsf{Rol}(Z,W)(A)R(X,Y) - \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A\\ &- \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A - \hat{R}(AX,AY)\mathsf{Rol}(Z,W)(A),\end{aligned}$$

and similarly for $\nu(U(A))|_q V$. □

For later use, we find it convenient to provide another expression for Proposition 4.13 and, for that purpose, we recall the following notation. For $A, B \in \mathfrak{so}(T|_x M)$, we define

$$[A,B]_{\mathfrak{so}} := A \circ B - B \circ A \in \mathfrak{so}(T|_x M).$$

Then, one has the following corollary.

Corollary 4.14 For any $q = (x, \hat{x}; A) \in Q$ and $X, Y, Z, W \in \mathrm{VF}(M)$ we have

$$\begin{aligned}&\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)), \nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q\\ =&A\big[R(X,Y),R(Z,W)\big]_{\mathfrak{so}} - \big[\hat{R}(AX,AY),\hat{R}(AZ,AW)\big]_{\mathfrak{so}}A\\ &- \hat{R}(\mathsf{Rol}(X,Y)(A)Z, AW)A - \hat{R}(AZ, \mathsf{Rol}(X,Y)(A)W)A\\ &+ \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A + \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A.\end{aligned} \tag{33}$$

4.3 Controllability Properties of $\mathcal{D}_{\mathrm{R}}$ and first results

Proposition 4.8 has the following simple consequence.

Corollary 4.15 The following cases are equivalent:

(i) The rolling distribution $\mathcal{D}_{\mathrm{R}}$ on Q is involutive.

(ii) For all $X, Y, Z \in T|_x M$ and $(x, \hat{x}; A) \in T^*(M) \otimes T(\hat{M})$

$$\mathsf{Rol}(X,Y)(A) = 0.$$

(iii) (M, g) and $(\hat{M}, \hat{g})$ both have constant and equal curvature.

The same result holds when one replaces Q by $T^*M \otimes T\hat{M}$.

Proof. (i) $\iff$ (ii) follows from Proposition 4.8.

(ii)$\Rightarrow$(iii) We use

$$\sigma_{(X,Y)} = g(R(X,Y)Y, X), \text{ and } \sigma_{(\hat{X},\hat{Y})} = \hat{g}(\hat{R}(\hat{X},\hat{Y})\hat{Y}, \hat{X}),$$

to denote the sectional curvature of M w.r.t orthonormal vectors $X, Y \in T|_x M$ and the sectional curvature of $\hat{M}$ w.r.t. orthonormal vectors $\hat{X}, \hat{Y} \in T|_{\hat{x}}\hat{M}$ respectively. The assumption that $\mathsf{Rol} = 0$ on Q then implies

$$\sigma_{(X,Y)} = \hat{\sigma}_{(AX,AY)}, \quad \forall (x, \hat{x}; A) \in Q,\ X, Y \in T|_x M. \tag{34}$$

If we fix $x \in M$ and g-orthonormal vectors $X, Y \in T|_x M$, then, for any $\hat{x} \in \hat{M}$ and any $\hat{g}$-orthonormal vectors $\hat{X}, \hat{Y} \in T|_{\hat{x}}\hat{M}$, we may choose $A \in Q|_{(x,\hat{x})}$ such that $AX = \hat{X}$, $AY = \hat{Y}$ (in the case $n = 2$ we may have to replace, say, $\hat{X}$ by $-\hat{X}$ but this does not change anything in the argument below). Hence the above equation (34) shows that the sectional curvatures at every point $\hat{x} \in \hat{M}$ and w.r.t every orthonormal pair $\hat{X}, \hat{Y}$ are all the same i.e., $\sigma_{(X,Y)}$. Thus $(\hat{M}, \hat{g})$ has constant sectional curvatures i.e., it has a constant curvature. Changing the roles of M and $\hat{M}$ we see that (M, g) also has constant curvature and the constants of curvatures are the same.

(iii)$\Rightarrow$(ii) Suppose that $M, \hat{M}$ have constant and equal curvatures. By a standard result (see [39] Lemma II.3.3), this is equivalent to the fact that there exists $k \in \mathbb{R}$ such that

$$\begin{aligned} &R(X,Y)Z = k\big(g(Y,Z)X - g(X,Z)Y\big), \quad X, Y, Z \in T|_x M,\ x \in M, \\ &\hat{R}(\hat{X},\hat{Y})\hat{Z} = k\big(\hat{g}(\hat{Y},\hat{Z})\hat{X} - \hat{g}(\hat{X},\hat{Z})\hat{Y}\big), \quad \hat{X}, \hat{Y}, \hat{Z} \in T|_{\hat{x}}\hat{M},\ \hat{x} \in \hat{M}. \end{aligned}$$

On the other hand, if $A \in Q$, $X, Y, Z \in T|_x M$, we would then have

$$\begin{aligned} &\hat{R}(AX, AY)(AZ) = k(\hat{g}(AY, AZ)AX - \hat{g}(AX, AZ)(AY)) \\ =&A(k(g(Y,Z)X - g(X,Z)Y) = A(R(X,Y)Z). \end{aligned}$$

This implies that $\mathsf{Rol}(X,Y)(A) = 0$ since Z was arbitrary.

□

In the situation of the previous corollary, the control system $(\Sigma)_R$ is as far away from being controllable as possible: all the orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, $q \in Q$, are integral manifolds of $\mathcal{D}_{\mathrm{R}}$. The next consequence of Proposition 4.8 can be seen as a (partial) generalization of the previous corollary and a special case of the Ambrose's theorem. The corollary gives a necessary and sufficient condition describing the case in which at least one $\mathcal{D}_{\mathrm{R}}$-orbit is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

Corollary 4.16 Suppose that (M, g) and $(\hat{M}, \hat{g})$ are complete. The following cases are equivalent:

(i) There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

(ii) There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that

$$\mathsf{Rol}(X,Y)(A) = 0, \quad \forall (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\ X, Y \in T|_x M.$$

(iii) There is a complete Riemannian manifold (N, h) and Riemannian covering maps $F : N \to M$, $G : N \to \hat{M}$. In particular, (M, g) and $(\hat{M}, \hat{g})$ are locally isometric.

Proof. (i) ⇒ (ii): Notice that the restrictions of vector fields $\mathscr{L}_{\mathrm{R}}(X)$, $X \in \mathrm{VF}(M)$, to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ are smooth vector fields of that orbit. Thus $[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]$ is also tangent to this orbit for any $X, Y \in \mathrm{VF}(M)$ and hence Proposition 4.8 implies the claim.

(ii) ⇒ (i): It follows, from Proposition 4.8, that $\mathcal{D}_{\mathrm{R}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$, the restriction of $\mathcal{D}_{\mathrm{R}}$ to the manifold $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, is involutive. Since maximal connected integral manifolds of an involutive distribution are exactly its orbits, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

(i) ⇒ (iii): Let $N := \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and $h := (\pi_{Q,M}|_N)^*(g)$ i.e., for $q = (x, \hat{x}; A) \in N$ and $X, Y \in T|_x M$, define

$$h(\mathscr{L}_{\mathrm{R}}(X)|_q, \mathscr{L}_{\mathrm{R}}(Y)|_q) = g(X, Y).$$

If $F := \pi_{Q,M}|_N$ and $G := \pi_{Q,\hat{M}}|_N$, we immediately see that F is a local isometry (note that $\dim(N) = n$) and the fact that G is a local isometry follows from the following computation: for $q = (x, \hat{x}; A) \in N$, $X, Y \in T|_x M$, one has

$$\hat{g}(G_*(\mathscr{L}_{\mathrm{R}}(X)|_q), G_*(\mathscr{L}_{\mathrm{R}}(Y)|_q)) = \hat{g}(AX, AY) = g(X, Y) = h(\mathscr{L}_{\mathrm{R}}(X)|_q, \mathscr{L}_{\mathrm{R}}(Y)|_q).$$

The completeness of (N, h) can be easily deduced from the completeness of M and $\hat{M}$ together with Proposition 3.20. Proposition II.1.1 in [39] proves that the maps F, G are in fact (surjective and) Riemannian coverings.

(iii) ⇒ (ii): Let $x_0 \in M$ and choose $z_0 \in N$ such that $F(z_0) = x_0$. Define $\hat{x}_0 = G(z_0) \in \hat{M}$ and $A_0 := G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$ which is an element of $Q|_{(x_0,\hat{x}_0)}$ since F, G were local isometries. Write $q_0 = (x_0, \hat{x}_0; A_0) \in Q$.

Let $\gamma : [0, 1] \to M$ be an a.c. curve with $\gamma(0) = x_0$. Since F is a smooth covering map, there is a unique a.c. curve $\Gamma : [0, 1] \to N$ with $\gamma = F \circ \Gamma$ and $\Gamma(0) = z_0$. Define $\hat{\gamma} = G \circ \Gamma$ and $A(t) = G_*|_{\Gamma(t)} \circ (F_*|_{\Gamma(t)})^{-1} \in Q$, $t \in [0, 1]$. It follows that, for a.e. $t \in [0, 1]$,

$$\dot{\hat{\gamma}}(t) = G_*|_{\Gamma(t)}\dot{\Gamma}(t) = A(t)\dot{\gamma}(t).$$

Since F, G are local isometries, $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A(\cdot) = 0$ for a.e. $t \in [0, 1]$. Thus $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ is the unique rolling curve along γ starting at q_0 and defined on $[0, 1]$ and therefore curves of Q formed in this fashion fill up the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Moreover, since F, G are local isometries, it follows that for every $z \in N$ and $X, Y \in T|_{F(z)}M$, $\mathsf{Rol}(X, Y)(G_*|_z \circ (F_*|_z)^{-1}) = 0$. These facts prove that the condition in (ii) holds and the proof is therefore finished. □

Remark 4.17 If one does not assume that (M, g) and $(\hat{M}, \hat{g})$ are complete in Corollary 4.16, then (iii) in the above corollary must be replaced by the following:

(iii)' There is a connected Riemannian manifold (N, h) (not necessarily complete) and Riemannian covering maps $F : N \to M^\circ$, $G : N \to \hat{M}^\circ$ where M°, $\hat{M}^\circ$ are open sets of M and $\hat{M}$ and there is a $z_0 \in N$ such that if $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ is defined by $A_0 := G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$, then $M^\circ = \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ and $\hat{M}^\circ = \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$.

In particular, M°, $\hat{M}^\circ$ are connected and (M°, g), $(\hat{M}^\circ, \hat{g})$ are locally isometric. Indeed, the argument in the implication (i) ⇒ (iii) goes through except for the completeness of (N, h), where $N = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (connected). Proposition 4.2 and Remark 4.2 show that

$F=\pi_{Q,M}|_N : N\to M^\circ$, $G=\pi_{Q,\hat{M}}|_N : N\to \hat{M}^\circ$ are bundles with discrete fibers. Now it is a standard (easy) fact that a bundle $\pi : X\to Y$ with connected total space X and discrete fibers is a covering map (this could have been used in the above proof instead of referring to [39]). On the other hand, in the argument of the implication (iii) $\Rightarrow$ (ii) we did not even use completeness of (N,h) but only the fact that $F:N\to M$ is a covering map to lift a curve γ in M to the curve Γ in Q. In this non-complete setting, we just have to consider using curves γ in M° and lift them to N by using $F:N\to M^\circ$. Indeed, if $q=(x,\hat{x};A)\in\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$, there is a curve $\gamma:[0,1]\to M$ such that $q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(1)=q$. For all t one has

$$\gamma(t)=\pi_{Q,M}(q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(t))\in\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))=M^\circ,$$

so γ is actually a curve in M°.

Finally, notice that the assumption in (iii)' that $\hat{M}^\circ=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$ follows from the others. Indeed, making only the other assumptions, it is first of all clear that if q and γ are as above, then

$$\pi_{Q,\hat{M}}(q)=\pi_{Q,\hat{M}}(q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(1))=G(\Gamma(1))\in\hat{M}^\circ,$$

so $\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))\subset\hat{M}^\circ$. Then if $\hat{x}\in\hat{M}^\circ$, one may take a path $\hat{\gamma}:[0,1]\to\hat{M}^\circ$ such that $\hat{\gamma}(0)=\hat{x}_0$, $\hat{\gamma}(1)=\hat{x}$ and lift it by the covering map G to a curve $\hat{\Gamma}(t)$ in N starting from z_0. Then if $\gamma(t):=F(\hat{\Gamma}(t))$, $t\in[0,1]$, we easily see that $\hat{\gamma}=\hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma,q_0)$, whence $\hat{x}=\hat{\gamma}(1)\in\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$.

On the opposite direction with respect to having the rolling curvature equal to zero, one gets the following proposition (cf. [16] for another proof).

Proposition 4.18 Suppose there is a point $q_0=(x_0,\hat{x}_0;A_0)\in Q$ and $\epsilon>0$ such that for every $X\in\mathrm{VF}(M)$ with $\|X\|_g<\epsilon$ on M one has

$$V|_{\Phi_{\mathscr{L}_{\rm R}(X)}(t,q_0)}(\pi_Q)\subset T\big(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\big),\quad |t|<\epsilon.$$

Then the orbit $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ is open in Q. As a consequence, we have the following characterization of complete controllability: the control system $(\Sigma)_R$ is completely controllable if and only if

$$\forall q\in Q,\quad V|_q(\pi_Q)\subset T|_q\mathcal{O}_{\mathcal{D}_{\rm R}}(q). \tag{35}$$

Proof. For the first part of the proposition, the assumptions and Lemma 4.20 given below imply that for every $X\in T|_{x_0}M$ we have $\mathscr{L}_{\rm NS}(Y,\hat{Y})|_{q_0}\in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ for every $Y\in X^\perp$, $\hat{Y}\in A_0X^\perp$. But since X is an arbitrary element of $T|_{x_0}M$, this means that $\mathcal{D}_{\rm NS}|_{q_0}\subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ and because $T|_{q_0}Q=\mathcal{D}_{\rm NS}|_{q_0}\oplus V|_{q_0}(\pi_Q)$, we get $T|_{q_0}Q=T|_{q_0}\big(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\big)$. This implies that $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ is open in Q. The last part of the proposition is an immediate consequence of this and the fact that Q is connected. □

Remark 4.19 The above corollary is intuitively obvious. Assumption given by Eq. (35) simply means that there is complete freedom for infinitesimal spinning, i.e., for reorienting one manifold with respect to the other one without moving in $M\times\hat{M}$. In that case, proving complete controllability is easy, by using a crab-like motion.

We end this section by providing a technical lemma needed for the argument of the previous proposition. It is actually a consequence of Proposition 3.36.

Lemma 4.20 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Suppose that, for some $X \in \mathrm{VF}(M)$ and a real sequence $(t_n)_{n=1}^{\infty}$ s.t. $t_n \neq 0$ for all n, $\lim_{n\to\infty} t_n = 0$, we have, for every $n \geq 0$,

$$V|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)}(\pi_Q) \subset T(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)). \tag{36}$$

Then $\mathscr{L}_{\mathrm{NS}}(Y, \hat{Y})\big|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for every $Y \in T|_{x_0}M$ that is g-orthogonal to $X|_{x_0}$ and every $\hat{Y} \in T|_{\hat{x}_0}\hat{M}$ that is $\hat{g}$-orthogonal to $A_0X|_{x_0}$. Hence the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has codimension at most 1 inside Q.

Proof. Letting n tend to infinity, it follows from (36) that $V|_{q_0}(\pi_Q) \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Recall, from Proposition 3.4, that every element of $V|_{q_0}(\pi_Q)$ is of the form $\nu(B)|_{q_0}$, with a unique $B \in A_0\mathfrak{so}(T|_{x_0}M)$. Fix such a B and define a smooth local section $\tilde{S}$ of $\mathfrak{so}(TM) \to M$ defined on an open set $W \ni x_0$ by

$$\tilde{S}|_x = P_0^1\big(t \mapsto \exp_{x_0}(t\exp_{x_0}^{-1}(x))\big)(A_0^{\overline{T}}B).$$

Then clearly, $\tilde{S}|_{x_0} = A_0^{\overline{T}}B$ and $\nabla_Y\tilde{S} = 0$ for all $Y \in T|_{x_0}M$ and it is easy to verify that $\tilde{S}|_x \in \mathfrak{so}(T|_xM)$ for all $x \in W$. We next define a smooth map $U : \pi_Q^{-1}(W \times \hat{M}) \to T^*M \otimes T\hat{M}$ by $U(x, \hat{x}; A) = A\tilde{S}|_x$. Obviously $\nu(U(x, \hat{x}; A)) \in V|_{(x,\hat{x};A)}(\pi_Q)$ for all $(x, \hat{x}; A)$. Then, choosing in Proposition 3.36, $\overline{T} = X + (\cdot)X$ (and the above U) and noticing that

$$\nu(U(A_0))|_{q_0}\overline{T} = U(A_0)X = BX,$$

one gets

$$[\mathscr{L}_{\mathrm{R}}(X), \nu(U(\cdot))]|_{q_0} = -\mathscr{L}_{\mathrm{NS}}(BX)|_{q_0} + \nu(\overline{\nabla}_{(X,A_0X)}(U(\tilde{A})))|_{q_0} \tag{37}$$

where $\tilde{A}|_{(x_0,\hat{x}_0)} = A_0$. By the choice of $\tilde{S}$ and $\tilde{A}$, we have, for all $\overline{Y} = (Y, \hat{Y}) \in T|_{(x_0,\hat{x}_0)}M \times \hat{M}$,

$$\nabla_{\overline{Y}}(U(\tilde{A})) = \nabla_{\overline{Y}}(\tilde{A}\tilde{S}) = (\nabla_{\overline{Y}}\tilde{A})\tilde{S}|_{(x_0,\hat{x}_0)} + \tilde{A}|_{(x_0,\hat{x}_0)}\nabla_Y\tilde{S} = 0,$$

and hence the last term on the right hand side of (37) actually vanishes.

By definition, the vector field $q \mapsto \mathscr{L}_{\mathrm{R}}(X)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and, by the assumption of Equation (36), the values of the map $q = (x, \hat{x}; A) \mapsto \nu(U(A))|_q$ are also tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at the points $\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n, q_0)$, $n \in \mathbb{N}$. Hence $\big((\Phi_{\mathscr{L}_{\mathrm{R}}(X)})_{-t_n}\big)_*\nu(U(\cdot))|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and therefore,

$$\begin{aligned}&[\mathscr{L}_{\mathrm{R}}(X), \nu(U(\cdot))]|_{q_0}\\ &= \lim_{n\to\infty}\frac{\big((\Phi_{\mathscr{L}_{\mathrm{R}}(X)})_{-t_n}\big)_*\nu(U(\cdot))|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)} - \nu(B)|_{q_0}}{t_n} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\end{aligned}$$

i.e., the left hand side of (37) must belong to $T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. But this implies that

$$\mathscr{L}_{\mathrm{NS}}(BX)|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), \quad \forall B \text{ s.t. } \nu(B) \in V|_{q_0}(\pi_Q)$$

i.e.,

$$\mathscr{L}_{\mathrm{NS}}(A_0\mathfrak{so}(T|_{x_0}M)X)|_{q_0} \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Notice next that $\mathfrak{so}(T|_{x_0}M)X$ is exactly the set $X|_{x_0}^{\perp}$ of vectors of $T|_{x_0}M$ that are g-perpendicular to $X|_{x_0}$. Since $A_0 \in Q$, it follows that the set $A_0\mathfrak{so}(T|_{x_0}M)X$ is equal to $A_0X|_{x_0}^{\perp}$ which is the set of vectors of $T|_{\hat{x}_0}\hat{M}$ that are $\hat{g}$-perpendicular to $A_0X|_{x_0}$. We conclude that $\mathscr{L}_{\mathrm{NS}}(Y)|_{q_0} = \mathscr{L}_{\mathrm{R}}(Y)|_{q_0} - \mathscr{L}_{\mathrm{NS}}(A_0Y)|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $Y \in X|_{x_0}^{\perp}$.

Finally notice that since the subspaces $X^{\perp} \times \{0\}$, $\mathbb{R}(X, A_0X)$ and $\{0\} \times (A_0X)^{\perp}$ of $T|_{(x_0,\hat{x}_0)}(M \times \hat{M})$ are linearly independent, their $\mathscr{L}_{\mathrm{NS}}$-lifts at q_0 are that also and hence these lifts span a $(n-1)+1+(n-1) = 2n-1$ dimensional subspace of $T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This combined with the fact that $V|_{q_0}(\pi_Q) \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ shows $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 2n-1+\dim V|_{q_0}(\pi_Q) = \dim(Q)-1$ i.e., the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has codimension at most 1 in Q. This finishes the proof.

□

5 Rolling Problem (R) in 3D

As mentioned in introduction, the goal of this chapter is to provide a local structure theorem of the orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ when M and $\hat{M}$ are 3-dimensional Riemannian manifolds. Recall that complete controllability of $(\Sigma)_R$ is equivalent to openess of *all* the orbits of $(\Sigma)_R$, thanks to the fact that Q is connected and $(\Sigma)_R$ is driftless. In case there is no complete controllability, then there exists a non open orbit which is an immersed manifold in Q of dimension at most eight. Moreover, as a fiber bundle over M, the fiber has dimension at most five.

5.1 Statement of the Results and Proof Strategy

Our first theorem provides all the possibilities for the local structure of a non open orbit for the rolling (R) of two 3D Riemannian manifolds.

Theorem 5.1 Let (M,g), $(\hat{M},\hat{g})$ be 3-dimensional Riemannian manifolds. Assume that $(\Sigma)_R$ is not completely controllable and let $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, for some $q_0 \in Q$, be a non open orbit. Then, there exists an open and dense subset O of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ so that, for every $q_1 = (x_1, \hat{x}_1; A_1) \in O$, there are neighbourhoods U of x_1 and $\hat{U}$ of $\hat{x}_1$ such that one of the following holds:

(a) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are (locally) isometric;

(b) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are both of class $\mathcal{M}_\beta$ for some $\beta > 0$;

(c) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are both isometric to warped products $(I \times N, h_f)$, $(I \times \hat{N}, \hat{h}_{\hat{f}})$ for some open interval $I \subset \mathbb{R}$ and warping functions $f, \hat{f}$ which moreover satisfy either

  (c-A) $f = \hat{f}$ or

  (c-B) there is a constant $K \in \mathbb{R}$ such that $\dfrac{f''(t)}{f(t)} = -K = \dfrac{\hat{f}''(t)}{\hat{f}(t)}$ for all $t \in I$.

For the definition and results on warped products and class $\mathcal{M}_\beta$, we refer to Appendix C.3 and Appendix C.2 respectively.

Remark 5.2 Regarding Item (c-A) above, what we actually establish in studying the appropriate case (see Proposition 5.27 below) is that, for every $t \in I$, $\frac{f'(t)}{f(t)} = \frac{\hat{f}'(t)}{\hat{f}(t)}$. Then by integrating the previous equation, one derives first that there exists a positive constant C such that $f = C\hat{f}$ and then one gets the conclusion of the theorem by eventually taking $C\hat{f}$ as warping function over $I \times \hat{N}$. Finally, the constant factor C can be absorbed into the metric $\hat{h}$.

Note that we do not address here to the issue of the global structure of a non open orbit for the rolling (R) of two 3D Riemmanian manifolds. For that, one would have to "glue" together the local information provided by Theorem 5.1. Instead, our second theorem below shows, in some sense, that the list of possibilities established in Theorem 5.1 is complete. We will exclude the case where $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold since in this case this orbit has dimension 3 and (M, g), $(\hat{M}, \hat{g})$ are locally isometric, see Corollary 4.16 and Remark 4.17.

Theorem 5.3 Let (M, g), $(\hat{M}, \hat{g})$ be 3D Riemannian manifolds, $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and suppose $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$. If one writes $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, then the following holds true.

(a) If (M, g), $(\hat{M}, \hat{g})$ are both of class $\mathcal{M}_\beta$ and if E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are adapted frames of (M, g) and $(\hat{M}, \hat{g})$, respectively, then one has:

- (A) If $A_0 E_2|_{x_0} = \pm\hat{E}_2|_{\hat{x}_0}$, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$;
- (B) If $A_0 E_2|_{x_0} \neq \pm\hat{E}_2|_{\hat{x}_0}$ and if (only) one of (M°, g) or $(\hat{M}^\circ, \hat{g})$ has constant curvature, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$;
- (C) Otherwise, $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

(b) If $(M, g) = (I \times N, h_f)$, $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$ are warped products, where $I, \hat{I} \subset \mathbb{R}$ are open intervals, and if $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, then one has

- (b-A) If $A_0 \frac{\partial}{\partial r}|_{(r_0,y_0)} = \frac{\partial}{\partial r}|_{(\hat{r}_0,\hat{y}_0)}$ and if there exists $C > 0$ such that, for every t such that $(t + r_0, t + \hat{r}_0) \in I \times \hat{I}$, it holds

$$f(t + r_0) = C\hat{f}(t + \hat{r}_0),$$

  then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;

- (b-B) Suppose there is a constant $K \in \mathbb{R}$ such that $\frac{f''(r)}{f(r)} = -K = \frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}$ for all $(r, \hat{r}) \in I \times \hat{I}$.
  - (b-B1) If $A_0 \frac{\partial}{\partial r}|_{(r_0,y_0)} = \pm\frac{\partial}{\partial r}|_{(\hat{r}_0,\hat{y}_0)}$ and $\frac{f'(r_0)}{f(r_0)} = \pm\frac{\hat{f}'(\hat{r}_0)}{\hat{f}(\hat{r}_0)}$, with $\pm$-cases correspondingly on both cases, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;
  - (b-B2) If (only) one of (M°, g), $(\hat{M}^\circ, \hat{g})$ has constant curvature, then one has $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;
  - (b-B3) Otherwise $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Here $(r,y)\mapsto \frac{\partial}{\partial r}|_{(r,y)}$, $(\hat{r},\hat{y})\mapsto \frac{\partial}{\partial r}|_{(\hat{r},\hat{y})}$, are the vector fields in $I\times N$ and $\hat{I}\times\hat{N}$ induced by the canonical, positively oriented vector field $r\mapsto \frac{\partial}{\partial r}\big|_r$ on $I,\hat{I}\subset\mathbb{R}$.

Remark 5.4 Similarly to Remark 5.2, we have the following. Regarding Item (b-A) above, what we actually assume (see Proposition 5.32 below) is that, for every t such that $(t+r_0,t+\hat{r}_0)\in I\times\hat{I}$, it holds $\frac{f'(t+r_0)}{f(t+r_0)}=\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}$, which is equivalent to the condition on the warping functions given in Item (b-A).

From now on(M,g), $(\hat{M},\hat{g})$ will be connected, oriented 3-dimensional Riemannian manifolds. The Hodge-duals of (M,g), $(\hat{M},\hat{g})$ are denoted by $\star:=\star_M$ and $\hat{\star}:=\star_{\hat{M}}$, respectively.

As a reminder, for $q_0=(x_0,\hat{x}_0;A_0)\in Q$, we will write

$$\begin{aligned}\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:=&\pi_Q|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to M\times\hat{M},\\ \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}:=&\mathrm{pr}_1\circ\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to M,\\ \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}:=&\mathrm{pr}_2\circ\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to \hat{M},\end{aligned}$$

where $\mathrm{pr}_1:M\times\hat{M}\to M$, $\mathrm{pr}_2:M\times\hat{M}\to\hat{M}$ are projections onto the first and second factor, respectively.

5.2 Proof of Theorem 5.1

In this subsection, we prove Theorem 5.1. We therefore fix for the rest of the paragraph a non open orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, for some $q_0\in Q$. By Proposition 4.2, one has that $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)<9=\dim Q$ and, by Corollary 4.14, one knows that the rank of Rol_q is less than or equal to two, for every $q\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

For $j=0,1,2$, let O_j be the set of points of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ where rank Rol_q is locally equal to j, i.e.,

$$O_j=\{q=(x,\hat{x};A)\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\mid \text{there exists an open neighbourhood } O \text{ of } q \text{ in } \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \text{ such that rank } \mathsf{Rol}_{q'}=j,\ \forall q'\in O\}.$$

Notice that the union of the O_j's, when $j=0,1,2$, is an open and dense subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ since each O_j is open in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (but might be empty). Clearly, Item (a) in Theorem 5.1 describes the local structures of (M,g) and $(\hat{M},\hat{g})$ at a point $q\in O_0$. The rest of the argument consists in addressing the same issue, first for $q\in O_2$ and then $q\in O_1$.

5.2.1 Local Structures for the Manifolds Around $q\in O_2$

Throughout the subsection, we assume, if not otherwise stated, that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q (i.e., $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)<9=\dim Q$) and, in the statements involving O_2, the latter is non empty. Note that O_2 is also equal to the set of points of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ where rank Rol_q is equal to 2.

Proposition 5.5 Let $q_0=(x_0,\hat{x}_0;A_0)\in Q$ so that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q. Then, for every $q=(x,\hat{x};A)\in O_2$, there exist an orthonormal pair $X_A,Y_A\in T|_xM$

such that if $Z_A := \star(X_A \wedge Y_A)$ then X_A, Y_A, Z_A is a positively oriented orthonormal pair with respect to which R and $\widetilde{\mathsf{Rol}}$ are written as

$$
\begin{aligned}
R(X_A \wedge Y_A) &= \begin{pmatrix} 0 & K(x) & 0 \\ -K(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix}, \\
R(Y_A \wedge Z_A) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K_1(x) \\ 0 & -K_1(x) & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_1(x) \\ 0 \\ 0 \end{pmatrix}, \\
R(Z_A \wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2(x) \\ 0 & 0 & 0 \\ K_2(x) & 0 & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2(x) \\ 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) = 0, \\
&\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} 0 & 0 & -\alpha(q) \\ 0 & 0 & K_1^{\mathsf{Rol}}(q) \\ \alpha(q) & -K_1^{\mathsf{Rol}}(q) & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} -K_1^{\mathsf{Rol}}(q) \\ -\alpha(q) \\ 0 \end{pmatrix}, \\
&\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}}(q) \\ 0 & 0 & \alpha(q) \\ K_2^{\mathsf{Rol}}(q) & -\alpha(q) & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} -\alpha(q) \\ -K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix},
\end{aligned}
$$

where K, K_1, K_2 are real valued functions defined on M.

Consequently, with respect to the orthonormal oriented basis AX_A, AY_A, AZ_A of $T|_{\hat{x}}\hat{M}$,

$$
\begin{aligned}
\star A^{\overline{T}}\hat{R}(AX_A \wedge AY_A)A &= \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix}, \\
\star A^{\overline{T}}\hat{R}(AY_A \wedge AZ_A)A &= \begin{pmatrix} -K_1(x) + K_1^{\mathsf{Rol}}(q) \\ \alpha(q) \\ 0 \end{pmatrix}, \\
\star A^{\overline{T}}\hat{R}(AZ_A \wedge AX_A)A &= \begin{pmatrix} \alpha(q) \\ -K_2(x) + K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix}.
\end{aligned} \tag{38}
$$

Before pursuing to the proof, we fix some additional notations provided in the following remark.

Remark 5.6 By the last proposition, $-K_1(x), -K_2(x), -K(x)$ are the eigenvalues of $R|_x$ corresponding to eigenvectors $\star X_A, \star Y_A, \star Z_A$ given by Proposition 5.5, for $q = (x, \hat{x}; A) \in O_2$. Recall that $Q(M, \hat{M}) \to Q(\hat{M}, M)$, $q = (x, \hat{x}; A) \mapsto \hat{q} = (\hat{x}, x; A^{\overline{T}})$ is an diffeomorphism which maps $\mathcal{D}_{\mathrm{R}}$ to $\widehat{\mathcal{D}_{\mathrm{R}}}$, where the latter is the rolling distribution on $Q(\hat{M}, M)$. Hence this map maps $\mathcal{D}_{\mathrm{R}}$-orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ to $\widehat{\mathcal{D}_{\mathrm{R}}}$-orbits $\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\hat{q})$, for all $q \in Q$. So the rolling problem (R) is completely symmetric w.r.t. the changing of the roles of (M, g) and $(\hat{M}, \hat{g})$. Hence Proposition 5.5 gives, when the roles of (M, g), $(\hat{M}, \hat{g})$ are changed, for every $q = (x, \hat{x}; A) \in O_2$ vectors $\hat{X}_A, \hat{Y}_A, \hat{Z}_A \in T|_{\hat{x}}\hat{M}$ such that $\widetilde{\mathsf{Rol}}_q((A^{\overline{T}}\hat{X}_A) \wedge (A^{\overline{T}}\hat{Y}_A)) = 0$ and that $\hat{\star}\hat{X}_A, \hat{\star}\hat{Z}_A, \hat{\star}\hat{Z}_A$ are eigenbasis of $\hat{R}|_{\hat{x}}$

with eigenvalues which we call $-\hat{K}_1(\hat{x}), -\hat{K}_2(\hat{x}), -\hat{K}(\hat{x})$, respectively. The condition $\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) = 0$ implies that $K(x) = \hat{K}(\hat{x})$ for every $q = (x, \hat{x}; A) \in O_2$ and also that $AZ_A = \hat{Z}_A$, since $\star(X_A \wedge Y_A) = Z_A$, $\hat{\star}(\hat{X}_A \wedge \hat{Y}_A) = \hat{Z}_A$.

We divide the proof of Proposition 5.5 into several lemmas.

Lemma 5.7 For every $q = (x, \hat{x}; A) \in O_2$ and any orthonormal pair (which exists) $X_A, Y_A \in T|_x M$ such that $\mathsf{Rol}(X_A \wedge Y_A) = 0$ and $X_A, Y_A, Z_A := \star(X_A \wedge Y_A)$ is an oriented orthonormal basis of $T|_x M$, one has with respect to the basis X_A, Y_A, Z_A,

$$R(X_A \wedge Y_A) = \begin{pmatrix} 0 & K_A & \eta_A \\ -K_A & 0 & -\beta_A \\ -\eta_A & \beta_A & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix},$$

$$R(Y_A \wedge Z_A) = \begin{pmatrix} 0 & -\beta_A & \xi_A \\ \beta_A & 0 & K_A^1 \\ -\xi_A & -K_A^1 & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix},$$

$$R(Z_A \wedge X_A) = \begin{pmatrix} 0 & -\eta_A & -K_A^2 \\ \eta_A & 0 & -\xi_A \\ K_A^2 & \xi_A & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} \xi_A \\ -K_A^2 \\ \eta_A \end{pmatrix},$$

$$\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) = 0,$$

$$\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & K_1^{\mathsf{Rol}} \\ \alpha & -K_1^{\mathsf{Rol}} & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} -K_1^{\mathsf{Rol}} \\ -\alpha \\ 0 \end{pmatrix},$$

$$\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}} \\ 0 & 0 & \alpha \\ K_2^{\mathsf{Rol}} & -\alpha & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} -\alpha \\ -K_2^{\mathsf{Rol}} \\ 0 \end{pmatrix}.$$

Here $\eta_A, \beta_A, \xi_A, \alpha, K_1^{\mathsf{Rol}}, K_2^{\mathsf{Rol}}$ depend *a priori* on the basis X_A, Y_A, Z_A and on the point q but the choice of these functions can be made *locally smoothly* on O_2 i.e., every $q \in O_2$ admits an open neighbourhood O_2' in O_2 such that the selection of these functions can be performed smoothly on O_2'.

Proof. Since rank $\mathsf{Rol}_q = 2 < 3$ for $q \in O_2$, it follows that there is a unit vector $\omega_A \in \wedge^2 T|_x M$ such that $\mathsf{Rol}_q(\omega_A) = 0$. But in dimension 3, as mentioned in Appendix, one then has an orthonormal pair $X_A, Y_A \in T|_x M$ such that $\omega_A = X_A \wedge Y_A$. Moreover, the assignments $q \mapsto \omega_A, X_A, Y_A$ can be made locally smoothly. Set $Z_A := \star(X_A \wedge Y_A)$. the fact that $\widetilde{\mathsf{Rol}}_q$ is a symmetric map implies that

$$g(\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A), X_A \wedge Y_A) = g(\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A), Y_A \wedge Z_A) = 0,$$
$$g(\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A), X_A \wedge Y_A) = g(\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A), Z_A \wedge X_A) = 0.$$

□

As a consequence of the previous result and because, for $X, Y \in T|_x M$, one gets

$$A^{\overline{T}} \hat{R}(AX \wedge AY)A = R(X \wedge Y) - \widetilde{\mathsf{Rol}}_q(X \wedge Y),$$

then we have that, w.r.t. the oriented orthonormal basis AX_A, AY_A, AZ_A of $T|_{\hat{x}}\hat{M}$,

$$\hat{\star} A^{\overline{T}}\hat{R}(AX_A \wedge AY_A)A = \begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix},$$
$$\hat{\star} A^{\overline{T}}\hat{R}(AY_A \wedge AZ_A)A = \begin{pmatrix} -K_A^1 + K_1^{\mathsf{Rol}} \\ \xi_A + \alpha \\ \beta_A \end{pmatrix},$$
$$\hat{\star} A^{\overline{T}}\hat{R}(AZ_A \wedge AX_A)A = \begin{pmatrix} \xi_A + \alpha \\ -K_A^2 + K_2^{\mathsf{Rol}} \\ \eta_A \end{pmatrix}. \tag{39}$$

The assumption that rank $\mathsf{Rol}_q = 2$ on O_2 is equivalent to the fact that for any choice of X_A, Y_A, Z_A as above, $\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A)$ and $\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A)$ are linearly independent for every $q = (x, \hat{x}; A) \in O_2$ i.e.

$$K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 \neq 0. \tag{40}$$

We next show that, with any (non-unique) choice of X_A, Y_A as in Lemma 5.7, then $\eta_A = \beta_A = 0$.

Lemma 5.8 Choose any $X_A, Y_A, Z_A = \star(X_A \wedge Y_A)$ as in Lemma 5.7. Then, for every $q = (x, \hat{x}; A) \in O_2$ and any vector fields $X, Y, Z, W \in \mathrm{VF}(M)$, one has

$$\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \in \nu(\mathrm{span}\{\star X_A, \star Y_A\})|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0). \tag{41}$$

Moreover, $\pi_Q|_{O_2}$ is an submersion (onto an open subset of $M \times \hat{M}$), $\dim V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) = 2$ for all $q \in O_2$ and $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Proof. First notice that by Lemma 5.7

$$\begin{pmatrix} \mathsf{Rol}_q(\star X_A) \\ \mathsf{Rol}_q(\star Y_A) \end{pmatrix} = \begin{pmatrix} -K_1^{\mathsf{Rol}} & -\alpha \\ -\alpha & -K_2^{\mathsf{Rol}} \end{pmatrix} \begin{pmatrix} \star X_A \\ \star Y_A \end{pmatrix}$$

for $q = (x, \hat{x}; A) \in O_2$ and since the determinant of the matrix on the right hand side is, at $q \in O_2$, $K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 \neq 0$, as noticed in (40) above, it follows that

$$\star X_A, \star Y_A \in \mathrm{span}\{\mathsf{Rol}_q(\star X_A), \mathsf{Rol}_q(\star Y_A)\}.$$

Next, from Proposition 4.8 we know that, for every $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and every $Z, W \in T|_x M$

$$\nu(\mathsf{Rol}_q(Z \wedge W))|_q \in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}) \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Hence, $\nu(\mathsf{Rol}_q(\star X_A)), \nu(\mathsf{Rol}_q(\star Y_A)) \in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})$ for every $q \in O_2$ and then

$$\nu(A \star X_A), \nu(A \star Y_A) \in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}), \tag{42}$$

for all $q = (x, \hat{x}; A) \in O_2$. We claim that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}$ is a submersion (onto an open subset of $M \times \hat{M}$). Indeed, for any vector field $W \in \mathrm{VF}(M)$ one has $\mathscr{L}_{\mathrm{R}}(W)|_q \in$

$T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for $q=(x,\hat{x};A)\in O_2$ and since the assignments $q\mapsto X_A,Y_A$ can be made locally smoothly, then also $[\mathscr{L}_{\mathrm{R}}(W),\nu(A\star X_A)]|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. But then Proposition 3.36 implies that

$$\begin{aligned}&(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*([\mathscr{L}_{\mathrm{R}}(W),\nu(A\star X_A)]|_q)\\ =&(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*\big(-\mathscr{L}_{\mathrm{NS}}(A(\star X_A)W)|_q+\nu(A\star\mathscr{L}_{\mathrm{R}}(W)|_qX_{(\cdot)})|_q\big)\\ =&(0,-A(\star X_A)W),\end{aligned}$$

where we wrote $X_{(\cdot)}$ as for the map $q\mapsto X_A$. Similarly,

$$(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*([\mathscr{L}_{\mathrm{R}}(W),\nu(A\star Y_A)]|_q)=(0,-A(\star Y_A)W).$$

This shows that for all $q=(x,\hat{x};A)\in O_2$ and $Z,W\in T|_xM$, we have

$$(0,-A(\star X_A)W),(0,-A(\star Y_A)W)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset T|_xM\times T|_{\hat{x}}\hat{M}.$$

Because $\star X_A,\star Y_A$ are linearly independent, this implies that

$$\{0\}\times T|_{\hat{x}}\hat{M}\subset(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Finally, because $\mathscr{L}_{\mathrm{R}}(W)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for any $q=(x,\hat{x};A)\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and any $W\in T|_xM$, and $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*\mathscr{L}_{\mathrm{R}}(W)|_q=(W,AW)$, one also has

$$(W,0)=(W,AW)-(0,AW)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),$$

which implies

$$T|_xM\times\{0\}\subset(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

This proves that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}=\pi_Q|_{O_2}$ is indeed a submersion.

Because O_2 is not open in Q (otherwise $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ would be an open subset of Q), it follows that $\dim O_2\le 8$ and since $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}$ has rank 6, being a submersion, we deduce that for all $q\in O_2$,

$$\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})=\dim O_2-6\le 2.$$

But because of (42) we see that $\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})\ge 2$ i.e.

$$\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})=2,$$

which shows that $\dim O_2=8$, hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=8$ and

$$\mathrm{span}\{\nu(A\star X_A)|_q,\nu(A\star Y_A)|_q\}=V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)),\quad\forall q=(x,\hat{x};A)\in O_2.$$

To conclude the proof, it is enough to notice that since for any $X,Y,Z,W\in\mathrm{VF}(M)$, $\nu(\mathsf{Rol}(X\wedge Y)(A))|_q,\nu(\mathsf{Rol}(Z\wedge W)(A))|_q\in V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, then

$$[\nu(\mathsf{Rol}(X\wedge Y)(\cdot)),\nu(\mathsf{Rol}(Z\wedge W)(\cdot))]|_q\in V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)).$$

□

Lemma 5.9 If one chooses any $X_A,Y_A,Z_A=\star(X_A\wedge Y_A)$ as in Lemma 5.7, then

$$\eta_A=\beta_A=0,\quad\forall q=(x,\hat{x};A)\in O_2.$$

Proof. Fix $q=(x,\hat{x};A)\in O_2$. Choosing in Corollary 4.14 $X,Y\in \mathrm{VF}(M)$ such that $X|_x=X_A$, $Y|_x=Y_A$, we get, since $\mathsf{Rol}_q(X_A\wedge Y_A)=0$,

$$\begin{aligned}
&\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X\wedge Y)(\cdot)),\nu(\mathsf{Rol}(Z\wedge W)(\cdot))\big]\big|_q\\
=&A\big[R(X_A\wedge Y_A),R(Z|_x\wedge W|_x)\big]_{\mathfrak{so}}-\big[\hat{R}(AX_A\wedge AY_A),\hat{R}(AZ|_x\wedge AW|_x)\big]_{\mathfrak{so}}A\\
&+\hat{R}(AX_A,A\widetilde{\mathsf{Rol}}_q(Z|_x\wedge W|_x)Y_A)A+\hat{R}(A\widetilde{\mathsf{Rol}}_q(Z|_x\wedge W|_x)X_A,AY_A)A.
\end{aligned}$$

We compute the right hand side of this formula in in two special cases (a)-(b) below.

(a) Take $Z,W\in \mathrm{VF}(M)$ such that $Z|_x=Y_A$, $W|_x=Z_A$.

In this case, computing the matrices in the basis $\star X_A,\star Y_A,\star Z_A$,

$$\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X\wedge Y)(\cdot)),\nu(\mathsf{Rol}(Z\wedge W)(\cdot))\big]\big|_q\\
=&\big[R(X_A\wedge Y_A),R(Y_A\wedge Z_A)\big]_{\mathfrak{so}}-A^{\overline{T}}\big[\hat{R}(AX_A\wedge AY_A),\hat{R}(AY_A\wedge AZ_A)\big]_{\mathfrak{so}}A\\
&+A^{\overline{T}}\hat{R}(AX_A,A\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A)Y_A)A+A^{\overline{T}}\hat{R}(A\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A)X_A,AY_A)A\\
=&\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}-K_A^1\\ \xi_A\\ \beta_A\end{pmatrix}-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}-K_A^1+K_1^{\mathsf{Rol}}\\ \xi_A+\alpha\\ \beta_A\end{pmatrix}\\
&+A^{\overline{T}}\hat{R}(AX_A,-K_1^{\mathsf{Rol}}AZ_A)A+A^{\overline{T}}\hat{R}(\alpha AZ_A,AY_A)A\\
=&-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}K_1^{\mathsf{Rol}}\\ \alpha\\ 0\end{pmatrix}+K_1^{\mathsf{Rol}}\begin{pmatrix}\xi_A+\alpha\\ -K_A^2+K_2^{\mathsf{Rol}}\\ \eta_A\end{pmatrix}-\alpha\begin{pmatrix}-K_A^1+K_1^{\mathsf{Rol}}\\ \xi_A+\alpha\\ \beta_A\end{pmatrix}\\
=&\begin{pmatrix}-\alpha K_A+K_1^{\mathsf{Rol}}(\xi_A+\alpha)-\alpha(-K_A^1+K_1^{\mathsf{Rol}})\\ K_AK_1^{\mathsf{Rol}}+K_1^{\mathsf{Rol}}(-K_A^2+K_2^{\mathsf{Rol}})-\alpha(\xi_A+\alpha)\\ -\alpha\beta_A+K_1^{\mathsf{Rol}}\alpha_A+K_1^{\mathsf{Rol}}\alpha_A-\alpha\beta_A\end{pmatrix}=\begin{pmatrix}\star\\ \star\\ 2(K_1^{\mathsf{Rol}}\eta_A-\alpha\beta_A)\end{pmatrix}.
\end{aligned}$$

By Lemma 5.8 the right hand side should belong to the span of $\star X_A,\star Y_A$ which implies

$$K_1^{\mathsf{Rol}}\eta_A-\alpha\beta_A=0. \tag{43}$$

(b) Take $Z,W\in \mathrm{VF}(M)$ such that $Z|_x=Z_A$, $W|_x=X_A$.

Again, computing w.r.t. the basis $\star X_A, \star Y_A, \star Z_A$, yields

$$
\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)),\nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\
=&\big[R(X_A,Y_A),R(Z_A,X_A)\big]_{\mathfrak{so}} - A^{\overline{T}}\big[\hat{R}(AX_A,AY_A),\hat{R}(AZ_A,AX_A)\big]_{\mathfrak{so}}A \\
&+A^{\overline{T}}\hat{R}(AX_A,A\widetilde{\mathsf{Rol}}_q(Z_A,X_A)Y_A)A + A^{\overline{T}}\hat{R}(A\widetilde{\mathsf{Rol}}_q(Z_A,X_A)X_A,AY_A)A \\
=&\Big(\begin{pmatrix}\beta_A\\ \alpha_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\xi_A\\ -K_A^2\\ \eta_A\end{pmatrix}-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\xi_A+\alpha\\ -K_A^2+K_2^{\mathsf{Rol}}\\ \eta_A\end{pmatrix}\Big) \\
&+A^{\overline{T}}\hat{R}(AX_A,-\alpha AZ_A)A + A^{\overline{T}}\hat{R}(K_2^{\mathsf{Rol}}AZ_A,AY_A)A \\
=&-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\alpha\\ K_2^{\mathsf{Rol}}\\ 0\end{pmatrix}+\alpha\begin{pmatrix}\xi_A+\alpha\\ -K_A^2+K_2^{\mathsf{Rol}}\\ \eta_A\end{pmatrix}-K_2^{\mathsf{Rol}}\begin{pmatrix}-K_A^1+K_1^{\mathsf{Rol}}\\ \xi_A+\alpha\\ \beta_A\end{pmatrix} \\
=&\begin{pmatrix}-K_AK_2^{\mathsf{Rol}}+\alpha(\xi_A+\alpha)-K_2^{\mathsf{Rol}}(-K_A^1+K_1^{\mathsf{Rol}})\\ \alpha K_A+\alpha(-K_A^2+K_2^{\mathsf{Rol}})-K_2^{\mathsf{Rol}}(\xi_A+\alpha)\\ -\beta_AK_2^{\mathsf{Rol}}+\alpha\eta_A+\alpha\eta_A-K_2^{\mathsf{Rol}}\beta_A\end{pmatrix}=\begin{pmatrix}\star\\ \star\\ 2(\alpha\eta_A-\beta_AK_2^{\mathsf{Rol}})\end{pmatrix}.
\end{aligned}
$$

Since the right hand side belongs to the span of $\star X_A, \star Y_A$, by Lemma 5.8, we obtain

$$\alpha\eta_A - K_2^{\mathsf{Rol}}\beta_A = 0. \tag{44}$$

Combining Equations (43) and (44) we get

$$\begin{pmatrix}K_1^{\mathsf{Rol}} & -\alpha\\ \alpha & -K_2^{\mathsf{Rol}}\end{pmatrix}\begin{pmatrix}\eta_A\\ \beta_A\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

According to Eq. (40) the determinant of the 2×2-matrix on the left hand side does not vanish, which implies that $\eta_A=\beta_A=0$. The proof is finished. □

Lemma 5.10 For every $q=(x,\hat{x};A)\in O_2$, there are orthonormal $X_A, Y_A\in T|_xM$ such that $X_A, Y_A, Z_A=\star(X_A\wedge Y_A)$ is an oriented orthonormal basis of $T|_xM$ with respect to which in Lemma 5.7 one has

$$\eta_A=\beta_A=\xi_A=0,$$

i.e., $\star X_A, \star Y_A, \star Z_A$ are eigenvectors of $R|_x$.

Proof. Fix $q=(x,\hat{x};A)\in O_2$, choose any $X_A, Y_A, Z_A=\star(X_A\wedge Y_A)$ as in Lemma 5.7 and suppose $\xi_A\neq 0$ (otherwise we are done). By Lemma 5.9, one has $\eta_A=\beta_A=0$, meaning that $\star Z_A$ is an eigenvector of $R|_x$. For $t\in\mathbb{R}$, set

$$\begin{pmatrix}X_A(t)\\ Y_A(t)\end{pmatrix}:=\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}X_A\\ Y_A\end{pmatrix}.$$

Then clearly $Z_A(t):=\star(X_A(t)\wedge Y_A(t))=\star(X_A\wedge Y_A)=Z_A$, and $X_A(t), Y_A(t), Z_A(t)$ is an orthonormal positively oriented basis of $T|_xM$. Since

$$\mathsf{Rol}_q(\star Z_A(t))=\mathsf{Rol}_q(\star Z_A)=0,$$

Lemma 5.9 implies that $\eta_A(t),\beta_A(t)=0$ if one writes $\eta_A(t),\beta_A(t),\xi_A(t)$ for the coefficients of matrices in Lemma 5.7 w.r.t $X_A(t), Y_A(t), Z_A(t)$. Our goal is to show that $\xi_A(t)=0$ for some $t\in\mathbb{R}$.

First of all $\star Z_A(t) = \star Z_A$ is a unit eigenvector of $R|_x$ which does not depend on t. On the other hand, $R|_x$ is a symmetric map $\wedge^2 T|_x M \to \wedge^2 T|_x M$, so it has two orthogonal unit eigenvectors, say, u_1, u_2 in $(\star Z_A)^\perp = \star(Z_A^\perp)$. Thus $u_1, u_2, \star Z_A$ forms an orthonormal basis of $\wedge^2 T|_x M$, which we may assume to be oriented (otherwise swap u_1, u_2). Then $\operatorname{span}\{u_1, u_2\} = \star Z_A^\perp = \operatorname{span}\{\star X_A, \star Y_A\}$ and there exists $t_0 \in \mathbb{R}$ such that $\star X_A(t_0) = u_1$, $\star Y_A(t_0) = u_2$ Since $R|_x(\star X_A(t_0)) = -K_1 \star X_A(t_0)$, $R|_x(\star Y_A(t_0)) = -K_2 \star Y_A(t_0)$, we have $\xi_A(t_0) = 0$ as well as $\eta_A(t_0) = \beta_A(t_0) = 0$. □

Remark 5.11 Notice that the choice of Z_A can be made locally smoothly on O_2 but, at this stage of the argument, it is not clear that one can choose X_A, Y_A, with $\xi_A = 0$, locally smoothly on O_2. However, it will be the case cf. Corollary 5.15.

We now aim to prove, roughly speaking, that the eigenvalue $-K$ must be double for both spaces (M, g), $(\hat{M}, \hat{g})$ if neither one of them has constant curvature.

Lemma 5.12 If the eigenspace at $x_1 \in \pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M}(O_2)$ corresponding to the eigenvalue $-K(x_1)$ of the curvature operator R is of dimension one, then $(\hat{M}, \hat{g})$ has constant curvature $K(x_1)$ on the open set $\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),\hat{M}}(\pi^{-1}_{\mathcal{O}_{\mathcal{D}_R}(q_0),M}(x_1))$ of $\hat{M}$. The claim also holds with the roles of (M, g) and $(\hat{M}, \hat{g})$ interchanged.

Proof. Suppose that at $x_1 \in \pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M}(O_2)$ the eigenvalue $-K(x_1)$ has multiplicity 1. By continuity, the $-K(\cdot)$-eigenspace of R is of dimension one on an open neighbourhood U of x_1. Since this eigenspace depends smoothly on a point of M, we may choose, taking U smaller around x_1 if needed, positively oriented orthonormal smooth vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on U such that $\star\tilde{Z} = \tilde{X} \wedge \tilde{Y}$ spans the $-K(\cdot)$-eigenspace of R at each point of U. Taking arbitrary $q' = (x', \hat{x}'; A') \in (\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M})^{-1}(U) \cap O_2$ and letting $X_{A'}, Y_{A'}, Z_{A'}$ be the vectors provided by Theorem 5.7 at q, we have that the $-K(x')$-eigenspace of $R|_{x'}$ is also spanned by $X_{A'} \wedge Y_{A'}$. By the orthonormality and orientability, $X_{A'} \wedge Y_{A'} = \tilde{X}|_{x'} \wedge \tilde{Y}|_{x'}$ from which $\tilde{Z}|_{x'} = Z_{A'}$ and $\mathsf{Rol}(\tilde{X}|_{x'} \wedge \tilde{Y}|_{x'})(A') = \mathsf{Rol}(X_{A'} \wedge Y_{A'})(A') = 0$. Now fix, for a moment, $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M})^{-1}(U) \cap O_2$. By replacing $\tilde{X}$ by $\cos(t)\tilde{X} + \sin(t)\tilde{Y}$ and $\tilde{Y}$ by $-\sin(t)\tilde{X} + \cos(t)\tilde{Y}$ on U for a certain constant $t = t_x \in \mathbb{R}$, we may assume that $\tilde{X}|_x = X_A$, $\tilde{Y}|_x = Y_A$. Since, as we just proved, for all $(x', \hat{x}'; A') \in (\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M})^{-1}(U) \cap O_2$, one has

$$\mathsf{Rol}(\tilde{X}|_{x'} \wedge \tilde{Y}|_{x'})(A') = 0,$$

then the vector field $\nu(\mathsf{Rol}(\tilde{X} \wedge \tilde{Y})(\cdot)) \in \mathrm{VF}(\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M})$ vanishes identically i.e. $\nu(\mathsf{Rol}(\tilde{X} \wedge \tilde{Y})(\cdot)) = 0$ on $(\pi_{\mathcal{O}_{\mathcal{D}_R}(q_0),M})^{-1}(U) \cap O_2$. Therefore, the computation in part (a) of the proof of Lemma 5.9 (replace $X \to \tilde{X}$, $Y \to \tilde{Y}$, $Z \to \tilde{Y}$, $W \to \tilde{Z}$ there; recall also that $\xi_A = 0$ by the choice of X_A, Y_A, Z_A) gives, by noticing also that here $K_A = K(x)$, $K_A^1 = K_1(x)$ and $K_A^2 = K_2(x)$,

$$\begin{aligned} 0 =& A^{\overline{T}} \nu|_q^{-1} \big[\nu(\mathsf{Rol}(\tilde{X}, \tilde{Y})(\cdot)), \nu(\mathsf{Rol}(\tilde{Y}, \tilde{Z})(\cdot))\big]\big|_q \\ =& \begin{pmatrix} -\alpha K_A + \alpha K_1^{\mathsf{Rol}} - \alpha(-K_A^1 + K_1^{\mathsf{Rol}}) \\ K_A K_1^{\mathsf{Rol}} + K_1^{\mathsf{Rol}}(-K_A^2 + K_2^{\mathsf{Rol}}) - \alpha^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(-K + K_1) \\ K_1^{\mathsf{Rol}}(K - K_2 + K_2^{\mathsf{Rol}}) - \alpha^2 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly, the computation in part (b) of the proof of Lemma 5.9 (now replace $X \to \tilde{X}$, $Y \to \tilde{Y}$, $Z \to \tilde{Z}$, $W \to \tilde{X}$ there) gives,

$$\begin{aligned} 0 =& A^{\overline{T}} \nu|_q^{-1} \big[\nu(\mathsf{Rol}(\tilde{X}, \tilde{Y})(\cdot)), \nu(\mathsf{Rol}(\tilde{Z}, \tilde{X})(\cdot))\big]\big|_q \\ =& \begin{pmatrix} -K_A K_2^{\mathsf{Rol}} + \alpha^2 - K_2^{\mathsf{Rol}}(-K_A^1 + K_1^{\mathsf{Rol}}) \\ \alpha K_A + \alpha(-K_A^2 + K_2^{\mathsf{Rol}}) - K_2^{\mathsf{Rol}}\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} K_2^{\mathsf{Rol}}(-K + K_1 - K_1^{\mathsf{Rol}}) + \alpha^2 \\ \alpha(K - K_2) \\ 0 \end{pmatrix}. \end{aligned}$$

By assumption, $-K(\cdot)$ is an eigenvalue of R distinct from the other eigenvalues $-K_1(\cdot)$, $-K_2(\cdot)$ on U. Hence we must have $\alpha(q) = 0$. Since $0 \neq K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 = K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q)$, we have $K_1^{\mathsf{Rol}}(q) \neq 0$ and $K_2^{\mathsf{Rol}}(q) \neq 0$, hence $K(x) - K_1(x) + K_1^{\mathsf{Rol}}(q) = 0$ and $K(x) - K_2(x) + K_2^{\mathsf{Rol}}(q) = 0$ for $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$. Since $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$ was arbitrary, we have proven that

$$\begin{aligned} \alpha(q) &= 0, \\ -K_1(x) + K_1^{\mathsf{Rol}}(q) &= -K(x), \\ -K_2(x) + K_2^{\mathsf{Rol}}(q) &= -K(x), \end{aligned}$$

for all $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$.

Looking at (38) reveals that for every $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$, the three 2-vectors $AX_A \wedge AY_A$, $AY_A \wedge AZ_A$ and $AZ_A \wedge AX_A$ are mutually orthonormal eigenvectors of $\hat{R}|_{\hat{x}}$ all corresponding to the eigenvalue $-K(x)$, i.e, $(\hat{M}, \hat{g})$ has constant curvature $-K(x)$ at $\hat{x}$. Since $x_1 \in U$, the Riemannian space $(\hat{M}, \hat{g})$ has constant curvature $-K(x_1)$ at all points $\hat{x}_1 \in \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}\big((\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1) \cap O_2\big)$.

Finally, we argue that $\hat{S} := \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}\big((\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1) \cap O_2\big)$ is an open subset of $\hat{M}$. It is enough to show that $\pi_{Q,\hat{M}}|_{\hat{O}_{x_1}} : \hat{O}_{x_1} \to \hat{M}$ is a submersion where $\hat{O}_{x_1} := (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1) \cap O_2$ is a submanifold of O_2. To begin with, recall that $\pi_Q|_{O_2}$ is an submersion onto an open subset of $M \times \hat{M}$ by Lemma 5.8. Let $q \in \hat{O}_{x_1}$ and write $q = (x_1, \hat{x}; A)$. Choose any frame $\hat{X}_1, \hat{X}_2, \hat{X}_3$ of $T|_{\hat{x}}\hat{M}$. Then there are $\hat{W}_i \in T|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $i = 1, 2, 3$, such that $(\pi_Q)_*(\hat{W}_i) = (0, \hat{X}_i)$. In particular, $(\pi_{Q,M})_*(\hat{W}_i) = 0$, so $\hat{W}_i \in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})$. But since $T|_q\hat{O}_{x_1} = V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})$, we have $\hat{W}_i \in T|_q\hat{O}_{x_1}$ and thus $\hat{X}_i = (\pi_{Q,\hat{M}})_*\hat{W}_i \in \mathrm{im}(\pi_{Q,\hat{M}}|_{\hat{O}_{x_1}})_*$, which proves the claim and finishes the proof. □

Remark 5.13 It is actually obvious that the eigenvalue $-K(\cdot)$ of R of (M, g) is constant, equal to $K(x_1)$ say, in a some neighbourhood of x_1 in M, if $-K(x_1)$ were a single eigenvalue of $R|_{x_1}$. Even more is true: One could show, even without questioning whether $-K(\cdot)$ is a single eigenvalue for R and/or $\hat{R}$ or not, that on $\pi_{Q,M}(O_2)$ and $\pi_{Q,\hat{M}}(O_2)$ this eigenvalue is actually locally constant (i.e. the function $K(\cdot)$ is locally constant). This fact will be observed e.g. in Lemma 5.16 below.

Lemma 5.14 The following holds:

(1) For any $q_1 = (x_1, \hat{x}_1; A_1) \in O_2$, $(\hat{M}, \hat{g})$ cannot have constant curvature at $\hat{x}_1$.

(2) There does not exist a $q_1 = (x_1, \hat{x}_1; A_1) \in O_2$ such that $-K(x_1)$ is a single eigenvalue of $R|_{x_1}$.

This also holds with the roles of (M,g) and $(\hat{M},\hat{g})$ interchanged.

Proof. (1) Suppose $(\hat{M},\hat{g})$ has a constant curvature $\hat{K}$ at $\hat{x}_1$. Let E_1,E_2,E_3 be an oriented orthonormal frame on a neighbourhood U of x_1 such that $\star E_1|_{x_1},\star E_2|_{x_1},\star E_3|_{x_1}$ are eigenvectors of R at x_1 with eigenvalues $-K_1(x_1)$, $-K_2(x_1)$, $-K(x_1)$, respectively, where these eigenvalues are as in Proposition 5.5. As we have noticed, $\hat{K}=K(x_1)$. Because $\hat{R}|_{\hat{x}_1}=-\hat{K}\mathrm{id}_{\wedge^2 T|_{\hat{x}_1}\hat{M}}$, one has

$$\begin{aligned}
\widetilde{\mathsf{Rol}}_{q_1}(\star E_1) =&(-K_1(x_1)+\hat{K})\star E_1|_{x_1},\\
\widetilde{\mathsf{Rol}}_{q_1}(\star E_2) =&(-K_2(x_1)+\hat{K})\star E_2|_{x_1},\\
\widetilde{\mathsf{Rol}}_{q_1}(\star E_3) =&(-K(x_1)+\hat{K})\star E_3|_{x_1}=0.
\end{aligned}$$

Since $\operatorname{rank}\widetilde{\mathsf{Rol}}_{q_1}=2$, we have $-K_1(x_1)+\hat{K}\neq 0$, $-K_2(x_1)+\hat{K}\neq 0$.

Because the vector fields $\nu(\mathsf{Rol}(\star E_1)(\cdot))$, $\nu(\mathsf{Rol}(\star E_2)(\cdot))$ are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ on $O_2':=O_2\cap\pi_{Q,M}^{-1}(U)$, so is their Lie bracket. According to Proposition 3.37, the value of this bracket at q_1 is equal to

$$[\nu(\mathsf{Rol}(\star E_1)(\cdot)),\nu(\mathsf{Rol}(\star E_2)(\cdot))]|_{q_1}=(-K_1(x_1)+\hat{K})(-K_2(x_1)+\hat{K})\nu(A\star E_3)|_{q_1}.$$

Hence $\nu(\mathsf{Rol}(\star E_1)(\cdot)$, $\nu(\mathsf{Rol}(\star E_2)(\cdot)$, $[\nu(\mathsf{Rol}(\star E_1)(\cdot)),\nu(\mathsf{Rol}(\star E_2)(\cdot))]$ are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and since they are linearly independent at q_1, hence they are linearly independent on an open neighbourhood of q_1 in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Therefore, from Corollary 4.18, it follows that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open in Q, which is a contradiction.

(2) Suppose $-K(x_1)$ is a single eigenvector of $R|_{x_1}$, where $q_1=(x_1,\hat{x}_1;A_1)\in O_2$. Then, by Lemma 5.12, the space $(\hat{M},\hat{g})$ would have a constant curvature in an open set which is a neighbourhood of $\hat{x}_1$. By Case (1), this leads to a contradiction. □

By the last two lemmas, we may thus assume that for every $q=(x,\hat{x};A)\in O_2$ the common eigenvalue $-K(x)=-\hat{K}(\hat{x})$ of $R|_x$, $\hat{R}|_{\hat{x}}$ has multiplicity two. It has the following consequence.

Corollary 5.15 The assignments $q\mapsto X_A,Y_A,Z_A$ and $q\mapsto K_1^{\mathsf{Rol}}(q),K_2^{\mathsf{Rol}}(q),\alpha(q)$ as in Proposition 5.5 can be made locally smoothly on O_2.

Proof. Let $q_1=(x_1,\hat{x}_1;A_1)\in O_2$. By Lemma 5.14, there are open neighbourhoods $U\ni x_1$ and $\hat{U}\ni\hat{x}_1$ such that the eigenvalues $-K_2(x)$ of $R|_x$ and $-\hat{K}_2(\hat{x})$ of $\hat{R}|_{\hat{x}}$ are both simple. Therefore the map $q\mapsto Y_A$ can be made locally smoothly on O_2 and this is also the case for the map $q\mapsto Z_A$ since it corresponds to the 1-dimensional kernel of $\widetilde{\mathsf{Rol}}_q$ and $X_A=\star(Y_A\wedge Z_A)$. □

Lemma 5.16 For every $q_1=(x_1,\hat{x}_1;A_1)\in O_2$, there are open neighbourhoods $U,\hat{U}$ of $x_1,\hat{x}_1$ and oriented orthonormal frames E_1,E_2,E_3 on M, $\hat{E}_1,\hat{E}_2,\hat{E}_3$ on $\hat{M}$ with respect to which the connections tables are of the form

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},\quad \hat{\Gamma}=\begin{pmatrix}\hat{\Gamma}^1_{(2,3)} & 0 & -\hat{\Gamma}^1_{(1,2)}\\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)}\\ \hat{\Gamma}^1_{(1,2)} & 0 & \hat{\Gamma}^1_{(2,3)}\end{pmatrix},$$

and

$$V(\Gamma^1_{(2,3)})=0,\quad V(\Gamma^1_{(1,2)})=0,\quad \forall V\in E_2|_x^\perp,\quad x\in U,$$
$$\hat{V}(\hat{\Gamma}^1_{(2,3)})=0,\quad \hat{V}(\hat{\Gamma}^1_{(1,2)})=0,\quad \forall \hat{V}\in \hat{E}_2|_{\hat{x}}^\perp,\quad \hat{x}\in \hat{U}.$$

Moreover, $\star E_1,\star E_2,\star E_3$ are eigenvectors of R with eigenvalues $-K,-K_2(\cdot),-K$ on U and similarly $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}\hat{E}_3$ are eigenvectors of $\hat{R}$ with eigenvalues $-K,-\hat{K}_2(\cdot),-K$ on $\hat{U}$, where $K\in\mathbb{R}$ is constant.

Proof. As we just noticed, for every $q=(x,\hat{x};A)\in O_2$, the common eigenvalue $-K(x)=-\hat{K}(\hat{x})$ of $R|_x$ and $\hat{R}|_{\hat{x}}$ has multiplicity equal to two.

Fix $q_1=(x_1,\hat{x}_1;A_1)\in O_2$ and let E_1,E_2,E_3 (resp. $\hat{E}_1,\hat{E}_2,\hat{E}_3$) be an orthonormal oriented frame of (M,g) defined on an open set $U\ni x_1$ (resp. $\hat{U}\ni\hat{x}_1$) such that $U\times\hat{U}\subset\pi_Q(O_2)$ and that $\star E_1,\star E_2,\star E_3$ (resp. $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}\hat{E}_3$) are eigenvectors with eigenvalues $-K_1(\cdot),-K_2(\cdot),-K(\cdot)$ (resp. $-\hat{K}_1(\cdot),-\hat{K}_2(\cdot),-\hat{K}_3(\cdot)$) on U (resp. $\hat{U}$) as given by Proposition 5.5. Since $-K$ has multiplicity two on U (resp. $-\hat{K}$ has multiplicity twoon $\hat{U}$), we assume that $K_1(\cdot)=K(\cdot)\neq K_2(\cdot)$ everywhere on U, (resp. $\hat{K}_1(\cdot)=\hat{K}(\cdot)\neq\hat{K}_2(\cdot)$ everywhere on $\hat{U}$) without loss of generality. Recall that $K(x)=\hat{K}(\hat{x})$ for all $q=(x,\hat{x};A)\in O_2$ by Proposition 5.5 (and the remark that follows it) and hence for all $x\in U$, $\hat{x}\in\hat{U}$, $K(x)=\hat{K}(\hat{x})$. Taking $U,\hat{U}$ to be connected, this immediately implies that both K and $\hat{K}$ are constant functions on U and $\hat{U}$. We denote the common constant value simply by K.

Let X_A,Y_A,Z_A be chosen as in Proposition 5.5 for every $q=(x,\hat{x};A)\in O_2$. Then, since $\star Y_A$ is a unit eigenvector of $R|_x$ corresponding to the single eigenvalue $-K_2(x)$, we must have $E_2|_x=\pm Y_A$ and since $\nu(A\star Y_A)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, by Lemma 5.8, it follows that for every $q=(x,\hat{x};A)\in O_2$, the vector $\nu(A\star E_2|_x)|_q$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This, together with Proposition C.18 (given in Appendix), proves the claim for (M,g). Symmetrically (working in $Q(\hat{M},M)$) the claim also holds for $(\hat{M},\hat{g})$. The proof is complete. □

We finally aim at proving that, using the notations of the previous lemma, $\Gamma^1_{(2,3)}(x)=\hat{\Gamma}^1_{(2,3)}(\hat{x})$ for all $(x,\hat{x})\in\pi_Q(O_2')$, where $O_2'=\pi_Q^{-1}(U\times\hat{U})\cap O_2$ and $U,\hat{U}$ are the domains of definition of orthonormal frames E_1,E_2,E_3 and $\hat{E}_1,\hat{E}_2,\hat{E}_3$ as given by Lemma 5.16 above.

To this end, we define $\theta:O_2'\to\mathbb{R}$ (restricting to smaller sets U, $\hat{U}$ if necessary) to be a smooth function such that for all $q=(x,\hat{x};A)\in O_2'$,

$$\begin{aligned}X_A=&\cos(\theta(q))E_1+\sin(\theta(q))E_3,\\ Z_A=&-\sin(\theta(q))E_1+\cos(\theta(q))E_3,\end{aligned}$$

where X_A,Z_A (and also Y_A) are chosen using Proposition 5.5. Indeed, this is well defined since X_A,Z_A lie in the plane $Y_A^\perp=E_2|_x^\perp$ as do also $E_1|_x,E_3|_x$, for all $q=(x,\hat{x};A)\in O_2'$. To simplify the notation, we write $c_\theta:=\cos(\theta(q))$ and $s_\theta:=\sin(\theta(q))$ as well as $\Gamma^i_{(j,k)}=\Gamma^i_{(j,k)}(x)$, when there is no room for confusion. We will be always working on O_2' if not mentioned otherwise. Moreover, it is convenient to denote the vector field E_2 of M by Y in the computations that follow (since $E_2|_x$ is parallel to Y_A for all $q\in O_2'$, this notation is justified). We will do computations on the "side of M" but the results are, by symmetry, always valid for $\hat{M}$ as well. We will make use of the following formulas which are easily verified (see Lemma 3.33),

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta - c_\theta\Gamma^1_{(3,1)} - s_\theta\Gamma^3_{(3,1)})Z_A + \Gamma^1_{(1,2)}Y,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)})Z_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta + s_\theta\Gamma^1_{(3,1)} - c_\theta\Gamma^3_{(3,1)})Z_A + \Gamma^1_{(2,3)}Y,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q Y &= -\Gamma^1_{(1,2)}X_A + \Gamma^1_{(2,3)}Z_A,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q Y &= 0,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Y &= -\Gamma^1_{(2,3)}X_A - \Gamma^1_{(1,2)}Z_A,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \Gamma^1_{(2,3)}Y,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)})X_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A + \Gamma^1_{(1,2)}Y.
\end{aligned} \tag{45}
$$

Remark 5.17 Notice that $\nu(A\star Z_A)|_q$ is not tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for any $q=(x,\hat{x};A)\in O_2'$. Indeed, otherwise there would be an open neighbourhood $O\subset O_2'$ of q such that for all $q'=(x',\hat{x}';A')$ the vectors $\nu(A'\star X_{A'})|_{q'},\nu(A'\star Y)|_{q'},\nu(A'\star Z_{A'})|_{q'}$ would span $V|_{q'}(\pi_Q)$ while being tangent to $T|_{q'}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which implies $V|_{q'}(\pi_Q)\subset T|_{q'}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Then Corollary 4.18 would imply that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open, which is not the case. We will use this fact frequently in what follows.

Taking U, $\hat{U}$ smaller if necessary, we may also assume that θ is actually defined not only on O_2' but on an open neighbourhood $\tilde{O}_2'$ of O_2 in Q. We will make this technical assumption to be able to write e.g. $\nu(A\star Z_A)|_q\theta$ whenever needed.

Lemma 5.18 For every $q=(x,\hat{x};A)\in O_2'$ we have

$$
\begin{aligned}
\nu(A\star Y)|_q\theta &= 1,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta &= c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)},\\
\mathscr{L}_{\mathrm{R}}(Y)|_q\theta &= \Gamma^2_{(3,1)} - \Gamma^1_{(2,3)}.
\end{aligned}
$$

Moreover, if one defines for $q=(x,\hat{x};A)\in O_2'$,

$$
\begin{aligned}
F_X|_q &:= \mathscr{L}_{\mathrm{NS}}(X_A)|_q - \Gamma^1_{(1,2)}\nu(A\star Z_A)|_q,\\
F_Z|_q &:= \mathscr{L}_{\mathrm{NS}}(Z_A)|_q - \Gamma^1_{(2,3)}\nu(A\star Z_A)|_q,
\end{aligned}
$$

then F_X, F_Z are smooth vector fields on O_2' tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

Proof. We begin by showing that $\nu(A\star Y)|_q\theta=1$. Indeed, we have for every $q=(x,\hat{x};A)\in O_2'$ that $\hat{g}(AZ_A,\hat{E}_2)=0$. Differentiating this w.r.t. $\nu(A\star Y)|_q$ yields

$$
0=\hat{g}(A(\star Y)Z_A,\hat{E}_2)-\nu(A\star Y)|_q\theta\hat{g}(AX_A,\hat{E}_2)=\hat{g}(AX_A,\hat{E}_2)(1-\nu(A\star Y)|_q\theta).
$$

We show that $\hat{g}(AX_A,\hat{E}_2)\neq 0$, whence $\nu(A\star Y)|_q\theta=1$. Indeed, if it were the case, then $AX_A\in E_2^\perp$ and hence $\hat{\star}(AX_A)$ would be an eigenvector of $\hat{R}|_{\hat{x}}$ with eigenvalue $-K$. This would then imply that

$$
\widetilde{\mathsf{Rol}}_q(\star X_A)=R(\star X_A)-A^{\overline{T}}\hat{R}(\hat{\star}(AX_A))A=-K\star X_A+KA^{\overline{T}}(\hat{\star}(AX_A)A=0.
$$

Because, $\widetilde{\mathsf{Rol}}_q(X_A \wedge Y) = 0$ as well, we see that $\widetilde{\mathsf{Rol}}_q$ has rank ≤ 1 as a map $\wedge^2 T|_x M \to \wedge^2 T|_x M$, which is a contradiction since $q \in O_2' \subset O_2$ and O_2 is, by definition, the set of points of the orbit where $\widetilde{\mathsf{Rol}}_q$ has rank 2. This contradiction establishes the above claim.

Next we compute the Lie brackets

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(Y), \nu((\cdot)\star X_{(\cdot)})]|_q =& -\mathscr{L}_{\mathrm{NS}}(A(\star X_A)Y)|_q + \nu(A\star \mathscr{L}_{\mathrm{R}}(Y)|_q X_{(\cdot)})|_q \\
=& -\mathscr{L}_{\mathrm{NS}}(AZ_A)|_q + (\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)})\nu(A\star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot)\star Y)]|_q =& -\mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q X_{(\cdot)})|_q - \nu(A\star Y)|_q\theta\mathscr{L}_{\mathrm{NS}}(A(\star Y)X_A)|_q \\
&+ \nu(A\star(c_\theta(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) + s_\theta(-\Gamma^1_{(2,3)}E_1 - \Gamma^1_{(1,2)}E_3)))|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q X_{(\cdot)})|_q + \mathscr{L}_{\mathrm{NS}}(AZ_A)|_q \\
&- \Gamma^1_{(1,2)}\nu(A\star X_A)|_q + \Gamma^1_{(2,3)}\nu(A\star Z_A)|_q,
\end{aligned}$$

which sum is equal to

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(Y), \nu((\cdot)\star X_{(\cdot)})]|_q + [\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot)\star Y)]|_q \\
=&(\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)} + \Gamma^1_{(2,3)})\nu(A\star Z_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q X_{(\cdot)})|_q \\
&- \Gamma^1_{(1,2)}\nu(A\star X_A)|_q.
\end{aligned}$$

Since this has to be tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, we get that the $\nu(A\star Z_A)|_q$-component vanished i.e.,

$$\mathscr{L}_{\mathrm{R}}(Y)|_q\theta = \Gamma^2_{(3,1)} - \Gamma^1_{(2,3)}.$$

Next compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot)\star X_{(\cdot)})]|_q =& -\nu(A\star X_A)|_q\theta\mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{NS}}(A\underbrace{(\star X_A)X_A}_{=0})|_q \\
&+ \nu\big(A\star\big((\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta - c_\theta\Gamma^1_{(3,1)} - s_\theta\Gamma^3_{(3,1)})Z_A) + \Gamma^1_{(1,2)}Y\big)\big|_q,
\end{aligned}$$

and so we must have again that the $\nu(A\star Z_A)|_q$-component is zero i.e.,

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta = c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}.$$

Since $\mathscr{L}_{\mathrm{NS}}(AZ_A)|_q = \mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{NS}}(Z_A)|_q$, $[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot)\star Y)]|_q$ can be written as

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot)\star Y)]|_q =& -F_Z|_q + \mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q X_{(\cdot)})|_q - \Gamma^1_{(1,2)}\nu(A\star X_A)|_q \\
=& -F_Z|_q - \Gamma^1_{(1,2)}\nu(A\star X_A)|_q,
\end{aligned}$$

which proves that F_Z, as defined in the statement, is indeed tangent to the orbit on O_2'. To show that F_X is also tangent to the orbit we compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}), \nu((\cdot)\star Y)]|_q =& -\mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q Z_{(\cdot)})|_q - \nu(A\star Y)|_q\theta\mathscr{L}_{\mathrm{NS}}(A(\star Y)Z_A)|_q \\
&+ \nu(A\star(-s_\theta(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) + c_\theta(-\Gamma^1_{(2,3)}E_1 - \Gamma^1_{(1,2)}E_3)))|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q Z_{(\cdot)})|_q - \mathscr{L}_{\mathrm{NS}}(AX_A)|_q \\
&- \Gamma^1_{(1,2)}\nu(A\star Z_A)|_q - \Gamma^1_{(2,3)}\nu(A\star X_A)|_q \\
=&F_X|_q - \mathscr{L}_{\mathrm{R}}(X_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A\star Y)|_q Z_{(\cdot)})|_q - \Gamma^1_{(2,3)}\nu(A\star X_A)|_q \\
=&F_X|_q - \Gamma^1_{(2,3)}\nu(A\star X_A)|_q,
\end{aligned}$$

which finishes the proof. □

Lemma 5.19 For all $(x, \hat{x}) \in \pi_Q(O_2')$ one has

$$\Gamma^1_{(2,3)}(x) = \hat{\Gamma}^1_{(2,3)}(\hat{x}).$$

Proof. We begin by observing that for all $q = (x, \hat{x}; A) \in O_2'$ one has $\hat{g}(AZ_A, \hat{E}_2) = 0$. Indeed, AZ_A and $\hat{E}_2|_{\hat{x}}$ are eigenvectors of $\hat{R}|_{\hat{x}}$ corresponding to non-equal eigenvalues $-K$ and $-\hat{K}_2(\hat{x})$, hence they must be orthogonal. Since $AZ_A \in \hat{E}_2|_{\hat{x}}^{\perp}$, there is a $\hat{\theta} = \hat{\theta}(q)$, for all $q = (x, \hat{x}; A) \in O_2'$, such that

$$AZ_A = -s_{\hat{\theta}}\hat{E}_1 + c_{\hat{\theta}}\hat{E}_3.$$

Because $AX_A, AY \in (AZ_A)^{\perp}$, there exits also a $\hat{\phi} = \hat{\phi}(q)$ such that

$$\begin{aligned} AX_A =& c_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + s_{\hat{\phi}}\hat{E}_2, \\ AY =& - s_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + c_{\hat{\phi}}\hat{E}_2. \end{aligned}$$

Moreover, Lemma 5.18 along with Eq. (45) implies that $\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)}$ simplifies to

$$\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)} = \Gamma^1_{(2,3)}(x) X_A.$$

Therefore, differentiating $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$, one obtains

$$\begin{aligned} 0 =& \mathscr{L}_{\mathrm{R}}(Y)|_q \hat{g}((\cdot)Z_{(\cdot)}, \hat{E}_2) = \hat{g}(A\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AY}\hat{E}_2) \\ =& \Gamma^1_{(2,3)}\hat{g}(AX_A, \hat{E}_2) + \hat{g}\big(AZ_A, -s_{\hat{\phi}}c_{\hat{\theta}}(-\hat{\Gamma}^1_{(1,2)}\hat{E}_1 + \hat{\Gamma}^1_{(2,3)}\hat{E}_3) - s_{\hat{\phi}}s_{\hat{\theta}}(-\hat{\Gamma}^1_{(2,3)}\hat{E}_1 - \hat{\Gamma}^1_{(1,2)}\hat{E}_3)\big) \\ =& s_{\hat{\phi}}\Gamma^1_{(2,3)} - s_{\hat{\phi}}\hat{g}\big(AZ_A, \hat{\Gamma}^1_{(2,3)}AZ_A - \hat{\Gamma}^1_{(1,2)}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3)\big) \\ =& s_{\hat{\phi}}(\Gamma^1_{(2,3)}(x) - \hat{\Gamma}^1_{(2,3)}(\hat{x})). \end{aligned}$$

We claim that $\sin(\hat{\phi}(q)) \neq 0$ for $q = (x, \hat{x}; A) \in O_2'$, which implies that $\Gamma^1_{(2,3)}(x) - \hat{\Gamma}^1_{(2,3)}(\hat{x}) = 0$ and finishes the proof. Indeed, $\sin(\hat{\phi}(q)) = 0$ would mean that $AX_A = \pm(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3)$, thus $AX_A \in \hat{E}_2^{\perp}$. By the argument at the beginning of the proof of Lemma 5.18, this would be a contradiction. □

Corollary 5.20 The following holds.

(i) If for some $(x_1, \hat{x}_1) \in \pi_Q(O_2')$, one has $\Gamma^1_{(2,3)}(x_1) \neq 0$ (or $\hat{\Gamma}^1_{(2,3)}(\hat{x}_1) \neq 0$), there are open neighbourhoods $U' \ni x_1$, $\hat{U}' \ni \hat{x}_1$ such that (U', g), $(\hat{U}', \hat{g})$ are both of class $\mathcal{M}_\beta$ for $\beta = \Gamma^1_{(2,3)}(x_1)$ (or $\beta = \hat{\Gamma}^1_{(2,3)}(\hat{x}_1)$).

(ii) If for some $(x_1, \hat{x}_1) \in \pi_Q(O_2')$, one has $\Gamma^1_{(2,3)}(x_1) = 0$ (or $\hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = 0$), there are open neighbourhoods $U' \ni x_1$, $\hat{U}' \ni \hat{x}_1$ such that $U' \times \hat{U}' \subset \pi_Q(O_2')$ and isometries $F : (I \times N, h_f) \to (U, g)$, $\hat{F} : (I \times \hat{N}, \hat{h}_{\hat{f}}) \to (\hat{U}, \hat{g})$, where $I \subset \mathbb{R}$ is an open interval, such that

$$\frac{f''(t)}{f(t)} = -K = \frac{\hat{f}''(t)}{\hat{f}(t)}, \quad \forall t \in I.$$

Proof. Let $U', \hat{U}'$ be connected neighbourhoods of $x_1, \hat{x}_1$ such that $U' \times \hat{U}' \subset \pi_Q(O_2')$ (recall that by Lemma 5.8, $\pi_Q(O_2')$ is open in $M \times \hat{M}$).

(i) Set $\beta = \Gamma^1_{(2,3)}(x_1) \neq 0$. By Lemma 5.19, one has for every $x \in U'$, $\hat{x} \in \hat{U}'$ that

$$\begin{aligned} \hat{\Gamma}^1_{(2,3)}(\hat{x}) &= \Gamma^1_{(2,3)}(x_1) = \beta, \\ \Gamma^1_{(2,3)}(x) &= \hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = \beta. \end{aligned}$$

By Proposition C.17 case (ii), it follows that (after shrinking $U', \hat{U}'$) (U, g) and $(\hat{U}, \hat{g})$ are both of class $\mathcal{M}_\beta$ This gives (i).

(ii) By Lemma 5.19, one has for every $x \in U'$, $\hat{x} \in \hat{U}'$ that

$$\begin{aligned} \hat{\Gamma}^1_{(2,3)}(\hat{x}) &= \Gamma^1_{(2,3)}(x_1) = 0, \\ \Gamma^1_{(2,3)}(x) &= \hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = 0, \end{aligned}$$

i.e. $\Gamma^1_{(2,3)}$ and $\hat{\Gamma}^1_{(2,3)}$ vanish on $U', \hat{U}'$, respectively. Then Proposition C.17 case (iii) gives (after shrinking $U', \hat{U}'$) the desired isometries $F, \hat{F}$. Moreover, Eq. (57) in that proposition gives, since $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$,

$$\begin{aligned} -K &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{f'(r)}{f(r)} + \big(-\frac{f'(r)}{f(r)}\big) - 0^2 = \frac{f''(r)}{f(r)}, \\ -K &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{\hat{f}'(r)}{\hat{f}(r)} + \big(-\frac{\hat{f}'(r)}{\hat{f}(r)}\big) - 0^2 = \frac{\hat{f}''(r)}{\hat{f}(r)}, \end{aligned}$$

where $r \in I$. This proves (ii). □

5.2.2 Local Structures for the Manifolds Around $q \in O_1$

In analogy to Proposition (5.5) we will first prove the following result. In the results below that concern O_1, we always assume that $O_1 \neq \emptyset$. For the next proposition, contrary to an analogous Proposition 5.5 of Subsubsection 5.2.1, we do not need to assume that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open. The subsequent result only relies on the fact that O_1 is not empty.

Proposition 5.21 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Then for every $q = (x, \hat{x}; A) \in O_1$ there exist an orthonormal pair $X_A, Y_A \in T|_x M$ such that if $Z_A := \star(X_A \wedge Y_A)$ then X_A, Y_A, Z_A is a positively oriented orthonormal pair with respect to which R and $\widetilde{\mathsf{Rol}}$

may be written as

$$\begin{aligned}
R(X_A\wedge Y_A) &= \begin{pmatrix} 0 & K(x) & 0\\ -K(x) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \star R(X_A\wedge Y_A)=\begin{pmatrix} 0\\ 0\\ -K(x)\end{pmatrix},\\
R(Y_A\wedge Z_A) &= \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & K(x)\\ 0 & -K(x) & 0\end{pmatrix}, \quad \star R(Y_A\wedge Z_A)=\begin{pmatrix} -K(x)\\ 0\\ 0\end{pmatrix},\\
R(Z_A\wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2(x)\\ 0 & 0 & 0\\ K_2(x) & 0 & 0\end{pmatrix}, \quad \star R(Z_A\wedge X_A)=\begin{pmatrix} 0\\ -K_2(x)\\ 0\end{pmatrix},\\
\widetilde{\mathsf{Rol}}_q(X_A\wedge Y_A) &= 0,\\
\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A) &= 0,\\
\widetilde{\mathsf{Rol}}_q(Z_A\wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}}(q)\\ 0 & 0 & 0\\ K_2^{\mathsf{Rol}}(q) & 0 & 0\end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A\wedge X_A)=\begin{pmatrix} 0\\ -K_2^{\mathsf{Rol}}(q)\\ 0\end{pmatrix},
\end{aligned} \tag{46}$$

where K, K_2 are real-valued functions defined on M.

With respect to X_A, Y_A, Z_A given by the theorem, we also have

$$\begin{aligned}
\star A^{T}\hat{R}(AX_A\wedge AY_A)A &= \begin{pmatrix} 0\\ 0\\ -K(x)\end{pmatrix},\\
\star A^{T}\hat{R}(AY_A\wedge AZ_A)A &= \begin{pmatrix} -K(x)\\ 0\\ 0\end{pmatrix},\\
\star A^{T}\hat{R}(AZ_A\wedge AX_A)A &= \begin{pmatrix} 0\\ -K_2(x)+K_2^{\mathsf{Rol}}(q)\\ 0\end{pmatrix}.
\end{aligned} \tag{47}$$

Relevant observations regarding the previous proposition are collected next.

Remark 5.22 (a) The last proposition says that $\star X_A, \star Y_A, \star Z_A$ are eigenvectors of $R|_x$, for every $q=(x,\hat{x};A)\in O_1$, with corresponding eigenvalues $-K(x)$, $-K_2(x)$ and $-K(x)$. Changing the roles of (M,g) and $(\hat{M},\hat{g})$, the proposition gives that, for every $q=(x,\hat{x};A)\in O_1$, eigenvectors $\hat{\star}\hat{X}_A, \hat{\star}\hat{Y}_A, \hat{\star}\hat{Z}_A$ are eigenvectors of $\hat{R}|_{\hat{x}}$, with corresponding eigenvalues $-\hat{K}(\hat{x})$, $-\hat{K}_2(\hat{x})$ and $-\hat{K}(\hat{x})$.

(b) The eigenvalues K and $\hat{K}$ coincide on the set of points that can be reached, locally, by the rolling. More precisely, Proposition 5.21 tells us that

$$-\hat{K}(\hat{x}) = -K(x), \quad \forall (x,\hat{x})\in \pi_Q(O_1),$$

and that this eigenvalue is at least a double eigenvalue for both $R|_x$ and $\hat{R}|_{\hat{x}}$.

(c) The above at-least-double eigenvalue cannot be a triple eigenvalue for both $R|_x$ and $\hat{R}|_{\hat{x}}$ at the same time, for $(x,\hat{x})\in\pi_Q(O_1)$. Indeed, if $K_2(x)=K(x)$ and $\hat{K}_2(\hat{x})=\hat{K}(\hat{x})$, then clearly this would imply that $\mathsf{Rol}_q=0$, which contradicts the fact that $q\in O_1$ implies $\mathrm{rank}\ \mathsf{Rol}_q=1$.

(d) It is not clear that the assignments $q \mapsto X_A, Z_A$ can be made locally smoothly on O_1. However, it is the case for the assignment $q \mapsto Y_A$. In addition, for every $q = (x, \hat{x}; A) \in O_1$, the choice of Y_A and $\hat{Y}_A$ are uniquely determined up to multiplication by -1. Indeed, $\star Y_A = Z_A \wedge X_A$ is a unit eigenvector of $\widetilde{\mathsf{Rol}}_q$ corresponding to the simple non-zero eigenvalue $-K_2^{\mathsf{Rol}}(q)$ (it is non-zero since $\mathrm{rank}\ \mathsf{Rol}_q = 1$, $q \in O_1$). By symmetry, the same holds of $\hat{Y}_A$ as well. Then

$$AY_A = \pm\hat{Y}_A, \quad \forall q = (x, \hat{x}; A) \in O_1.$$

We begin by the following simple lemma.

Lemma 5.23 For every $q = (x, \hat{x}; A) \in O_1$ and any orthonormal pair (which exists) $X_A, Y_A \in T|_x M$ such that $X_A, Y_A, Z_A := \star(X_A \wedge Y_A)$ is an oriented orthonormal basis of $T|_x M$ and $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$, $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$, one has with respect to the basis X_A, Y_A, Z_A,

$$
\begin{aligned}
R(X_A \wedge Y_A) &= \begin{pmatrix} 0 & K_A & \alpha_A \\ -K_A & 0 & -\beta_A \\ -\alpha_A & \beta_A & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} \beta_A \\ \alpha_A \\ -K_A \end{pmatrix}, \\
R(Y_A \wedge Z_A) &= \begin{pmatrix} 0 & -\beta_A & \xi_A \\ \beta_A & 0 & K_A^1 \\ -\xi_A & -K_A^1 & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix}, \\
R(Z_A \wedge X_A) &= \begin{pmatrix} 0 & -\alpha_A & -K_A^2 \\ \alpha_A & 0 & -\xi_A \\ K_A^2 & \xi_A & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} \xi_A \\ -K_A^2 \\ \alpha_A \end{pmatrix}, \\
\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) &= 0, \\
\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) &= 0, \\
\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}} \\ 0 & 0 & 0 \\ K_2^{\mathsf{Rol}} & 0 & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2^{\mathsf{Rol}} \\ 0 \end{pmatrix}.
\end{aligned}
$$

Moreover, the choice of the above quantities can be made locally smoothly on O_1.

Proof. We only need to prove the existence of an oriented orthonormal basis X_A, Y_A and Z_A such that $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$, $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$. Indeed, when this has been established, one may use Lemma 5.7, where we now have $K_1^{\mathsf{Rol}}(q) = 0$, $\alpha(q) = 0$ because $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$, to conclude.

Since for a given $q = (x, \hat{x}; A) \in O_1$, $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_x M \to \wedge^2 T|_x M$ is symmetric linear map that has rank 1, it follows that its eigenspaces are orthogonal and its kernel has dimension exactly 2. Thus there is an orthonormal basis $\omega_1, \omega_2, \lambda$ of $\wedge^2 T|_x M$ such that $\widetilde{\mathsf{Rol}}_q(\omega_i) = 0$, $i = 1, 2$. Taking $X_A = \star\omega_1$, $Z_A = \star\omega_2$ and $Y_A = \star\lambda$ we get, up to replacing X_A with $-X_A$ if necessary, an oriented orthonormal basis of $T|_x M$ such that $\mathsf{Rol}(X_A \wedge Y_A) = 0$, $\mathsf{Rol}(Y_A \wedge Z_A) = 0$. □

As a consequence of the lemma and because $A^T \hat{R}(AX, AY)A = R(X, Y) - \widetilde{\mathsf{Rol}}_q(X, Y)$ for $X, Y \in T|_x M$, we have that w.r.t. the oriented orthonormal basis

X_A, Y_A, Z_A,

$$\star A^{\overline{T}}\hat{R}(AX_A, AY_A)A = \begin{pmatrix} \beta_A \\ \alpha_A \\ -K_A \end{pmatrix},$$
$$\star A^{\overline{T}}\hat{R}(AY_A, AZ_A)A = \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix},$$
$$\star A^{\overline{T}}\hat{R}(AZ_A, AX_A)A = \begin{pmatrix} \xi_A \\ -K_A^2 + K_2^{\mathsf{Rol}} \\ \alpha_A \end{pmatrix}. \tag{48}$$

The assumption that rank $\mathsf{Rol}_q = 1$ is equivalent to the fact that for every $q = (x, \hat{x}; A) \in O_1$,

$$K_2^{\mathsf{Rol}}(q) \neq 0. \tag{49}$$

This implies that Y_A is uniquely determined up to multiplication by -1 (see also Remark 5.22 above). Hence, in particular, for every $q = (x, \hat{x}; A) \in O_1$,

$$\nu\big(\mathsf{Rol}_q(\wedge^2 TM)(A)\big)\big|_q = \mathrm{span}\{\nu(A(Z_A \wedge X_A))|_q\} = \mathrm{span}\{\nu(A \star Y_A)|_q\}.$$

We will now show that, with any (non-unique) choice of a pair X_A, Y_A as in Lemma 5.23, one has that $\alpha_A = 0$ and $K_A = K_A^1$.

Lemma 5.24 If one chooses any $X_A, Y_A, Z_A = \star(X_A \wedge Y_A)$ as in Lemma 5.23, then

$$\beta_A = 0, \quad K_A = K_A^1, \quad \forall q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Proof. Fix $q = (x, \hat{x}; A) \in O_1$. Choosing in Corollary 4.14 $X, Y \in \mathrm{VF}(M)$ such that $X|_x = X_A$, $Y|_x = Y_A$, we get, since $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$,

$$\begin{aligned} &\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X, Y)(\cdot)), \nu(\mathsf{Rol}(Z, W)(\cdot))\big]\big|_q \\ =&A\big[R(X_A \wedge Y_A), R(Z|_x \wedge W|_x)\big]_{\mathfrak{so}} - \big[\hat{R}(AX_A \wedge AY_A), \hat{R}(AZ|_x \wedge AW|_x)\big]_{\mathfrak{so}}A \\ &+ \hat{R}(AX_A \wedge A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)Y_A)A + \hat{R}(A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)X_A, AY_A)A. \end{aligned}$$

Since $q' = (x', \hat{x}'; A') \mapsto \nu(\mathsf{Rol}(\wedge^2 T|_{x'}M)(A'))|_{q'} = \mathrm{span}\{\nu(A' \star Y_{A'})\}$ is a smooth rank one distribution on O_1, it follows that it is involutive and hence for all $X, Y, Z, W \in \mathrm{VF}(M)$,

$$\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \in \mathrm{span}\{\nu(A \star Y_A)|_q\},$$

where we used that $\mathsf{Rol}(\wedge^2 TM)(A) = \mathrm{span}\{A \star Y_A\}$ as observed above.

We compute the right hand side of this formula in different cases. We begin by taking any smooth vector fields X, Y, Z, W with $X|_x = X_A$, $Y|_x = Y_A$, $Z|_x = Z_A$,

$W|_x = X_A$. One gets

$$\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)),\nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\
=&\big[R(X_A\wedge Y_A),R(Z_A\wedge X_A)\big]_{\mathfrak{so}} - \big[A^{\overline{T}}\hat{R}(AX_A\wedge AY_A)A, A^{\overline{T}}\hat{R}(AZ_A\wedge AX_A)A\big]_{\mathfrak{so}} \\
&+A^{\overline{T}}\hat{R}(AX_A\wedge \mathsf{Rol}(Z_A\wedge X_A)(A)Y_A)A + A^{\overline{T}}\hat{R}(\mathsf{Rol}(Z_A\wedge X_A)(A)X_A\wedge AY_A)A \\
=&\begin{pmatrix}\beta_A\\ \alpha_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}0\\ -K_2^{\mathsf{Rol}}\\ 0\end{pmatrix} + A^{\overline{T}}\hat{R}(AX_A\wedge 0)A + A^{\overline{T}}\hat{R}(K_2^{\mathsf{Rol}}AZ_A\wedge AY_A)A \\
=&\begin{pmatrix}-K_AK_2^{\mathsf{Rol}}\\ 0\\ -\beta_AK_2^{\mathsf{Rol}}\end{pmatrix} - K_2^{\mathsf{Rol}}\begin{pmatrix}-K_A^1\\ \xi_A\\ \beta_A\end{pmatrix} = \begin{pmatrix}K_2^{\mathsf{Rol}}(-K_A+K_A^1)\\ K_2^{\mathsf{Rol}}\xi_A\\ -2\beta_AK_2^{\mathsf{Rol}}\end{pmatrix}\in\mathrm{span}\{\nu(A\star Y_A)|_q\}.
\end{aligned}$$

Because $K_2^{\mathsf{Rol}}(q)\neq 0$, this immediately implies that

$$-K_A+K_A^1=0,\quad \beta_A=0.$$

This completes the proof. □

We will now rotate X_A, Y_A, Z_A in such a way that we can set α_A equal to zero.

Lemma 5.25 For every $q=(x,\hat{x};A)\in O_1$ there are orthonormal $X_A,Y_A\in T|_xM$ such that $X_A,Y_A,Z_A=\star(X_A\wedge Y_A)$ is an oriented orthonormal basis of $T|_xM$ with respect to which in Lemma 5.23 one has $\alpha_A=0$.

Proof. Fix $q=(x,\hat{x};A)\in O_1$, choose any $X_A,Y_A,Z_A=\star(X_A\wedge Y_A)$ as in Lemma 5.23 and suppose $\alpha_A\neq 0$ (otherwise we are done). For $t\in\mathbb{R}$, set

$$\begin{pmatrix}X_A(t)\\ Z_A(t)\end{pmatrix}=\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}X_A\\ Z_A\end{pmatrix}.$$

Then clearly $Y_A(t):=\star(X_A(t)\wedge Z_A(t))=\star(X_A\wedge Z_A)=Y_A$ and $X_A(t),Y_A(t),Z_A(t)$ is an orthonormal positively oriented basis of $T|_xM$. Since $\widetilde{\mathsf{Rol}}_q$ is a symmetric map $\wedge^2T|_xM\to\wedge^2T|_xM$ and since $\star X_A,\star Z_A$ are its eigenvectors corresponding to the eigenvalue 0, it follows that $\star X_A(t),\star Z_A(t)$, which are just rotated $\star X_A,\star Z_A$ in the plane that they span, are eigenvectors of Rol_q corresponding to the eigenvalue 0, i.e. $\mathsf{Rol}_q(X_A(t)\wedge Y_A)=0$, $\mathsf{Rol}_q(Y_A\wedge Z_A(t))=0$ for all $t\in\mathbb{R}$.

Hence the conclusion of Lemma 5.23 holds for the basis $X_A(t),Y_A,Z_A(t)$ and we write $\xi_A(t),\alpha_A(t)$, $\beta_A(t)$, $K_A(t)$, $K_A^1(t)$, $K_A^2(t)$ for the coefficients of the matrices of R given there w.r.t. $X_A(t),Y_A,Z_A(t)$. Then Lemma 5.24 implies that $\beta_A(t)=0$, $K_A(t)=K_A^1(t)$ for all $t\in\mathbb{R}$. We now compute

$$\begin{aligned}
\alpha_A(t)=&g(R(X_A(t)\wedge Y_A)Z_A(t),X_A(t))=g(R(Z_A(t)\wedge X_A(t))X_A(t),Y_A(t)) \\
=&-g(R(Z_A\wedge X_A)Y_A,X_A(t)) \\
=&-g(-\alpha_AX_A+\xi_AZ_A,\cos(t)X_A+\sin(t)Z_A) \\
=&-\alpha_A\cos(t)+\xi_A\sin(t).
\end{aligned}$$

Thus choosing $t_0\in\mathbb{R}$ such that

$$\cot(t_0)=\frac{\xi_A}{\alpha_A},$$

we get that $\alpha_A(t_0)=0$. As already observed, we also have $\beta_A(t_0)=0$, $K_A^1(t_0)=K_A(t_0)$ and $\mathsf{Rol}_q(X_A(t_0)\wedge Y_A)=0$, $\mathsf{Rol}_q(Y_A\wedge Z_A(t_0))=0$. □

Since α_A and β_A vanish w.r.t X_A, Y_A, Z_A, as chosen by the previous lemma, we have that $-K_A$ is an eigenvalue of $R|_x$ with eigenvector $X_A \wedge Y_A$, where $q = (x, \hat{x}; A) \in O_1$. Knowing this, we may prove that even ξ_A is zero as well and that (automatically) $-K_A$ is a at least a double eigenvalue of $R|_x$. This is given in the lemma that follows.

Lemma 5.26 If $q = (x, \hat{x}; A) \in O_1$ and X_A, Y_A, Z_A as in Lemma 5.25, then $\xi_A = 0$.

Proof. Since for any $q = (x, \hat{x}; A) \in O_1$, $-K_A$ is an eigenvalue of $R|_x$, we know that its value only depends on the point x of M and hence we consider it as a smooth function $-K(x)$ on M. We claim that that $-K(x)$ is at least a double eigenvalue of $R|_x$. Suppose it is not. Then in a neighbourhood U of x we have that $-K(y)$ is a simple eigenvalue of $R|_y$ for all $y \in U$. In that case, we may choose smooth vector fields X, Y on U, taking U smaller if necessary, such that $X|_y \wedge Y|_y$ is a (non-zero) eigenvector of $R|_y$ corresponding to $-K(y)$ and $X|_x = X_A$, $Y|_x = Y_A$. Write $O := \pi_{Q,M}^{-1}(U) \cap O_1$. For any $(y, \hat{y}; B) \in O$, we know that $X_B \wedge Y_B$ is a unit eigenvector of $R|_y$ corresponding to $-K(y)$ and hence, modulo replacing X by $-X$, we have $X_B \wedge Y_B = X|_y \wedge Y|_y$. Then, for all $(y, \hat{y}; B) \in O$ with $y \in U$, one has

$$\nu(\mathsf{Rol}(X|_y \wedge Y|_y)(B))|_{(y,\hat{y};B)} = \nu(\mathsf{Rol}(X_B \wedge Y_B)(B))|_{(y,\hat{y};B)} = 0,$$

i.e., $\nu(\mathsf{Rol}(X \wedge Y)(\cdot))$ is a zero vector field on the open subset O of the orbit. If we also take some smooth vector fields Z, W such that $Z|_x = Z_A$, $W|_x = X_A$, we get by the fact that $\nu(\mathsf{Rol}(X \wedge Y)(\cdot)) = 0$ and from the computations in the proof of Lemma 5.24 that

$$0 = \nu|_q^{-1}\big[\nu(\mathsf{Rol}(X, Y)(\cdot)), \nu(\mathsf{Rol}(Z, W)(\cdot))\big]\big|_q = \begin{pmatrix} K_2^{\mathsf{Rol}}(-K_A + K_A^1) \\ K_2^{\mathsf{Rol}}\xi_A \\ -2\beta_A K_2^{\mathsf{Rol}} \end{pmatrix} = \begin{pmatrix} 0 \\ K_2^{\mathsf{Rol}}\xi_A \\ 0 \end{pmatrix}.$$

Since $K_2^{\mathsf{Rol}}(q) \neq 0$ we get $\xi_A = 0$. This implies, along with the results obtained in the previous lemma (i.e. $K = K_A^1$, $\beta_A = \alpha_A = 0$), that w.r.t. the basis X_A, Y_A, Z_A, one has

$$\star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K_A \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ 0 \\ 0 \end{pmatrix},$$

which means that $X_A \wedge Y_A$ and $Y_A \wedge Z_A$ are linearly independent eigenvectors of $R|_x$ corresponding to the eigenvalue $-K_A = -K(x)$. This is in contradiction to what we assumed at the beginning of the proof. Hence we have that $-K_A$ is, for every $q = (x, \hat{x}; A) \in O_1$, an eigenvalue of $R|_x$ of multiplicity at least 2. Finally, since we know that w.r.t. X_A, Y_A, Z_A,

$$\star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K_A \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} \xi_A \\ -K_A^2 \\ 0 \end{pmatrix},$$

and since $R|_x$ is a symmetric linear map having double eigenvalue $-K_A$, then there exists a unit eigenvector ω of $R|_x$ corresponding to $-K_A$ which belongs to the plane orthogonal to $X_A \wedge Y_A$ (in $\wedge^2 T|_x M$). Hence, $\omega = \cos(t) Y_A \wedge Z_A + \sin(t) Z_A \wedge X_A$ for some $t \in \mathbb{R}$ and

$$-K_A \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} = -K_A \star \omega = \star R(\omega) = \cos(t) \star R(Y_A \wedge Z_A) + \sin(t) \star R(Z_A \wedge X_A)$$
$$= \cos(t) \begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} \xi_A \\ -K_A^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -K_A \cos(t) + \xi_A \sin(t) \\ \xi_A \cos(t) - K_A^2 \sin(t) \\ 0 \end{pmatrix},$$

where the matrices are formed w.r.t. X_A, Y_A, Z_A. From the first row, we get $\xi_A \sin(t) = 0$. So either $\xi_A = 0$ and we are done or $\sin(t) = 0$, implying that $\omega = 1_\pm Y_A \wedge Z_A$ with $1_\pm \in \{-1, +1\}$ and hence

$$\begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix} = \star R(Y_A \wedge Z_A) = 1_\pm \star R(\omega) = -K_A(1_\pm \star \omega) = -K_A \star (Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ 0 \\ 0 \end{pmatrix},$$

which gives $\xi_A = 0$ anyhow.

□

The previous lemma implies Proposition 5.21, since now $-K_A = -K_A^1$, $-K_A^2$ are eigenvalues of $R|_x$ for every $(x, \hat{x}; A) \in O_1$ and hence, defining $K(x) := K_A$, $K_2(x) := K_A^2$, we obtain well defined functions $K, K_2 : M \to \mathbb{R}$.

The following Proposition is the last result of this subsection. Notice that it does need the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open while the previous results do not need this assumption.

Proposition 5.27 Suppose $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q. Then there is an open dense subset O_1° of O_1 such that for every $q_1 = (x_1, \hat{x}_1; A_1) \in O_1^\circ$ there are neighbourhoods U and $\hat{U}$ of x_1 and $\hat{x}_1$, respectively, such that either

(i) both $(U, g|_U)$, $(\hat{U}, \hat{g}|_{\hat{U}})$ are of class $\mathcal{M}_\beta$ or

(ii) both $(U, g|_U)$, $(\hat{U}, \hat{g}|_{\hat{U}})$ are isometric to warped products $(I \times N, h_f)$, $(I \times \hat{N}, \hat{h}_{\hat{f}})$ and $\frac{f'(r)}{f(r)} = \frac{\hat{f}'(r)}{\hat{f}(r)}$, for all $r \in I$.

Moreover, there is an oriented orthonormal frame E_1, E_2, E_3 (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$) defined on U (resp. on $\hat{U}$) respectively, such that $\star E_1, \star E_3$ (resp. $\star \hat{E}_1, \star \hat{E}_3$) are eigenvectors of $\hat{R}$ with common eigenvalue $-K(\cdot)$ (resp. $-\hat{K}(\cdot)$) and one has

$$A_1 E_2|_{x_1} = \hat{E}_2|_{\hat{x}_1}.$$

Proof. Let $q_1 = (x_1, \hat{x}_1; A_1) \in O_1$. As observed in Remark 5.22, either $R|_{x_1}$ or $\hat{R}|_{\hat{x}_1}$ has $-K_2(x_1)$ or $-\hat{K}_2(\hat{x}_1)$, respectively, as a single eigenvalue. By symmetry of the problem in (M, g), $(\hat{M}, \hat{g})$, we assume that this is the case for $R|_{x_1}$. Hence there is a neighbourhood U of x_1 such that $K_2(x) \neq K(x)$ for all $x \in U$. Then, there is an open dense subset O_1' of $O_1 \cap \pi_{Q,M}^{-1}(U)$ such that, for every $q = (x, \hat{x}; A) \in O_1'$, there exists an open neighbourhood $\hat{V}$ of $\hat{x}$ where either $\hat{K}_2 = \hat{K}$ on $\hat{V}$ or $\hat{K}_2(\hat{y}) \neq \hat{K}(\hat{y})$ for $\hat{y} \in \hat{V}$. For the rest of the argument, we assume that q_1 belongs to O_1'. By shrinking U around x_1 and taking a small enough neighbourhood $\hat{U}$ of $\hat{x}_1$, we assume there are oriented orthonormal frames E_1, E_2, E_3 on U (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$ on $\hat{U}$) such that $\star E_1, \star E_2, \star E_3$ (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$) are eigenvectors of R (resp. $\hat{R}$) with eigenvalues $-K(\cdot), -K_2(\cdot), -K(\cdot)$ (resp. $-\hat{K}(\cdot), -\hat{K}_2(\cdot), -\hat{K}(\cdot)$), where these eigenvalues correspond to those in Proposition 5.21. Taking U, $\hat{U}$ smaller if necessary, we take X_A, Y_A, Z_A as given by Proposition 5.21 for M and $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$ for $\hat{M}$ on $\pi_Q^{-1}(U \times \hat{U}) \cap O_1'$, which we still denote by O_1'. Since $\star Y_A$ and $\star E_2|_x$ are both eigenvalues of $R|_x$, for $q = (x, \hat{x}; A) \in O_1'$, corresponding to single eigenvalue $-K_2(x)$, we moreover assume that $Y_A = E_2|_x$, $\forall q = (x, \hat{x}; A) \in O_1'$. Then because $\nu(\mathsf{Rol}_q(Z_A \wedge X_A))|_q = -K_2^{\mathsf{Rol}}(q)\nu(A \star E_2)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at the points $q = (x, \hat{x}; A) \in O_1'$, we conclude from Proposition C.18 (given in Appendix) that

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)} \end{pmatrix},$$

where Γ and $\Gamma^i_{(j,k)}$ are as defined there.

We will now divide the proof in two parts (cases I and II below), depending whether $(\hat{M}, \hat{g})$ has, in certain areas, constant curvature or not.

Case I: Suppose, after shrinking $\hat{U}$ around x_1, that $\hat{K}_2(\hat{x}) = \hat{K}(\hat{x})$ for all $\hat{x} \in \hat{U}$. We also assume that $\hat{U}$ is connected. This implies by Schur Lemma (see [39], Proposition II.3.6) that $\hat{K}_2 = \hat{K}$ is constant on $\hat{U}$ and we write simply $\hat{K}$ for this constant. Again by possibly shrinking $\hat{U}$, we assume that $(\hat{U}, \hat{g}|_{\hat{U}})$ is isometric to an open subset of a 3-sphere of curvature $\hat{K}$.

Assume first that $\Gamma^1_{(2,3)} \neq 0$ on U. Then Proposition C.17, case (ii), implies that $\Gamma^1_{(1,2)} = 0$ on U and $(\Gamma^1_{(2,3)})^2 = K(x)$ is constant on U, which must be $\hat{K}$. Hence if $\beta := \Gamma^1_{(2,3)}$, which is constant on U, then $(U, g|_U)$ is of class $\mathcal{M}_\beta$ as is $(\hat{U}, \hat{g}|_{\hat{U}})$ and we are done (recall that $\mathcal{M}_{-\beta} = \mathcal{M}_\beta$) i.e., this is case (i). On the other hand, if $\Gamma^1_{(2,3)} = 0$ on U, then we have that $(U, g|_U)$, after possibly shrinking U, is isometric, by some F, to a warped product $(I \times N, h_f)$ by Proposition C.17 case (iii). At the same time, the space of constant curvature $(\hat{U}, \hat{g}|_{\hat{U}})$, again after shrinking $\hat{U}$ if necessary, can be presented, isometrically by certain $\hat{F}$, as a warped product $(\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, where $\hat{N}$ is a 2-dimensional space of constant curvature. Because for all $x \in U$ we have $K(x) = \hat{K}$, we get that for all $(r, y) \in I \times N$, $\hat{r} \in \hat{I}$,

$$-\frac{f''(r)}{f(r)} = K(F(r, y)) = \hat{K} = -\frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}.$$

It is not hard to see that we may choose $\hat{f}$ such that $\hat{f}(0) = f(0)$ and $\hat{f}'(0) = f'(0)$, which then implies that $\hat{f}(r) = f(r)$, for all $r \in I$. This leads us to case (ii)

Case II: We assume here that $\hat{K}_2(\hat{x}) \neq \hat{K}(\hat{x})$ for all $\hat{x} \in \hat{U}$. The same way as for (M, g) above, this implies that $\hat{Y}_A = \hat{E}_2|_{\hat{x}}$ and that w.r.t. the frame $\hat{E}_1, \hat{E}_2, \hat{E}_3$, Proposition C.18 yields

$$\hat{\Gamma} = \begin{pmatrix} \hat{\Gamma}^1_{(2,3)} & 0 & -\hat{\Gamma}^1_{(1,2)} \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & 0 & \hat{\Gamma}^1_{(2,3)} \end{pmatrix},$$

where $\hat{\Gamma}^i_{(j,k)} = \hat{g}(\hat{\nabla}_{\hat{E}_i}\hat{E}_j, \hat{E}_k)$, $1 \leq i,j,k \leq 3$.

We now claim that for all $(x,\hat{x}) \in \pi_Q(O_1')$, we have

$$\Gamma^1_{(2,3)}(x) = \hat{\Gamma}^1_{(2,3)}(\hat{x})$$
$$\Gamma^1_{(1,2)}(x) = \hat{\Gamma}^1_{(1,2)}(\hat{x}).$$

By Remark 5.22, we have $AY_A = \pm\hat{Y}_A$ for $q = (x,\hat{x};A) \in O_1'$, and so we get $AE_2|_x = \pm\hat{E}_2|_{\hat{x}}$. Without loss of generality, we assume that the '+' -case holds here. In particular, if $X \in \mathrm{VF}(M)$, one may differentiate the identity $AE_2 = \hat{E}_2$ w.r.t. $\mathscr{L}_{\mathrm{R}}(X)|_q$ to obtain

$$A\nabla_X E_2 = \hat{\nabla}_{AX}\hat{E}_2, \quad \forall q = (x,\hat{x};A) \in O_1'.$$

Since $AE_1, AE_3, \hat{E}_1, \hat{E}_3 \in (AE_2)^\perp = \hat{E}_2^\perp$, there exists, for every $q \in O_1'$, $\varphi = \varphi(q) \in \mathbb{R}$ such that

$$AE_1|_x = \cos(\varphi(q))\hat{E}_1|_{\hat{x}} + \sin(\varphi(q))\hat{E}_3|_{\hat{x}}$$
$$AE_3|_x = -\sin(\varphi(q))\hat{E}_1|_{\hat{x}} + \cos(\varphi(q))\hat{E}_3|_{\hat{x}}.$$

As usual, we write below $\cos(\varphi(q)) = c_\varphi$, $\sin(\varphi(q)) = s_\varphi$ Having this, we compute

$$\begin{aligned} A\nabla_{E_1}E_2 =& A(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) \\ =& (-c_\varphi\Gamma^1_{(1,2)} - s_\varphi\Gamma^1_{(2,3)})\hat{E}_1 + (-s_\varphi\Gamma^1_{(1,2)} + c_\varphi\Gamma^1_{(2,3)})\hat{E}_3, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \hat{\nabla}_{AE_1}\hat{E}_2 =& c_\varphi(-\hat{\Gamma}^1_{(1,2)}\hat{E}_1 + \hat{\Gamma}^1_{(2,3)}\hat{E}_3) + s_\varphi(-\hat{\Gamma}^1_{(2,3)}\hat{E}_1 - \hat{\Gamma}^1_{(1,2)}\hat{E}_3) \\ =& (-c_\varphi\hat{\Gamma}^1_{(1,2)} - s_\varphi\hat{\Gamma}^1_{(2,3)})\hat{E}_1 + (c_\varphi\hat{\Gamma}^1_{(2,3)} - s_\varphi\hat{\Gamma}^1_{(1,2)})\hat{E}_3. \end{aligned}$$

Taking $X = E_1$ above and using the last two formulas, we get

$$\begin{aligned} &(-c_\varphi\Gamma^1_{(1,2)} - s_\varphi\Gamma^1_{(2,3)})\hat{E}_1 + (-s_\varphi\Gamma^1_{(1,2)} + c_\varphi\Gamma^1_{(2,3)})\hat{E}_3 = A\nabla_{E_1}E_2 \\ =&\hat{\nabla}_{AE_1}\hat{E}_2 = (-c_\varphi\hat{\Gamma}^1_{(1,2)} - s_\varphi\hat{\Gamma}^1_{(2,3)})\hat{E}_1 + (c_\varphi\hat{\Gamma}^1_{(2,3)} - s_\varphi\hat{\Gamma}^1_{(1,2)})\hat{E}_3, \end{aligned}$$

from which

$$c_\varphi(-\Gamma^1_{(1,2)} + \hat{\Gamma}^1_{(1,2)}) + s_\varphi(\Gamma^1_{(2,3)} - \hat{\Gamma}^1_{(2,3)}) = 0.$$

Next we notice that differentiating the identity $AE_1 = c_\varphi\hat{E}_1 + s_\varphi\hat{E}_3$ w. r. t. $\nu(A \star E_2)|_q$ gives

$$A(\star E_2)E_1 = (\nu(A \star E_2)|_q\varphi)(-s_\varphi\hat{E}_1 + c_\varphi\hat{E}_3),$$

which simplifies to

$$-AE_3 = (\nu(A \star E_2)|_q\varphi)AE_3,$$

and hence yields

$$\nu(A \star E_2)|_q\varphi = -1, \quad \forall q = (x,\hat{x};A) \in O_1'.$$

Thus, if $(t,q)\mapsto\Phi(t,q)$ is the flow of $\nu((\cdot)\star E_2)$ in O'_2 with initial position at $t=0$ at $q\in O'_1$, the above implies that $\varphi(\Phi(t,q))=\varphi(q)+t$ for all t such that $|t|$ is small enough. Since sin and cos are linearly independent functions on any non-empty open real interval, the above relation implies that

$$\begin{aligned}-\Gamma^1_{(1,2)}(x)+\hat{\Gamma}^1_{(1,2)}(\hat{x})&=0,\\ \Gamma^1_{(2,3)}(x)-\hat{\Gamma}^1_{(2,3)}(\hat{x})&=0,\end{aligned}$$

which establishes the claim.

We may now finish the proof of the proposition. Indeed, if $\Gamma^1_{(2,3)}\neq 0$ on U, Proposition C.17 implies that $\Gamma^1_{(2,3)}=:\beta$ is constant and $\Gamma^1_{(1,2)}=0$ on U. If $\hat{x}$ belongs to the open subset $\pi_{Q,\hat{M}}(O'_1)$ of $\hat{M}$, there is a $q=(x,\hat{x};A)\in O'_1$ where $(x,\hat{x})\in U\times\hat{U}$, by the definition of O'_1. The above implies

$$\hat{\Gamma}^1_{(1,2)}(\hat{x})=\Gamma^1_{(1,2)}(x)=0,\quad \hat{\Gamma}^1_{(2,3)}(\hat{x})=\Gamma^1_{(2,3)}(x)=\beta.$$

Thus shrinking $\hat{U}$ if necessary, this shows that $\hat{\Gamma}^1_{(1,2)}$ vanishes on $\hat{U}$ and $\hat{\Gamma}^1_{(2,3)}$ is constant $=\beta$ on $\hat{U}$. We conclude that $(U,g|_U)$ and $(\hat{U},\hat{g}|_{\hat{U}})$ both belong to the class $\mathcal{M}_\beta$ and we are in case (i).

Similarly, if $\Gamma^1_{(2,3)}=0$ on U, the above argument implies that, after taking smaller $\hat{U}$, that $\hat{\Gamma}^1_{(2,3)}=0$ on $\hat{U}$. Proposition C.17 implies that there is, taking smaller $U,\hat{U}$ if needed, open interval $I=\hat{I}\subset\mathbb{R}$, smooth functions $f,\hat{f}:I=\hat{I}\to\mathbb{R}$, 2-dimensional Riemannian manifolds (N,h), $(\hat{N},\hat{h})$ and isometries $F:(I\times N,h_f)\to(U,g|_U)$, $\hat{F}:(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})\to\hat{U}$ such that

$$\begin{aligned}\frac{f'(r)}{f(r)}&=\Gamma^1_{(1,2)}(F(r,y)),\quad\forall(r,y)\in I\times N,\\ \frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}&=\hat{\Gamma}^1_{(1,2)}(\hat{F}(\hat{r},\hat{y})),\quad\forall(\hat{r},\hat{y})\in\hat{I}\times\hat{N}.\end{aligned}$$

Clearly we may assume that $0\in I=\hat{I}$ and $F(0,y_1)=x_1$, $\hat{F}(0,\hat{y}_1)=\hat{x}_1$ for some $y_1\in N$, $\hat{y}_1\in\hat{N}$. Since $t\mapsto(t,y_1)$ and $t\mapsto(t,\hat{y}_1)$ are geodesics in $(I\times N,h_f)$, $(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})$, respectively, $\gamma(t):=F(t,y_1)$ and $\hat{\gamma}(t)=\hat{F}(t,\hat{y}_1)$ are geodesics on M and $\hat{M}$. In addition,

$$\hat{\gamma}'(0)=\hat{E}_2|_{\hat{x}_1}=A_1E_2|_{x_1}=A_1\gamma'(0),$$

so $\hat{\gamma}(t)=\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_1)(t)$ for all t. This means that

$$(F(t,y_1),\hat{F}(t,\hat{y}_1))=(\gamma(t),\hat{\gamma}(t))\in\pi_Q(O'_1),$$

and therefore

$$\frac{f'(t)}{f(t)}=\Gamma^1_{(1,2)}(F(t,y_1))=\hat{\Gamma}^1_{(1,2)}(\hat{F}(t,\hat{y}_1))=\frac{\hat{f}'(t)}{\hat{f}(t)},$$

for all $t\in I=\hat{I}$. We then belong to case (ii) and the proof of the proposition is concluded. □

We have studied the case where q belongs to $O_1 \cup O_2$. As for the points of O_0, one uses Corollary 4.16 and Remark 4.17 to conclude that for every $q_0 = (x_0, \hat{x}_0; A_0) \in O_0$, there are open neighbourhoods $U \ni x_0$ and $\hat{U} \ni \hat{x}_0$ such that $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are locally isometric. With the choice of the set O as the union of $O_0 \cup O_1^\circ \cup O_2$, (where O_1° was introduced in Proposition 5.27), one concludes the proof of Theorem 5.1.

5.3 Proof of Theorem 5.3

Only Items (b) and (c) are addressed and they are treated in separate subsections.

5.3.1 Case where both Manifolds are of Class $\mathcal{M}_\beta$

Consider two manifolds (M, g) and $(\hat{M}, \hat{g})$ of class $\mathcal{M}_\beta$, $\beta \geq 0$ and oriented orthonormal frames E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ which are adapted frames for of (M, g) and $(\hat{M}, \hat{g})$ respectively. We will prove that in this situation, the rolling problem is not completely controllable.

We define on Q two subsets

$$Q_0 := \{q = (x, \hat{x}; A) \in Q \mid AE_2 \neq \pm \hat{E}_2\},$$
$$Q_1 := \{q = (x, \hat{x}; A) \in Q \mid AE_2 = \pm \hat{E}_2\}.$$

Proposition 5.28 Let (M, g), $(\hat{M}, \hat{g})$ be of class $\mathcal{M}_\beta$ for $\beta \in \mathbb{R}$. Then for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q_1$ one has $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q_1$. Moreover, Q_1 is a closed 7-dimensional submanifold of Q and hence in particular $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 7$.

Proof. Define $h_1, h_2 : Q \to \mathbb{R}$ by

$$h_1(q) = \hat{g}(AE_1, \hat{E}_2), \quad h_2(q) = \hat{g}(AE_3, \hat{E}_2),$$

when $q = (x, \hat{x}; A) \in Q$. Set $h = (h_1, h_2) : Q \to \mathbb{R}^2$, then $Q_1 = h^{-1}(0)$. We will first show that h is regular at the points of Q_1, which then implies that Q_1 is a closed submanifold of Q of codimension 2 i.e., $\dim Q_1 = 7$ as claimed. Before proceeding, we divide Q_1 into two disjoint subsets

$$Q_1^+ = \{q = (x, \hat{x}; A) \in Q \mid AE_2 = +\hat{E}_2\},$$
$$Q_1^- = \{q = (x, \hat{x}; A) \in Q \mid AE_2 = -\hat{E}_2\},$$

whence $Q_1 = Q_1^+ \cup Q_1^-$. These are the components of Q_1

and we prove the claims only for Q_1^+, the considerations for Q_1^- being completely similar. First, since for every $q = (x, \hat{x}; A) \in Q_1^+$ one has $AE_2 = \hat{E}_2$, it follows that $AE_1, AE_3 \in \hat{E}_2^\perp$ and hence there is a smooth $\phi : Q_1^+ \to \mathbb{R}$ such that

$$AE_1 = \cos(\phi)\hat{E}_1 + \sin(\phi)\hat{E}_3 =: \hat{X}_A,$$
$$AE_3 = -\sin(\phi)\hat{E}_1 + \cos(\phi)\hat{E}_3 =: \hat{Z}_A.$$

In the sequel, we set $c_\phi = \cos(\phi(q))$, $s_\phi = \sin(\phi(q))$. For $q = (x, \hat{x}; A) \in Q_1^+$, one has

$$\nu(A \star E_3)|_q h_1 = \hat{g}(A(\star E_3)E_1, \hat{E}_2) = \hat{g}(AE_2, \hat{E}_2) = 1,$$
$$\nu(A \star E_1)|_q h_1 = \hat{g}(A(\star E_1)E_1, \hat{E}_2) = 0,$$
$$\nu(A \star E_3)|_q h_2 = \hat{g}(A(\star E_3)E_3, \hat{E}_2) = 0,$$
$$\nu(A \star E_1)|_q h_2 = \hat{g}(A(\star E_1)E_3, \hat{E}_2) = -\hat{g}(AE_2, \hat{E}_2) = -1,$$

which shows that indeed h is regular on Q_1^+. We next show that the vectors $\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q$ are all tangent to Q_1^+ and hence to Q_1. This is equivalent to the fact that $\mathscr{L}_{\mathrm{R}}(E_i)|_q h = 0$ for $i = 1, 2, 3$. We compute for $q = (x, \hat{x}; A) \in Q_1^+$, recalling that $AE_1 = \hat{X}_A$, $AE_2 = \hat{E}_2$, $AE_3 = \hat{Z}_A$,

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_1 =& \hat{g}(A\nabla_{E_1}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{X}_A}\hat{E}_2) \\
=& -\Gamma^1_{(3,1)}\hat{g}(AE_3, \hat{E}_2) + \hat{g}(\hat{X}_A, \beta c_\phi \hat{E}_3 - \beta s_\phi \hat{E}_1) \\
=& -\Gamma^1_{(3,1)}\hat{g}(\hat{Z}_A, \hat{E}_2) + \hat{g}(\hat{X}_A, \beta\hat{Z}_A) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_2 =& \hat{g}(A\nabla_{E_1}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{X}_A}\hat{E}_2) \\
=& \hat{g}(A(\Gamma^1_{(3,1)}E_1 - \beta E_2), \hat{E}_2) + \hat{g}(\hat{Z}_A, \beta\hat{Z}_A) \\
=& \hat{g}(\Gamma^1_{(3,1)}\hat{X}_A - \beta\hat{E}_2, \hat{E}_2) + \beta = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_1 =& \hat{g}(A\nabla_{E_2}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{E}_2}\hat{E}_2) = -\Gamma^2_{(3,1)}\hat{g}(\hat{Z}_A, \hat{E}_2) + 0 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_2 =& \hat{g}(A\nabla_{E_2}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{E}_2}\hat{E}_2) = \Gamma^2_{(3,1)}\hat{g}(\hat{X}_A, \hat{E}_2) + 0 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_1 =& \hat{g}(A\nabla_{E_3}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{Z}_A}\hat{E}_2) \\
=& \hat{g}(A(\beta E_2 - \Gamma^3_{(3,1)}E_3), \hat{E}_2) + \hat{g}(\hat{X}_A, -\beta s_\phi \hat{E}_3 - \beta c_\phi \hat{E}_1) \\
=& \hat{g}(\beta\hat{E}_2 - \Gamma^3_{(3,1)}\hat{Z}_A), \hat{E}_2) - \beta\hat{g}(\hat{X}_A, \hat{X}_A) = \beta - \beta = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_2 =& \hat{g}(A\nabla_{E_3}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{Z}_A}\hat{E}_2) \\
=& \Gamma^3_{(3,1)}\hat{g}(AE_1, \hat{E}_2) + \hat{g}(\hat{Z}_A, -\beta\hat{X}_A) = \Gamma^3_{(3,1)}\hat{g}(\hat{X}_A, \hat{E}_2) + 0 = 0.
\end{aligned}$$

Thus $\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q$ and hence $\mathcal{D}_{\mathrm{R}}$ are tangent to Q_1^+, which implies that any orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ through a point $q \in Q_1^+$, is also a subset of Q_1^+. The same observation obviously holds for Q_1^- and therefore the proof is complete. □

Next we will show that if (M, g) and $(\hat{M}, \hat{g})$ are of class $\mathcal{M}_\beta$ with the same $\beta \in \mathbb{R}$, then the rolling problem of M against $\hat{M}$ is not controllable. We begin by completing the proposition in the sense that we show that the orbit can be of dimension exactly 7, if (M, g), $(\hat{M}, \hat{g})$ are not locally isometric.

Proposition 5.29 Let (M, g), $(\hat{M}, \hat{g})$ be Riemannian manifolds of class $\mathcal{M}_\beta$, $\beta \neq 0$, and let $q_0 = (x_0, \hat{x}_0; A_0) \in Q_1$. Then if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, one has $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$.

Proof. Without loss of generality, we assume that $A_0E_2|_{x_0} = \hat{E}_2|_{\hat{x}_0}$. Then Proposition 5.28 and continuity imply that $AE_2|_x = \hat{E}_2|_{\hat{x}}$ for all $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and hence that $AE_1|_x, AE_3|_x \in \mathrm{span}\{\hat{E}_1|_{\hat{x}}, \hat{E}_3|_{\hat{x}}\}$. This combined with Lemma C.8 implies

$$\widetilde{\mathsf{Rol}}_q(\star E_1) = 0, \quad \widetilde{\mathsf{Rol}}_q(\star E_2) = (-K_2(x) + \hat{K}_2(\hat{x}))(\star E_2), \quad \widetilde{\mathsf{Rol}}_q(\star E_3) = 0,$$

for $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $-K_2(x), -\hat{K}_2(\hat{x})$ are eigenvalues of $R|_x, \hat{R}|_{\hat{x}}$ corresponding to eigenvectors $\star E_2|_x, \hat{\star}\hat{E}_2|_{\hat{x}}$, respectively. Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, there is a point $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $-K_2(x_1) + \hat{K}_2(\hat{x}_1) \neq 0$ (see Corollary 4.16 and Remark 4.17). Then there are open neighbourhoods U and $\hat{U}$ of x_1 and $\hat{x}_1$ in M and $\hat{M}$, respectively, such that $-K_2(x) + \hat{K}_2(\hat{x}) \neq 0$ for all $x \in U, \hat{x} \in \hat{U}$. Define $O := \pi_Q^{-1}(U \times \hat{U}) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which is an open subset of $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ containing q_0. Because for all $q=(x,\hat{x};A)\in O$ one has $\nu(\mathsf{Rol}_q(\star E_2))|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ and $-K_2(x)+\hat{K}_2(\hat{x})\neq 0$, it follows that

$$\nu(A\star E_2)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),\quad \forall q=(x,\hat{x};A)\in O.$$

Moreover, $\Gamma^1_{(1,2)}=0$ and $\Gamma^1_{(2,3)}=\beta$ is constant and hence one may use Proposition C.20, case (i), to conclude that the vector fields defined by

$$\begin{aligned}L_1|_q=&\mathscr{L}_{\rm NS}(E_1)|_q-\beta\nu(A\star E_1)|_q,\\ L_2|_q=&\beta\mathscr{L}_{\rm NS}(E_2)|_q,\\ L_3|_q=&\mathscr{L}_{\rm NS}(E_3)|_q-\beta\nu(A\star E_2)|_q,\end{aligned}$$

are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$. Therefore the linearly independent vectors

$$\mathscr{L}_{\rm R}(E_1)|_q,\mathscr{L}_{\rm R}(E_2)|_q,\mathscr{L}_{\rm R}(E_3)|_q,\nu(A\star E_2)|_q,L_1|_q,L_2|_q,L_3|_q,$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ for all $q\in O$, which implies that $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\geq 7$. By Proposition 5.28, we conclude that $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)=7$. □

We are left to study the case of an $\mathcal{D}_{\rm R}$-orbit passing through some $q_0\in Q_0$.

Proposition 5.30 Let (M,g) and $(\hat{M},\hat{g})$ be two Riemannian manifolds of class $\mathcal{M}_\beta$, $\beta\neq 0$, and let $q_0=(x_0,\hat{x}_0;A_0)\in Q_0$. Write $M^\circ:=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$, $\hat{M}^\circ:=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$, which are open connected subsets of M, $\hat{M}$. Then we have:

- (i) If only one of (M°,g) or $(\hat{M}^\circ,\hat{g})$ has constant curvature, then $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)=7$.
- (ii) Otherwise $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)=8$.

Proof. As before, we let E_1,E_2,E_3 and $\hat{E}_1,\hat{E}_2,\hat{E}_3$ to be some adapted frames of (M,g) and $(\hat{M},\hat{g})$ respectively. We will not fix the choice of q_0 in Q_0 (and hence do not define M°, $\hat{M}^\circ$) until the last half of the proof (where we introduce the sets $M_0,M_1,\hat{M}_0,\hat{M}_1$ below). Notice that Proposition 5.28 implies that $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\subset Q_0$, for every $q_0\in Q_0$.

The fact that $AE_2|_x\neq\pm\hat{E}_2|_{\hat{x}}$ for $q=(x,\hat{x};A)\in Q_0$ is equivalent to the fact that the intersection $(AE_2^\perp|_x)\cap\hat{E}_2^\perp|_{\hat{x}}$ is non-trivial for all $q=(x,\hat{x};A)\in Q_0$. Therefore, for a small enough open neighbourhood $\tilde{O}$ of q_0 inside Q_0, we may find a smooth functions $\theta,\hat{\theta}:\tilde{O}\to\mathbb{R}$ such that this intersection is spanned by $AZ_A=\hat{Z}_A$, where

$$\begin{aligned}Z_A:=&-\sin(\theta(q))E_1|_x+\cos(\theta(q))E_3|_x,\\ \hat{Z}_A:=&-\sin(\hat{\theta}(q))\hat{E}_1|_{\hat{x}}+\cos(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}.\end{aligned}$$

We also define

$$\begin{aligned}X_A:=&\cos(\theta(q))E_1|_x+\sin(\theta(q))E_3|_x,\\ \hat{X}_A:=&\cos(\hat{\theta}(q))\hat{E}_1|_{\hat{x}}+\sin(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}.\end{aligned}$$

To unburden the formulas, we set $s_\tau:=\sin(\tau(q))$, $c_\tau:=\cos(\tau(q))$ if $\tau:\tilde{O}\to\mathbb{R}$ is some function and the point $q\in\tilde{O}$ is clear from the context. Since $X_A,E_2|_x,Z_A$

(resp. $\hat{X}_A, \hat{E}_2|_{\hat{x}}, \hat{Z}_A$) form an orthonormal frame for every $q = (x, \hat{x}; A) \in \tilde{O}$ and because $A(Z_A^\perp) = \hat{Z}_A^\perp$, it follows that there is a smooth $\phi : O' \to \mathbb{R}$ such that

$$\begin{aligned}
AX_A =& c_{\hat{\phi}}\hat{X}_A + s_{\hat{\phi}}\hat{E}_2 = c_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + s_{\hat{\phi}}\hat{E}_2,\\
AE_2 =& - s_{\hat{\phi}}\hat{X}_A + c_{\hat{\phi}}\hat{E}_2 = -s_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + c_{\hat{\phi}}\hat{E}_2,\\
AZ_A =& \hat{Z}_A.
\end{aligned}$$

In particular, for all $q = (x, \hat{x}; A) \in \tilde{O}$, one has $\hat{g}(AZ_A, \hat{E}_2) = 0$. Note that for all $q = (x, \hat{x}; A) \in \tilde{O}$, since $A \star Z_A = \hat{\star}\hat{Z}_A A$,

$$\widetilde{\mathsf{Rol}}_q(\star Z_A) = R(\star Z_A) - A^{\overline{T}}\hat{R}(\hat{\star}\hat{Z}_A)A = -K \star Z_A + KA^{\overline{T}}\hat{\star}\hat{Z}_A A = 0,$$

and hence, since $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_x M \to \wedge^2 T|_x M$ is a symmetric map,

$$\begin{aligned}
\widetilde{\mathsf{Rol}}_q(\star X_A) =& - K_1^{\mathsf{Rol}}(q) \star X_A - \alpha \star E_2,\\
\widetilde{\mathsf{Rol}}_q(\star E_2) =& - \alpha \star X_A - K_2^{\mathsf{Rol}}(q) \star E_2,
\end{aligned}$$

for some smooth real-valued functions $K_1^{\mathsf{Rol}}, K_2^{\mathsf{Rol}}, \alpha$ defined on $\tilde{O}$.

We begin by considering the smooth 5-dimensional distribution Δ on the open subset $\tilde{O}$ of Q_0 spanned by

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q, \nu(A \star E_2)|_q, \nu(A \star X_A)|_q.$$

What will be shown is that $\mathrm{Lie}(\Delta)$ spans at every point $q \in \mathcal{O}$ a smooth distribution $\mathrm{Lie}(\Delta)|_q$ of dimension 8 which, by construction, is involutive. We consider $\mathrm{VF}^k_{\mathcal{D}_{\mathrm{R}}}, \mathrm{VF}^k_{\Delta}, \mathrm{Lie}(\Delta)$ as $C^\infty(\tilde{O})$-modules. Since $X_A = c_\theta E_1 + s_\theta E_3$, in order to compute brackets of the first 4 vector fields above against $\nu(A \star X_A)|_q$, we need to know some derivatives of θ. We begin by computing the following.

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \beta E_2,\\
\mathscr{L}_{\mathrm{R}}(E_2)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta + \Gamma^2_{(3,1)})X_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A.
\end{aligned}$$

Differentiating $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$ gives,

$$\begin{aligned}
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)}, \hat{Y}) + \hat{g}(AZ_A, \hat{\nabla}_{AX_A}\hat{E}_2)\\
=& \hat{g}(A(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \beta E_2), \hat{E}_2)\\
&+ \hat{g}(AZ_A, c_{\hat{\phi}}c_{\hat{\theta}}\beta\hat{E}_3 - c_{\hat{\phi}}s_{\hat{\theta}}\beta\hat{E}_1)\\
=& s_{\hat{\phi}}(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}) - \beta c_{\hat{\phi}} + c_{\hat{\phi}}s^2_{\hat{\theta}}\beta + c_{\hat{\phi}}c^2_{\hat{\theta}}\beta\\
=& s_{\hat{\phi}}(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}).
\end{aligned}$$

Since $s_{\hat{\phi}} \neq 0$ (because otherwise $AE_2 = \pm\hat{E}_2$), we get

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta = c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}.$$

In a similar way, differentiating $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(Z_A)|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q$, one finds

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta =& -s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)},\\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta =& -\beta + \Gamma^2_{(3,1)}.
\end{aligned}$$

Finally, applying $\nu(A\star E_2)|_q$ on the equation $\hat{g}(AZ_A,\hat{E}_2)=0$ gives,

$$\begin{aligned}0=&\hat{g}(\nu(A\star E_2)|_q((\cdot)Z_{(\cdot)},\hat{E}_2)=\hat{g}(A(\star E_2)Z_A-(\nu(A\star E_2)|_q\theta)AX_A,\hat{E}_2)\\=&(1-\nu(A\star E_2)|_q\theta)\hat{g}(AX_A,\hat{E}_2),\end{aligned}$$

and since $\hat{g}(AX_A,\hat{E}_2)=s_{\hat{\phi}}\neq 0$, $\nu(A\star E_2)|_q\theta=1$. Using the definition of X_A and Z_A, we may now summarize

$$\begin{aligned}&\mathscr{L}_{\mathrm{R}}(E_1)|_q\theta=\Gamma^1_{(3,1)}, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_q\theta=-\beta+\Gamma^2_{(3,1)},\\&\mathscr{L}_{\mathrm{R}}(E_3)|_q\theta=\Gamma^3_{(3,1)}, \quad \nu(A\star E_2)|_q\theta=1.\end{aligned}$$

By Proposition C.20 and the fact that $\beta\neq 0$, we see that VF^2_Δ contains the vector fields given by

$$\begin{aligned}L_1|_q=&\mathscr{L}_{\mathrm{NS}}(E_1)|_q-\beta\nu(A\star E_1)|_q,\\\tilde{L}_2|_q=&\mathscr{L}_{\mathrm{NS}}(E_2)|_q,\\L_3|_q=&\mathscr{L}_{\mathrm{NS}}(E_3)|_q-\beta\nu(A\star E_3)|_q,\end{aligned}$$

i.e., $\tilde{L}_2=\frac{1}{\beta}L_2$. Computing

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star X_{(\cdot)})]|_q=&-s_\theta\mathscr{L}_{\mathrm{R}}(E_2)|_q+s_\theta\tilde{L}_2|_q-s_\theta\beta\nu(A\star E_2)|_q,\\[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star X_{(\cdot)})]|_q=&-\mathscr{L}_{\mathrm{R}}(Z_A)|_q-s_\theta L_1|_q+c_\theta L_3|_q,\\[\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star X_{(\cdot)})]|_q=&c_\theta\mathscr{L}_{\mathrm{R}}(E_2)|_q-c_\theta\tilde{L}_2|_q-c_\theta\beta\nu(A\star E_2)|_q,\\[\nu((\cdot)\star E_2),\nu((\cdot)\star X_{(\cdot)})]|_q,=&0\end{aligned}$$

and since one also has

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_1),\mathscr{L}_{\mathrm{R}}(E_2)]|_q=&\mathscr{L}_{\mathrm{R}}([E_1,E_2])|_q-s_\theta K_1^{\mathsf{Rol}}\nu(A\star X_A)|_q-s_\theta\alpha\nu(A\star E_2)|_q,\\[\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(E_3)]|_q=&\mathscr{L}_{\mathrm{R}}([E_2,E_3])|_q-c_\theta K_1^{\mathsf{Rol}}\nu(A\star X_A)|_q-c_\theta\alpha\nu(A\star E_2)|_q,\\[\mathscr{L}_{\mathrm{R}}(E_3),\mathscr{L}_{\mathrm{R}}(E_1)]|_q=&\mathscr{L}_{\mathrm{R}}([E_3,E_1])|_q-\alpha\nu(A\star X_A)|_q-K_2^{\mathsf{Rol}}\nu(A\star E_2)|_q,\end{aligned}$$

we see using in addition Proposition C.20, case (ii) (the first three Lie brackets there), that VF^2_Δ is generated by the following 8 linearly independent vector fields defined on $\tilde{O}$ by

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,\nu(A\star X_A)|_q,L_1|_q,\tilde{L}_2|_q,L_3|_q.$$

We now proceed to show that $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$. According to Proposition C.20 case (ii) and the previous computations, we know that all the brackets between $\mathscr{L}_{\mathrm{R}}(E_1)$, $\mathscr{L}_{\mathrm{R}}(E_2)$, $\mathscr{L}_{\mathrm{R}}(E_3)$, $\nu((\cdot)\star E_2)$ and L_1,L_3 and also $[L_1,L_3]$ belong to VF^2_Δ, so we are left to compute the bracket of $\nu((\cdot)\star X_{(\cdot)}),\tilde{L}_2$ against L_1,L_3 and also $\tilde{L}_2$ against $\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,\nu((\cdot)\star X_{(\cdot)})|_q$. To do that, we need to know more derivatives of θ. Since $[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star E_2)]=\mathscr{L}_{\mathrm{R}}(E_3)|_q-L_3|_q$, we get

$$L_3|_q\theta=\mathscr{L}_{\mathrm{R}}(E_3)|_q\theta-\mathscr{L}_{\mathrm{R}}(E_1)|_q\big(\underbrace{\nu((\cdot)\star E_2)\theta}_{=1}\big)+\nu(A\star E_2)|_q\big(\underbrace{\mathscr{L}_{\mathrm{R}}(E_1)\theta}_{=\Gamma^1_{(3,1)}}\big)=\Gamma^3_{(3,1)},$$

and similarly, by using $[\mathscr{L}_{\rm R}(E_3),\nu((\cdot)\star E_2)]=-\mathscr{L}_{\rm R}(E_1)|_q+L_1|_q$, one gets $L_1|_q\theta=\Gamma^1_{(3,1)}$. On the other hand, $\mathscr{L}_{\rm NS}(E_2)|_q Z_{(\cdot)}=(-\mathscr{L}_{\rm NS}(E_2)|_q\theta+\Gamma^2_{(3,1)})X_A$, and to compute $\tilde{L}_2|_q\theta=\mathscr{L}_{\rm NS}(E_2)|_q\theta$, operate by $\mathscr{L}_{\rm NS}(E_2)|_q$ onto equation $\hat{g}(AZ_A,\hat{E}_2)=0$ to get $\tilde{L}_2|_q\theta=\Gamma^2_{(3,1)}$. With these derivatives of θ being available, we easily see that

$$\begin{aligned}
[L_1,\nu((\cdot)\star X_{(\cdot)})]|_q=&0,\\
[L_1,\tilde{L}_2]|_q=&(\Gamma^2_{(3,1)}+\beta)L_3|_q,\\
[L_3,\nu((\cdot)\star X_{(\cdot)})]|_q=&0,\\
[L_3,\tilde{L}_2]|_q=&-(\Gamma^2_{(3,1)}+\beta)L_1|_q,\\
[\mathscr{L}_{\rm R}(E_1),\tilde{L}_2]|_q=&\beta L_3|_q-\mathscr{L}_{\rm R}(\nabla_{E_2}E_1)|_q,\\
[\mathscr{L}_{\rm R}(E_2),\tilde{L}_2]|_q=&0,\\
[\mathscr{L}_{\rm R}(E_3),\tilde{L}_2]|_q=&-\beta L_1|_q-\mathscr{L}_{\rm R}(\nabla_{E_2}E_3)|_q,\\
[\nu((\cdot)\star E_2),\tilde{L}_2]|_q=&0,\\
[\nu((\cdot)\star X_{(\cdot)}),\tilde{L}_2]|_q=&0.
\end{aligned}$$

Hence we have proved that VF^2_Δ is involutive and hence

$$\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta.$$

There being 8 linearly independent generators for $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$, we conclude that the distribution $\mathcal{D}$ spanned pointwise on $\tilde{O}$ by $\mathrm{Lie}(\Delta)$ is integrable by Frobenius theorem. The choice of $q_0\in Q_0$ was arbitrary and we thus can build an 8-dimensional smooth involutive distribution $\mathcal{D}$ by the above construction on the whole Q_0. Since $\mathcal{D}_{\rm R}\subset\Delta\subset\mathcal{D}$, we have $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\subset\mathcal{O}_{\mathcal{D}}(q_0)$ for all $q_0\in Q_0$ and thus $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\leq 8$.

We will show when the equality holds here and show when actually $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)=7$. Define

$$\begin{aligned}
M_0=&\{x\in M\mid \beta^2\neq K_2(x)\},\\
M_1=&\{x\in M\mid \exists\text{ open }V\ni x\text{ s.t. }\forall x'\in V,\ \beta^2=K_2(x')\},\\
\hat{M}_0=&\{\hat{x}\in\hat{M}\mid \beta^2\neq \hat{K}_2(\hat{x})\},\\
\hat{M}_1=&\{\hat{x}\in\hat{M}\mid \exists\text{ open }\hat{V}\ni \hat{x}\text{ s.t. }\forall \hat{x}'\in \hat{V},\ \beta^2=\hat{K}_2(\hat{x}')\},
\end{aligned}$$

and notice that $M_0\cup M_1$ (resp. $\hat{M}_0\cup\hat{M}_1$) is an open dense subset of M (resp. $\hat{M}$). At this point we also fix $q_0\in Q_0$ and write $M^\circ=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$, $\hat{M}^\circ=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$ as in the statement of this proposition. Let $q_1=(x_1,\hat{x}_1;A_1)\in\pi_Q^{-1}(M_0\times\hat{M}_0)\cap Q_0$. Take an open neighbourhood $\tilde{O}$ of q_1 in Q_0 as above (now for q_1 instead of q_0 which we fixed) such that $\pi_Q(\tilde{O})\subset M_0\times\hat{M}_0$, and introduce on $\tilde{O}$ the vectors $X_A,Z_A,\hat{X}_A,\hat{Z}_A$ along with the angles $\theta,\hat{\theta},\hat{\phi}$, again as above. For $q\in\tilde{O}$, one has

$$\begin{aligned}
\begin{pmatrix}\widetilde{\mathsf{Rol}}_q(\star X_A)\\ \widetilde{\mathsf{Rol}}_q(\star E_2)\end{pmatrix}=&\begin{pmatrix}s^2_{\hat{\phi}}(-\beta^2+\hat{K}_2) & c_{\hat{\phi}}s_{\hat{\phi}}(-\beta^2+\hat{K}_2)\\ (-\beta^2+\hat{K}_2)s_{\hat{\phi}}c_{\hat{\phi}} & -K_2+s^2_{\hat{\phi}}\beta^2+c^2_{\hat{\phi}}\hat{K}_2\end{pmatrix}\begin{pmatrix}\star X_A\\ \star E_2\end{pmatrix},\\
\widetilde{\mathsf{Rol}}_q(\star Z_A)=&0.
\end{aligned}$$

The determinant $d(q)$ of the above matrix is equal to

$$d(q)=-s^2_{\hat{\phi}}(-K_2+\beta^2)(-\hat{K}_2+\beta^2),$$

so $d(q) \neq 0$ since $q \in \tilde{O} \subset \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap Q_0$. Since $\nu(\mathsf{Rol}(\star E_2)(A))|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$, we obtain that $\nu(A_1 \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. If $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_0 \times \overline{\hat{M}_0}) \cap Q_0$, then one can take a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ such that $q'_n \to q_1$ while $\hat{x}'_n \in \hat{M}_0$. Since M_0 and Q_0 are open, we have for large enough n that $q'_n \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap Q_0$, hence $\nu(A'_n \star E_2)|_{q'_n} \in T|_{q'_n}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ and by taking the limit as $n \to \infty$, we have $\nu(A_1 \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Suppose next $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_0 \times \hat{M}_1) \cap Q_0$. Then $\widetilde{\mathsf{Rol}}_{q_1}(\star E_1) = \widetilde{\mathsf{Rol}}_{q_1}(\star E_3) = 0$, $\widetilde{\mathsf{Rol}}_{q_1}(\star E_2) = (-K_2(x_1) + \beta^2) \star E_2$ with $K_2(x_1) \neq \beta^2$ and hence $\nu(A \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Thus we have proven that

$$\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

Changing the roles of M and $\hat{M}$ we also have

$$\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

On Q, define two 3-dimensional distributions $D, \hat{D}$ as follows, for $q \in Q$ let $\hat{D}|_q$ be the span of

$$\begin{aligned}
\hat{K}_1|_q &= \mathscr{L}_{\mathrm{NS}}(AE_1)|_q + \beta\nu(A \star E_1)|_q,\\
\hat{K}_2|_q &= \mathscr{L}_{\mathrm{NS}}(AE_2)|_q,\\
\hat{K}_3|_q &= \mathscr{L}_{\mathrm{NS}}(AE_3)|_q + \beta\nu(A \star E_3)|_q,
\end{aligned}$$

and $D|_q$ be the span of

$$\begin{aligned}
K_1|_q &= \mathscr{L}_{\mathrm{NS}}(A^T\hat{E}_1)|_q - \beta\nu((\hat{\star}\hat{E}_1)A)|_q,\\
K_2|_q &= \mathscr{L}_{\mathrm{NS}}(A^T\hat{E}_2)|_q,\\
K_3|_q &= \mathscr{L}_{\mathrm{NS}}(A^T\hat{E}_3)|_q - \beta\nu((\hat{\star}\hat{E}_3)A)|_q.
\end{aligned}$$

We claim that for any $q_1 = (x_1, \hat{x}_1; A_1) \in Q$ and any smooth paths $\gamma : [0,1] \to M$, $\hat{\gamma} : [0,1] \to \hat{M}$ with $\gamma(0) = x_1$, $\hat{\gamma}(0) = \hat{x}_1$ there are unique curves $\Gamma, \hat{\Gamma} : [0,1] \to Q$ of the same regularity as $\gamma, \hat{\gamma}$ such that Γ is tangent to D, $\Gamma(0) = q_1$ and $\pi_{Q,M}(\Gamma(t)) = \gamma$ and similarly $\hat{\Gamma}$ is tangent to $\hat{D}$, $\hat{\Gamma}(0) = q_1$ and $\pi_{Q,\hat{M}}(\hat{\Gamma}(t)) = \hat{\gamma}$. The key point here is that $\Gamma, \hat{\Gamma}$ are defined on $[0,1]$ and not only on a smaller interval $[0,T]$ with $T \leq 1$. We write these curves as $\Gamma = \Gamma(\gamma, q_1)$ and $\hat{\Gamma} = \hat{\Gamma}(\hat{\gamma}, q_1)$, respectively. Notice that since $(\pi_{Q,\hat{M}})_*D = 0$ and $(\pi_{Q,M})_*\hat{D} = 0$, one has

$$\pi_{Q,\hat{M}}(\Gamma(\gamma, q_1)(t)) = \hat{x}_1, \quad \pi_{Q,M}(\hat{\Gamma}(\hat{\gamma}, q_1)(t)) = x_1, \quad \forall t \in [0,1].$$

We only prove the above claim for D since the proof for $\hat{D}$ is similar. Uniqueness and local existence are straigthforward. Take some extension of γ to an interval $]-\epsilon, 1+\epsilon[=: I$ and write $\Gamma_1 := \Gamma(\gamma, q_1)$. Consider a trivialization (global since we assumed the frames E_i, $\hat{E}_i$, $i = 1,2,3$ to be global) of π_Q given by

$$\Phi : Q \to M \times \hat{M} \times \mathrm{SO}(n), \quad (x, \hat{x}; A) \mapsto (x, \hat{x}, \mathcal{M}_{F,\hat{F}}(A)),$$

where $F = (E_1, E_2, E_3)$, $\hat{F} = (\hat{E}_1, \hat{E}_2, \hat{E}_3)$. For every $(s, C) \in I \times \mathrm{SO}(n)$ one has

$$\Phi(\Gamma(\gamma(s + \cdot), \Phi^{-1}(\gamma(s), \hat{x}_1; C))(t)) = (\gamma(s+t), \hat{x}_1, B_{(s,C)}(t)),$$

where $B_{(s,C)}(t) \in \mathrm{SO}(n)$ and t in an open interval containing 0. On $I \times \mathrm{SO}(n)$, define a vector field

$$\mathcal{X}|_{(s,C)} := (\frac{\partial}{\partial t}, \dot{B}_{(s,C)}(0)).$$

If $\Phi(\Gamma(\gamma, q_1)(t)) = (\gamma(t), \hat{x}_1; C_1(t))$, then since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\Gamma_1(s)) =& \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\Phi(\Gamma(\gamma, q_1)(t+s)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\Phi\big(\Gamma(\gamma(s+\cdot), \Gamma(\gamma, q_1)(s))(t)\big)\\ =& \frac{\mathrm{d}}{\mathrm{d}t}\big|_0(\gamma(t+s), \hat{x}_1, B_{(s,C_1(s))}(t)) = (\dot{\gamma}(s), 0, (\mathrm{pr}_2)_*\mathcal{X}|_{(s,C_1(s))}),\end{aligned}$$

we see that $s \mapsto (s, (\mathrm{pr}_3 \circ \Phi \circ \Gamma_1)(s)) = (s, C_1(s))$ is the integral curve of $\mathcal{X}$ starting from $(0, C_1(0))$. Conversely, if $\Lambda_1(t) = (t, C(t))$ is the integral curve of $\mathcal{X}$ starting from $(0, C_1(0))$, then $\tilde{\Gamma}_1(t) := \Phi^{-1}(\gamma(t), \hat{x}_1, C(t))$ gives an integral curve of D starting from q_1 and $\pi_{Q,M}(\tilde{\Gamma}_1(t)) = \gamma(t)$.

Hence the maximal positive interval of definition of Γ_1 is the same as that of the integral curve Λ_1 of $\mathcal{X}$ starting from $(0, C_1)$. If it is of the form $[0, t_0[$ for some $t_0 < 1 + \epsilon$, then, because $[0, 1] \times \mathrm{SO}(n)$ is a compact subset of $I \times \mathrm{SO}(n)$, there is a $t_1 \in [0, t_0[$ with $\Lambda_1(t_1) \notin [0, 1] \times \mathrm{SO}(n)$ i.e. $t_1 \notin [0, 1]$ which is only possible if $t_1 > 1$, and thus $t_0 > 1$. We have shown that the existence of $\Gamma_1(t) = \Gamma(\gamma, q_1)(t)$ is guaranteed on the whole interval $[0, 1]$.

Since for all $q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M})$, which is an open subset of Q, one has $\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, it follows from Proposition C.20 that

$$\begin{aligned}L_1|_q &= \mathscr{L}_{\mathrm{NS}}(E_1)|_q - \beta\nu(A \star E_1)|_q,\\ \tilde{L}_2|_q &= \mathscr{L}_{\mathrm{NS}}(E_2)|_q,\\ L_3|_q &= \mathscr{L}_{\mathrm{NS}}(E_3)|_q - \beta\nu(A \star E_3)|_q,\end{aligned}$$

are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ and hence so are $\mathscr{L}_{\mathrm{R}}(E_1)|_q - L_1|_q = \hat{K}_1|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q - \tilde{L}_2 = \hat{K}_2|_q$ and $\mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q = \hat{K}_3|_q$ i.e.,

$$\hat{D}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

A similar argument shows that

$$D|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

Assume now that $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and that $M_0 \neq \emptyset$. Choose any $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ with $(x_1, \hat{x}_1) \in M_1 \times \hat{M}_0$ and take any curve $\gamma : [0, 1] \to M$ with $\gamma(0) = x_1$, $\gamma(1) \in M_0$. Then since $\pi_{Q,\hat{M}}(\Gamma(\gamma, q_1)(t)) = \hat{x}_1$, we have $\pi_Q(\Gamma(\gamma, q_1)(t)) \in M \times \hat{M}_0$ for all $t \in [0, 1]$ and since also $D|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_Q^{-1}(M \times \hat{M}_0)$, we have that $\Gamma(\gamma, q_1)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $t \in [0, 1]$. Indeed, suppose there is a $0 \leq t < 1$ with $\Gamma(\gamma, q_1)(t) \notin \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and define $t_1 = \inf\{t \in [0, 1] \mid \Gamma(\gamma, q_1)(t) \notin \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\}$. Clearly $t_1 > 0$. Because $q_2 := \Gamma(\gamma, q_1)(t_1) \in \pi_Q^{-1}(M \times \hat{M}_0)$, it follows that for $|t|$ small one has $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_2)$, whence if $t < 0$ small, $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_2) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which means that $q_2 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and thus for $t \geq 0$ small $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, a contradiction. Hence one has $\pi_Q(\Gamma(\gamma, q_1)(1)) \in (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. In other words we have the implication:

$$(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset, \quad M_0 \neq \emptyset \implies (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset.$$

By a similar argument, using $\hat{D}$ instead of D, one has that

$$(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset, \quad \hat{M}_0 \neq \emptyset \implies (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset.$$

Suppose now that there exists $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. We already know that $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ contains vectors

$$\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1}, \mathscr{L}_{\mathrm{R}}(E_2)|_{q_1}, \mathscr{L}_{\mathrm{R}}(E_3)|_{q_1}, \\
&\nu(A \star E_2)|_{q_1}, \nu((\hat{\star}\hat{E}_2)A)|_{q_1}, \\
&L_1|_{q_1}, \tilde{L}_2|_{q_1}, L_3|_{q_1},
\end{aligned}$$

which are linearly independent since $q_1 \in (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Indeed, if one introduces X_A, Z_A and an angle ϕ as before, we have $\sin(\phi(q_1)) \neq 0$ as $q_1 \in Q_0$ and

$$\nu((\hat{\star}\hat{E}_2)A_1)|_{q_1} = \nu(A_1 \star (A_1^{\overline{T}}\hat{E}_2))|_{q_1} = \sin(\phi(q_1))\nu(A_1 \star X_{A_1})|_{q_1} + \cos(\phi(q_1))\nu(A_1 \star E_2)|_{q_1}.$$

Therefore $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 8$ and since we have also shown that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$, we have that

$$(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset \implies \dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$$

Write $Q^\circ := \pi_Q^{-1}(M^\circ \times \hat{M}^\circ)$, which is an open subset of Q and clearly $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q^\circ$. To finish the proof, we proceed case by case.

a) Suppose $(\hat{M}^\circ, \hat{g})$ has constant curvature i.e. $\hat{M}_0 \cap \hat{M}^\circ = \emptyset$. By assumption then, (M°, g) does not have constant curvature, which means that $M_0 \cap M^\circ \neq \emptyset$.

   At every $q = (x, \hat{x}; A) \in Q^\circ$, one has $\widetilde{\mathsf{Rol}}_q(\star E_1) = \widetilde{\mathsf{Rol}}_q(\star E_3) = 0$ and $\widetilde{\mathsf{Rol}}_q(\star E_2) = (-K_2(x) + \beta^2) \star E_2$ and therefore

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(E_1), \mathscr{L}_{\mathrm{R}}(E_2)]|_q = \mathscr{L}_{\mathrm{R}}([E_1, E_2])|_q, \quad [\mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(E_3)]|_q = \mathscr{L}_{\mathrm{R}}([E_2, E_3])|_q, \\
&[\mathscr{L}_{\mathrm{R}}(E_3), \mathscr{L}_{\mathrm{R}}(E_1)]|_q = \mathscr{L}_{\mathrm{R}}([E_3, E_1])|_q + (-K_2(x) + \beta^2)\nu(A \star E_2)|_q.
\end{aligned}$$

   From these, Proposition C.20 case (ii) and from the brackets (as above)

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(E_1), \tilde{L}_2]|_q =& \beta L_3|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_2}E_1)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_3), \tilde{L}_2]|_q =& -\beta L_1|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_2}E_3)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), \tilde{L}_2]|_q =& 0, \\
[\nu((\cdot) \star E_2), \tilde{L}_2]|_q =& 0, \\
[L_1, \tilde{L}_2]|_q =& (\Gamma^2_{(3,1)} + \beta)L_3|_q, \\
[L_3, \tilde{L}_2]|_q =& -(\Gamma^2_{(3,1)} + \beta)L_1|_q,
\end{aligned}$$

we see that the distribution $\tilde{\mathcal{D}}$on Q° spanned by the 7 linearly independent vector fields

$$\mathscr{L}_{\mathrm{R}}(E_1), \mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(E_3), \nu((\cdot) \star E_2), L_1, \tilde{L}_2, L_3,$$

with $L_1, \tilde{L}_2, L_3$ as above, is involutive. Moreover $\tilde{\mathcal{D}}$ contains $\mathcal{D}_{\mathrm{R}}|_{Q^\circ}$, which implies $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}|_{Q^\circ}}(q_0) \subset \mathcal{O}_{\tilde{\mathcal{D}}}(q_0)$ and hence $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 7$.

To show the equality here, notice that since $M_0 \cap M^\circ \neq \emptyset$, one has that $O := \pi_{Q,M}^{-1}(M_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an open non-empty subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Moreover, because $K_2(x) \neq \beta^2$ on $M_0 \cap M^\circ$, we get that $\nu(A \star E_2)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q \in O$, from which one deduces by Proposition C.20, case (i) that $\tilde{\mathcal{D}}|_q \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which then implies $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 7$. This proves one half of case (i) in the statement of this proposition.

b) If (M°, g) has constant curvature, one proves as in case a), by simply changing the roles of M and $\hat{M}$, that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$. This finishes the proof of case (i) of this proposition.

For the last case, we assume that neither (M°, g) nor $(\hat{M}^\circ, \hat{g})$ have constant curvature i.e. we have $M^\circ \cap M_0 \neq \emptyset$ and $\hat{M}^\circ \cap \hat{M}_0 \neq \emptyset$.

c) Since $M^\circ \cap M_0 \neq \emptyset$, there is a $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $x_1 \in M_0$. If $\hat{x}_1 \in \hat{M}_0$, we have $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and which implies, as we have shown, that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Suppose then that $\hat{x}_1 \in \overline{\hat{M}_1}$. Then one may choose a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $q'_n \to q_1$ and $\hat{x}'_n \in \hat{M}_1$. Because M_0 is open, for n large enough one has $(x'_n, \hat{x}'_n) \in (M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Hence $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and $\emptyset \neq \hat{M}^\circ \cap \hat{M}_0 \subset \hat{M}_0$, which has been shown to imply that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and again $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

The proof is complete. □

Remark 5.31 One could adapt the proofs of Propositions 5.28, 5.29 and 5.30 to deal also with the case $\beta = 0$. For example, Proposition 5.28 as formulated already is valid in this case, but the conclusion when $\beta = 0$ could be strengthened to $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 6$. However, since a Riemannian manifold of class $\mathcal{M}_0$ is also locally a Riemannian product, and hence locally a warped product, we prefer to view this special case $\beta = 0$ as part of the subject of Subsection 5.3.2.

5.3.2 Case where both manifolds are Warped Products

Suppose $(M, g) = (I \times N, h_f)$ and $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, where $I, \hat{I} \subset \mathbb{R}$ are open intervals, (N, h) and $(\hat{N}, \hat{h})$ are connected, oriented 2-dimensional Riemannian manifolds and the warping functions $f, \hat{f}$ are smooth and positive everywhere. We write $\frac{\partial}{\partial r}$ for the canonical, positively directed unit vector field on $(\mathbb{R}, s_1)$ and consider it as a vector field on (M, g) and $(\hat{M}, \hat{g})$ as is usual in direct products. Notice that then $\frac{\partial}{\partial r}$ is a g-unit (resp. $\hat{g}$-unit) vector field on M (resp. $\hat{M}$) which is orthogonal to $T|_y N$ (resp. $T|_{\hat{y}} \hat{N}$) for every $(r, y) \in M$ (resp. $(\hat{r}, \hat{y}) \in \hat{N}$). We will prove that starting from any point point $q_0 \in Q = Q(M, \hat{M})$ and if the warping functions $f, \hat{f}$ satisfy extra conditions relative to each other, then the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is either 6- or 8-dimensional. The first case is formulated in the following proposition.

Proposition 5.32 Let $(M, g) = (I \times N, h_f)$, $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$ be warped products of dimension 3, with $I, \hat{I} \subset \mathbb{R}$ open intervals. Also, let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ be such that if one writes $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, then

$$A_0 \frac{\partial}{\partial r}\big|_{(r_0,y_0)} = \frac{\partial}{\partial r}\big|_{(\hat{r}_0,y_0)}. \tag{50}$$

holds and

$$\frac{f'(t+r_0)}{f(t+r_0)} = \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}, \quad \forall t \in (I - r_0) \cap (\hat{I} - \hat{r}_0). \tag{51}$$

Then if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, one has $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$.

Proof. For convenience we write $\kappa(r) := \frac{f'(r+r_0)}{f(r+r_0)} = \frac{\hat{f}'(r+\hat{r}_0)}{\hat{f}(r+\hat{r}_0)}$, $r \in (I-r_0)\cap(\hat{I}-\hat{r}_0) =: J$. Let γ be a smooth curve in M defined on some interval containing 0 and such that $\gamma(0) = x_0$ and let $(\gamma(t), \hat{\gamma}(t); A(t)) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)$ be the rolling curve generated by γ starting at q_0 and defined on some (possible smaller) maximal interval containing 0. Write $\gamma(t) = (r(t), \gamma_1(t))$ and $\hat{\gamma}(t) = (\hat{r}(t), \hat{\gamma}_1(t))$ corresponding to the direct products $M = I \times N$ and $\hat{M} = \hat{I} \times \hat{N}$. Define also,

$$\begin{aligned}
\zeta(t) &:= r(t) - r_0, \quad S(t) := \frac{\partial}{\partial r}\big|_{\gamma(t)}, \\
\hat{\zeta}(t) &:= \hat{r}(t) - \hat{r}_0, \quad \hat{S}(t) := A(t)^{-1}\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)},
\end{aligned}$$

which are vector fields on M along γ. Notice that

$$\begin{aligned}
\dot{\zeta}(t) = \dot{r}(t) =& g\big(\dot{\gamma}(t), \frac{\partial}{\partial r}\big|_{\gamma(t)}\big) = g(\dot{\gamma}(t), S(t)), \\
\dot{\hat{\zeta}}(t) = \dot{\hat{r}}(t) =& \hat{g}\big(\dot{\hat{\gamma}}(t), \frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big) = \hat{g}\big(A(t)\dot{\gamma}(t), \frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big) = g(\dot{\gamma}(t), \hat{S}(t)).
\end{aligned}$$

By Proposition 35, Chapter 7, p. 206 in [35], we have

$$\begin{aligned}
\nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial r} =& \frac{f'(r(t))}{f(r(t))}\big(\dot{\gamma}(t) - \dot{r}(t)\frac{\partial}{\partial r}\big|_{\gamma(t)}\big), \\
=& \kappa(\zeta(t))\big(\dot{\gamma}(t) - \dot{\zeta}(t)\frac{\partial}{\partial r}\big|_{\gamma(t)}\big), \\
\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\frac{\partial}{\partial r} =& \frac{\hat{f}'(\hat{r}(t))}{\hat{f}(\hat{r}(t))}\big(\dot{\hat{\gamma}}(t) - \dot{\hat{r}}(t)\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big), \\
=& \kappa(\hat{\zeta}(t))\big(\dot{\hat{\gamma}}(t) - \dot{\hat{\zeta}}(t)\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\nabla_{\dot{\gamma}(t)}S(t) =& \kappa(\zeta(t))(\dot{\gamma}(t) - \dot{\zeta}(t)S(t)), \\
\nabla_{\dot{\gamma}(t)}\hat{S}(t) =& A(t)^{-1}\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\frac{\partial}{\partial r} = \kappa(\hat{\zeta}(t))\big(A(t)^{-1}\dot{\hat{\gamma}}(t) - \dot{\hat{\zeta}}(t)A(t)^{-1}\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big), \\
=& \kappa(\hat{\zeta}(t))(\dot{\gamma}(t) - \dot{\hat{\zeta}}(t)\hat{S}(t)).
\end{aligned}$$

Let $\rho \in C^\infty(\mathbb{R})$ and $t \mapsto X(t)$ be a vector field along γ and consider a first order ODE

$$\begin{cases} \dot{\rho}(t) = g(\dot{\gamma}(t), X(t)), \\ \nabla_{\dot{\gamma}(t)} X = \kappa(\rho(t))(\dot{\gamma}(t) - \dot{\rho}(t)X(t)). \end{cases}$$

By the above we see that the pairs $(\rho, X) = (\zeta, S)$ and $(\rho, X) = (\hat{\zeta}, \hat{S})$ both solve this ODE. Moreover, by assumption $\zeta(0) = 0 = \hat{\zeta}(0)$ and $\hat{S}(0) = A(0)^{-1}\frac{\partial}{\partial r}\big|_{\hat{x}_0} = \frac{\partial}{\partial r}\big|_{x_0} = S(0)$ so these pairs have the same initial conditions and hence $(\zeta, S) = (\hat{\zeta}, \hat{S})$ on the interval where they are both defined. Then,

$$\begin{aligned} r(t) - r_0 =& \hat{r}(t) - \hat{r}_0, \\ A(t)\frac{\partial}{\partial r}\big|_{\gamma(t)} =& \frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}, \end{aligned}$$

for all t in the interval where the rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)$ is defined. Define

$$Q_+^* = \big\{q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q \mid r - r_0 = \hat{r} - \hat{r}_0,\ A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}\big\}.$$

By the above considerations, as long as the curve is defined,

$$q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) \in Q_+^*,$$

which implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q_+^*$. We show that Q_+^* is a 6-dimensional submanifold of Q. Let $q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q$ such that $A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}$. Then for all $\alpha \in \mathbb{R}, X' \in T|_y N$ one has

$$\|X'\|_g^2 + \alpha^2 = \Big\|X' + \alpha\frac{\partial}{\partial r}\big|_x\Big\|_g^2 = \Big\|A(X' + \alpha\frac{\partial}{\partial r}\big|_x)\Big\|_{\hat{g}}^2 = \|AX'\|_{\hat{g}}^2 + 2\hat{g}\big(AX', \alpha\frac{\partial}{\partial r}\big|_{\hat{x}}\big) + \alpha^2.$$

This implies that

$$\begin{aligned} &\|X'\|_g^2 = \|AX'\|_{\hat{g}}^2, \\ &\hat{g}\big(AX', \frac{\partial}{\partial r}\big|_{\hat{x}}\big) = 0, \end{aligned}$$

for all $X' \in T|_y N$. Thus $AT|_y N \perp \frac{\partial}{\partial r}\big|_{\hat{x}}$ and also $A\frac{\partial}{\partial r}\big|_x \perp T|_{\hat{y}}\hat{N}$ by assumption. Define

$$Q_1^+ = \big\{q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q \mid A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}\big\},$$

and let $q_1 = (x_1, \hat{x}_1; A_1) = ((r_1, y_1), (\hat{r}_1, \hat{y}_1); A_1) \in Q_1^+$. Choose a local oriented h- and $\hat{h}$-orthonormal frames X_1', X_2' in N around y_1 and $\hat{X}_1', \hat{X}_2'$ in $\hat{N}$ around $\hat{y}_1$. Let the corresponding domains be U' and $\hat{U}'$. Writing $E_1 = \frac{\partial}{\partial r}$, $E_2 = \frac{1}{f}X_1'$, $E_3 = \frac{1}{f}X_2'$ on M and $\hat{E}_1 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{1}{\hat{f}}\hat{X}_1'$, $\hat{E}_3 = \frac{1}{\hat{f}}\hat{X}_2'$ on $\hat{M}$, we see that E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are g- and $\hat{g}$-orthonormal oriented frames and we define

$$\begin{aligned} &\Psi : V := \pi_Q^{-1}((\mathbb{R} \times U') \times (\mathbb{R} \times \hat{U}')) \to \mathrm{SO}(3), \\ &\Psi(x, \hat{x}; A) = [(\hat{g}(AE_i, \hat{E}_j))_i^j]. \end{aligned}$$

This is a chart of Q and clearly

$$\Psi(V\cap Q_1^+)=(\mathbb{R}\times U')\times(\mathbb{R}\times\hat{U}')\times\Big\{\begin{pmatrix}1 & 0\\ 0 & A'\end{pmatrix}\mid A'\in\mathrm{SO}(2)\Big\}.$$

This shows that $Q_1^+\cap V$ is a 7-dimensional submanifold of Q and hence Q_1^+ is a closed 7-dimensional submanifold of Q. Defining $F:Q_1^+\to\mathbb{R}$ by $F((r,y),(\hat{r},\hat{y});A)=(r-r_0)-(\hat{r}-\hat{r}_0)$, we see that $Q_+^*=F^{-1}(0)$. Once we show that F is a submersion, it follows that Q_+^* is a closed codimension 1 submanifold of Q_1^+ (i.e. $\dim Q_+^*=7-1=6$) and thus it is a 6-dimensional submanifold of Q. Indeed, let $q=(x,\hat{x};A)\in Q_1^+$ and let $\gamma(t)$ be an integral curve of $\frac{\partial}{\partial r}$ starting from x and $\hat{\gamma}(t)=\hat{x}$ a constant path. Let $q(t)=(\gamma(t),\hat{\gamma}(t);A(t))$ be the $\mathcal{D}_{\mathrm{NS}}$-lift of $(\gamma,\hat{\gamma})$ starting from q. Then $\dot{\gamma}(t)=\frac{\partial}{\partial r}\big|_{\gamma(t)}$, $\dot{\hat{\gamma}}(t)=0$ and since $\frac{\partial}{\partial r}$ is a unit geodesic field on M, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{g}\big(A(t)\frac{\partial}{\partial r}\big|_{\gamma(t)},\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=\hat{g}\big(A(t)\nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial r},\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)+\hat{g}\big(A(t)\frac{\partial}{\partial r},\hat{\nabla}_0\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=0.$$

This shows that $q(t)\in Q_1^+$ for all t and in particular, $\mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}\big|_x)|_q=\dot{q}(0)\in T|_qQ_1^+$. Then if one writes $\gamma(t)=(r(t),\gamma_1(t))$, $\hat{\gamma}(t)=\hat{x}=(\hat{r},\hat{y})$=constant, one has $\dot{r}(t)=1$ and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0F(q(t))=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big((r(t)-r_0)-(\hat{r}-\hat{r}_0)\big)=1,$$

i.e., $F_*\mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}\big|_x)|_q=1$, which shows that F is submersive. (Alternatively, one could have uses the charts Ψ as above to prove this fact.) Since we have shown that $\dim Q_+^*=6$ and $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q_+^*$, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\leq 6$. To prove the equality here, we will use the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$. Take local frames $E_i,\hat{E}_i$ as above near x_1 and $\hat{x}_1$, where $q_1=(x_1,\hat{x}_1;A_1)=((r_1,y_1),(\hat{r}_1,\hat{y}_1);A_1)\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. The assumption that $\frac{f'(t+r_0)}{f(t+r_0)}=\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}$ for all $t\in J$ easily imply that $\frac{f''(t+r_0)}{f(t+r_0)}=\frac{\hat{f}''(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}=:\kappa_2(t)$ for all $t\in J$ as well. Respect to the frames $\star E_1,\star E_2,\star E_3$ and $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}E_3$ one has (see Proposition 42, Chapter 7, p. 210 of [35])

$$R|_{(r,y)}=\begin{pmatrix}-\frac{\sigma(y)}{f(r)^2}+\kappa(r-r_0)^2 & 0 & 0\\ 0 & \kappa_2(r-r_0) & 0\\ 0 & 0 & \kappa_2(r-r_0)\end{pmatrix},$$

$$\hat{R}|_{(\hat{r},\hat{y})}=\begin{pmatrix}-\frac{\hat{\sigma}(\hat{y})}{\hat{f}(\hat{r})^2}+\kappa(\hat{r}-\hat{r}_0)^2 & 0 & 0\\ 0 & \kappa_2(\hat{r}-\hat{r}_0) & 0\\ 0 & 0 & \kappa_2(\hat{r}-\hat{r}_0)\end{pmatrix},$$

where $\sigma(y)$ and $\hat{\sigma}(\hat{y})$ are the unique sectional curvatures of (N,h) and $(\hat{N},\hat{h})$ at points $y,\hat{y}$. Write

$$-K_2(r,y)=-\frac{\sigma(y)}{f(r)^2}+\kappa(r-r_0),\quad -\hat{K}_2(\hat{r},\hat{y})=-\frac{\hat{\sigma}(\hat{y})}{\hat{f}(\hat{r})^2}+\kappa(\hat{r}-\hat{r}_0).$$

Since $A_1\frac{\partial}{\partial r}\big|_{x_1}=\frac{\partial}{\partial r}\big|_{\hat{x}_1}$, we already know that $A_1E_2|_{x_1}$ and $A_1E_3|_{x_1}$ are in the plane $\mathrm{span}\{\hat{E}_2|_{\hat{x}_1},\hat{E}_3|_{\hat{x}_1}\}$. This and the fact that $r_1-r_0=\hat{r}_1-\hat{r}_0$ imply that

$$\widetilde{\mathsf{Rol}}_{q_1}=\begin{pmatrix}-K_2(x_1)+\hat{K}_2(\hat{x}_1) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

w.r.t. $\star E_1|_{x_1}, \star E_2|_{x_1}, \star E_3|_{x_1}$. Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, it follows from Corollary 4.16 and Remark 4.17 that there is a $q_1 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $\widetilde{\mathsf{Rol}}_{q_1} \neq 0$. Hence there is a neighbourhood O of q_1 in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $\widetilde{\mathsf{Rol}}_q \neq 0$. With respect to local frames $E_i, \hat{E}_i$ as above (taking O smaller if necessary), this means that $K_2(x) \neq \hat{K}_2(\hat{x})$ for all $q = (x, \hat{x}; A) \in O$ and since $\nu(\mathsf{Rol}_q(\star E_1))|_q = (-K_2(x) + \hat{K}_2(\hat{x}))\nu(A \star E_1)|_q$, we have

$$\nu(A \star E_1)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), \quad \forall q \in O.$$

Hence applying Proposition C.20 case (i) to the frame $F_1 := E_2$, $F_2 := E_1$, $F_3 := E_3$ implies that the 6 linearly independent vectors (notice that we have $\Gamma^1_{(2,3)} = 0$ in that proposition)

$$\mathscr{L}_{\mathrm{R}}(F_1)|_q, \mathscr{L}_{\mathrm{R}}(F_2)|_q, \mathscr{L}_{\mathrm{R}}(F_3)|_q, \nu(A \star F_2)|_q, L_1|_q, L_3|_q,$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at $q \in O$, where

$$L_1 = \mathscr{L}_{\mathrm{NS}}(F_1)|_q - \Gamma^1_{(1,2)}(x)\nu(A \star F_3)|_q,$$
$$L_3 = \mathscr{L}_{\mathrm{NS}}(F_3)|_q + \Gamma^1_{(1,2)}(x)\nu(A \star F_1)|_q,$$

with $\Gamma^1_{(1,2)}(x) = g(\nabla_{F_1} F_1 F_2) = g(\nabla_{E_2} E_2, E_1) = -\frac{f'(r)}{f(r)}$ if $x = (r, y)$. Hence $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 6$. □

Remark 5.33 The condition $\mathsf{Rol}_{q_1} \neq 0$ in the proof of the previous proposition was equivalent to the condition $K_2(x_1) \neq \hat{K}_2(\hat{x}_1)$ which again means that if $x_1 = (r_1, y_1)$, $\hat{x}_1 = (\hat{x}_1, \hat{y}_1)$,

$$\frac{\sigma(y_1)}{f(r_1)^2} \neq \frac{\hat{\sigma}(\hat{y}_1)}{\hat{f}(r_1)^2},$$

where $\sigma(y)$ (resp. $\hat{\sigma}(\hat{y})$) is the sectional curvature of (N, h) at $y \in N$ (resp. of $(\hat{N}, \hat{h})$ at $\hat{y} \in \hat{N}$).

Remark 5.34 To show that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 6$ under the assumptions of the proposition, we showed that if $q = (x, \hat{x}; A) \in Q^*_+$, then $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(t) \in Q^*_+$ for any path γ starting from x. For this we basically used the uniqueness of the solutions of an ODE. Alternatively, one could have proceeded exactly in the same way as in the proof of Proposition 5.28. To this end, one defines as there $h_1, h_2 : Q \to \mathbb{R}$ and also $F : Q \to \mathbb{R}$ as above as

$$h_1(q) = \hat{g}(AE_1, \hat{E}_2), \quad h_2(q) = \hat{g}(AE_3, \hat{E}_2), \quad F(q) = (r - r_0) - (\hat{r} - \hat{r}_0).$$

Write $H = (h_1, h_2, F) : Q \to \mathbb{R}^3$, $Q^* := H^{-1}(0)$ and $Q = Q^*_+ \cup Q^*_-$ where Q^*_+ (resp. Q^*_-) consists of all $q = (x, \hat{x}; A) \in Q^*$ where $A\frac{\partial}{\partial r} = +\frac{\partial}{\partial r}$ (resp. $A\frac{\partial}{\partial r} = -\frac{\partial}{\partial r}$). Then, for all $q \in Q^*_+$,

$$H_*\nu(A \star E_1)|_q = (0, -1, 0), \quad H_*\nu(A \star E_3)|_q = (1, 0, 0), \quad H_*\mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}, 0)|_q = (0, 0, 1),$$

which shows (again) that Q^*_+ is a 6-dimensional closed submanifold of Q (and so is Q^*) while w.r.t. orthonormal bases E_1, E_2, E_3, $\hat{E}_1, \hat{E}_2, \hat{E}_3$, where $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$, one has for $q = (x, \hat{x}; A) \in Q^*_+$, since $x = (r, y)$, $\hat{x} = (\hat{r}, \hat{y})$ with $r - r_0 = \hat{r} - \hat{r}_0 =: t$,

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_1 &= \hat{g}(A(\Gamma^1_{(1,2)}E_2 - \Gamma^1_{(3,1)}E_3), \hat{E}_2) + \hat{g}(AE_1, -\hat{\Gamma}^1_{(1,2)}AE_1) \\
&= -\frac{f'(r)}{f(r)} + \frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})} = -\frac{f'(t+r_0)}{f(t+r_0)} + \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)} = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_2 &= \Gamma^1_{(3,1)}\hat{g}(AE_1, \hat{E}_2) + \hat{g}(AE_3, -\hat{\Gamma}^1_{(1,2)}AE_1) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_1 &= -\Gamma^2_{(3,1)}\hat{g}(AE_3, \hat{E}_2) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_2 &= \Gamma^2_{(3,1)}\hat{g}(AE_1, \hat{E}_2) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_1 &= \mathscr{L}_{\mathrm{R}}(E_3)|_q h_2 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q F &= \mathscr{L}_{\mathrm{R}}(E_2)|_q F = \mathscr{L}_{\mathrm{R}}(E_3)|_q F = 0,
\end{aligned}$$

hence $\mathcal{D}_{\mathrm{R}}|_q \subset T|_q Q^*_+$ for all $q \in Q^*_+$. This obviously implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \subset Q^*_+$ for all $q \in Q^*_+$ and thus $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \leq \dim Q^*_+ = 6$.

For the following proposition we introduce some more notations,

$$\begin{aligned}
Q_0 :=&\{q = ((r,y), (\hat{r}, \hat{y}); A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} \neq \pm\frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1^+ :=&\{q = (x, \hat{x}; A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} = +\frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1^- :=&\{q = (x, \hat{x}; A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} = -\frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1 :=&Q_1^+ \cup Q_1^-, \\
S_1^+ :=&\{q = ((r,y), (\hat{r}, \hat{y}); A) \in Q_1^+ \mid \frac{f'(r)}{f(r)} = +\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}\}, \\
S_1^- :=&\{q = ((r,y), (\hat{r}, \hat{y}); A) \in Q_1^- \mid \frac{f'(r)}{f(r)} = -\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}\}, \\
S_1 :=&S_1^+ \cup S_1^-.
\end{aligned}$$

We have that Q decomposes into the disjoint union

$$Q = S_1 \cup (Q \backslash S_1) = S_1 \cup (Q_1 \backslash S_1) \cup Q_0.$$

Proposition 5.35 Let $(M, g) = (I \times N, h_f)$ and $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, be warped products with $I, \hat{I} \subset \mathbb{R}$ open intervals and suppose that there is a constant $K \in \mathbb{R}$ such that

$$\frac{f''(r)}{f(r)} = -K = \frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}, \quad \forall (r, \hat{r}) \in I \times \hat{I}.$$

Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and write $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Assuming that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, we have the following cases:

(i) If $q_0 \in S_1$, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;

(ii) If $q_0 \in Q\backslash S_1$ and if only one of (M°, g) or $(\hat{M}^\circ, \hat{g})$ has constant curvature, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;

(iii) Otherwise $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Proof. As in the proof of Proposition 5.32 (see also Remark 5.34) it is clear that Q_1 is a closed 7-dimensional closed submanifolds of Q and Q_1^-, Q_1^+ are disjoint open and closed submanifolds of Q_1. Also, S_1, S_1^+, S_1^- are closed subsets of Q_1. Let us begin with the case where $q_0 \in S_1^+$. Writing $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$ and defining $w(t) := \frac{f'(t+r_0)}{f(t+r_0)} - \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}$, we see that for all $t \in (I - r_0) \cap (\hat{I} - \hat{r}_0)$,

$$w'(t) = \underbrace{\frac{f''(t+r_0)}{f(t+r_0)}}_{=-K} - \Big(\frac{f'(t+r_0)}{f(t+r_0)}\Big)^2 - \underbrace{\frac{\hat{f}''(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}}_{=-K} + \Big(\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}\Big)^2,$$

i.e.,

$$w'(t) = -w(t)\Big(\frac{f'(t+r_0)}{f(t+r_0)} + \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}\Big), \quad w(0) = 0.$$

This shows that $w(t) = 0$ for all $t \in (I - r_0) \cap (\hat{I} - \hat{r}_0)$ and hence the assumptions of Proposition 5.32 have been met. Thus $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$. On the other hand, if $q_0 = (x_0, \hat{x}_0; A_0) \in S_1^-$ and $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, define $\hat{f}^\vee(t) := \hat{f}(-t)$, $\hat{I}^\vee := -\hat{I}$ and notice that $\varphi : (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}}) \to (\hat{I}^\vee \times \hat{N}, \hat{h}_{\hat{f}^\vee}) =: (\hat{M}^\vee, \hat{g}^\vee)$ given by $(\hat{y}, \hat{r}) \mapsto (\hat{y}, -\hat{r})$ is an isometry, which induces a diffeomorphism $\Phi : Q \to Q(M, \hat{M}^\vee)$ by $(x, \hat{x}; A) \mapsto (x, \varphi(\hat{x}); \varphi_*|_{\hat{x}} \circ A)$ which preserves the respective rolling distributions and orbits: $\Phi_*(\mathcal{D}_{\mathrm{R}}|_q) = \mathcal{D}_{\mathrm{R}}^\vee|_{\Phi(q)}$, $\Phi(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}^\vee}(\Phi(q))$, the notation being clear here. But now $\Phi(A_0) = \varphi_*(A_0 \frac{\partial}{\partial r}) = -\varphi_* \frac{\partial}{\partial r} = \frac{\partial}{\partial r}$ and since $q_0^\vee := \Phi(q_0) = ((r_0, y_0), (-\hat{r}_0, \hat{y}_0); \varphi_* \circ A_0)$,

$$\frac{(\hat{f}^\vee)'(-\hat{r}_0)}{\hat{f}^\vee(-\hat{r}_0)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \hat{f}^\vee(t - \hat{r}_0)}{\hat{f}(\hat{r}_0)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \hat{f}(\hat{r}_0 - t)}{\hat{f}(\hat{r}_0)} = -\frac{\hat{f}'(\hat{r}_0)}{\hat{f}(\hat{r}_0)} = \frac{f'(r_0)}{f(r_0)}.$$

Thus $\Phi(q_0)$ belongs to the set S_1^+ of $Q(M, \hat{M}^\vee)$ (which corresponds by Φ to S_1^- of Q) and thus the above argument implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}^\vee}(\Phi(q_0)) = 6$ and therefore $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$. Hence we have proven (i). We next deal with the case where $q_0 \in Q\backslash S_1$. Up until the second half of the proof, where we introduce the sets $M_0, M_1, \hat{M}_0, \hat{M}_1$, we assume that the choice of $q_0 \in Q\backslash S_1$ is not fixed (and hence M°, $\hat{M}^\circ$ are not defined yet). So let $q_0 = (x_0, \hat{x}_0; A_0) = ((r_0, y_0), (\hat{r}_0, \hat{y}_0); A_0) \in Q\backslash S_1$ and choose some orthonormal frame X_1, X_3 (resp. $\hat{X}_1, \hat{X}_3$) on N (resp. $\hat{N}$) defined on an open neighbourhood U' of y_0 (resp. $\hat{U}'$ of $\hat{y}_0$) and consider them, in the natural way, as vector fields on M (resp. $\hat{M}$). Moreover, assume that $X_1, \frac{\partial}{\partial r}, X_3$ (resp. $\hat{X}_1, \frac{\partial}{\partial r}, \hat{X}_3$) is oriented. Writing $E_1 = \frac{1}{f} X_1$, $E_2 = \frac{\partial}{\partial r}$, $E_3 = \frac{1}{f} X_3$, and $\hat{E}_1 = \frac{1}{\hat{f}} \hat{X}_1$, $\hat{E}_2 = \frac{\partial}{\partial r}$, $\hat{E}_3 = \frac{1}{\hat{f}} \hat{X}_3$, we get positively oriented orthonormal frames of M and $\hat{M}$, defined on $U := I \times U'$, $\hat{U} := \hat{I} \times \hat{U}'$, respectively. Then we have, by [35], Chapter 7, Proposition 42 (one should pay attention that there the definition of the curvature tensor differs by sign to the definition used here) that with respect to the frames $\star E_1, \star E_2, \star E_3$ and $\hat{\star}\hat{E}_1, \hat{\star}\hat{E}_2, \hat{\star}\hat{E}_3$,

$$R = \begin{pmatrix} -K & 0 & 0 \\ 0 & \frac{-\sigma+(f')^2}{f^2} & 0 \\ 0 & 0 & -K \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} -K & 0 & 0 \\ 0 & \frac{-\hat{\sigma}+(\hat{f}')^2}{\hat{f}^2} & 0 \\ 0 & 0 & -K \end{pmatrix},$$

where $\sigma(y)$ and $\hat{\sigma}(\hat{y})$ are the unique sectional (or Gaussian) curvatures of (N, h) and $(\hat{N}, \hat{h})$ at points $y, \hat{y}$. Write $-K_2 := \frac{-\sigma+(f')^2}{f^2}$ and $-\hat{K}_2 := \frac{-\hat{\sigma}+(\hat{f}')^2}{\hat{f}^2}$. We now take an open neighbourhood $\tilde{O}$ of q_0 in Q according to the following cases:

(a) If $q_0 \in Q_0$, we assume that $\tilde{O} \subset Q_0 \cap \pi_Q^{-1}(U \times \hat{U})$.

(b) If $q_0 \in Q_1^+ \backslash S_1$ (resp. $q_0 \in Q_1^- \backslash S_1$) we assume that $\tilde{O} \subset \pi_Q^{-1}(U \times \hat{U}) \backslash (S_1 \cup Q_1^-)$ (resp. $\tilde{O} \subset \pi_Q^{-1}(U \times \hat{U}) \backslash (S_1 \cup Q_1^+)$).

Write $\tilde{O}_0 := \tilde{O} \cap Q_0$. Thus in case (a) one has $\tilde{O} = \tilde{O}_0 \ni q_0$ while in case (b) one has $\tilde{O} = \tilde{O}_0 \cup (\tilde{O} \cap (Q_1^{\pm} \backslash S_1))$, as a disjoint union, and $q_0 \notin \tilde{O}_0$, the "$\pm$" depending on the respective situation. Moreover, if the case (b) occurs, we assume that $q_0 \in Q_1^+ \backslash S_1$ since the case where $q_0 \in Q_1^- \backslash S_1$ is handled in a similar way. We will still shrink $\tilde{O}$ around q_0 whenever convenient and always keep in mind that $\tilde{O}_0 = \tilde{O} \cap Q_0$ even after the shrinking. Notice that this shrinking does not change the properties in (a) and (b) above. Moreover, [35], Chapter 7, Proposition 35 implies that if Γ, $\hat{\Gamma}$ are connection tables w.r.t. E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$, respectively,

$$\Gamma = \begin{pmatrix} 0 & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & 0 \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} 0 & 0 & -\hat{\Gamma}^1_{(1,2)} \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} W(\Gamma^1_{(1,2)}) &= 0, \quad \forall W \in E_2^{\perp}, \\ \hat{W}(\hat{\Gamma}^1_{(1,2)}) &= 0, \quad \forall \hat{W} \in \hat{E}_2^{\perp}, \end{aligned}$$

since $\Gamma^1_{(1,2)}(r, y) = -\frac{f'(r)}{f(r)}$ and $\hat{\Gamma}^1_{(1,2)}(\hat{r}, \hat{y}) = -\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}$. Actually one even has $\Gamma^2_{(3,1)} = 0$ and $\hat{\Gamma}^2_{(3,1)} = 0$, but we do not use this fact; one could for example rotate E_1, E_3 (resp. $\hat{E}_1, \hat{E}_3$) between them, in a non-constant way, to destroy this property. The fact that $AE_2|_x \neq \pm \hat{E}_2|_{\hat{x}}$ for $q = (x, \hat{x}; A) \in Q_0$ is equivalent to the fact that the intersection $(AE_2^{\perp}|_x) \cap \hat{E}_2^{\perp}|_{\hat{x}}$ is non-trivial for all $q = (x, \hat{x}; A) \in Q_0$. Therefore, by shrinking $\tilde{O}$ around q_0 if necessary, we may find a smooth functions $\theta, \hat{\theta} : \tilde{O}_0 \to \mathbb{R}$ such that this intersection is spanned by $AZ_A = \hat{Z}_A$, where

$$\begin{aligned} Z_A &:= -\sin(\theta(q)) E_1|_x + \cos(\theta(q)) E_3|_x, \\ \hat{Z}_A &:= -\sin(\hat{\theta}(q)) \hat{E}_1|_{\hat{x}} + \cos(\hat{\theta}(q)) \hat{E}_3|_{\hat{x}}. \end{aligned}$$

We also define

$$\begin{aligned} X_A &:= \cos(\theta(q)) E_1|_x + \sin(\theta(q)) E_3|_x, \\ \hat{X}_A &:= \cos(\hat{\theta}(q)) \hat{E}_1|_{\hat{x}} + \sin(\hat{\theta}(q)) \hat{E}_3|_{\hat{x}}. \end{aligned}$$

To unburden the formulas, we write from now on usually $s_\tau := \sin(\tau(q))$, $c_\tau := \cos(\tau(q))$ if $\tau : \tilde{V} \to \mathbb{R}$ is some function, $\tilde{V} \subset Q$, and the point $q \in \tilde{V}$ is clear from the context. Since $X_A, E_2|_x, Z_A$ (resp. $\hat{X}_A, \hat{E}_2|_{\hat{x}}, \hat{Z}_A$) form an orthonormal frame for every $q = (x, \hat{x}; A) \in \tilde{O}_0$ and because $A(Z_A^\perp) = \hat{Z}_A^\perp$, it follows that there is a smooth $\phi : \tilde{O}_0 \to \mathbb{R}$ such that

$$\begin{aligned}
AX_A =& c_\phi \hat{X}_A + s_\phi \hat{E}_2 = c_\phi(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + s_\phi \hat{E}_2, \\
AE_2 =& - s_\phi \hat{X}_A + c_\phi \hat{E}_2 = -s_\phi(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + c_\phi \hat{E}_2, \\
AZ_A =& \hat{Z}_A.
\end{aligned}$$

In particular, for all $q = (x, \hat{x}; A) \in \tilde{O}_0$, one has $\hat{g}(AZ_A, \hat{E}_2) = 0$. Formulas in Eq. (45) on page 53 hold with $\Gamma^1_{(2,3)} = 0$ and $Y = \hat{E}_2$. Since they are very useful in computations, we will now derive three relations, two of which simplify Eq. (45), and all of which play an important role later on in the proof. Differentiating the identity $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q$ and $\mathscr{L}_{\mathrm{R}}(Z_A)|_q$, one at a time, yields on $\tilde{O}_0$,

$$\begin{aligned}
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(X_A)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AX_A}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, \hat{E}_2) + \hat{g}(\hat{Z}_A, -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) \\
=& s_\phi(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}), \\
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(E_2)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AE_2}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)})\hat{g}(AX_A, \hat{E}_2) + \hat{g}(\hat{Z}_A, s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) \\
=& s_\theta(-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)}), \\
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(Z_A)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AZ_A}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, \hat{E}_2) \\
& + \Gamma^1_{(1,2)}\hat{g}(AE_2, \hat{E}_2) + \hat{g}(\hat{Z}_A, -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A) \\
=& s_\phi(-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)}) + c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}.
\end{aligned}$$

Define

$$\lambda(q) := \mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta + s_\theta\Gamma^1_{(3,1)} - c_\theta\Gamma^3_{(3,1)}, \quad q \in \tilde{O}_0,$$

which is a smooth function on $\tilde{O}_0$. Since $\sin(\phi(q)) = 0$ would imply that $AE_2 = \pm\hat{E}_2$, we have $\sin(\phi(q)) \neq 0$ on $\tilde{O}_0 \subset Q_0$ and hence we get

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta &= c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta &= \Gamma^2_{(3,1)}, \\
s_\phi\lambda &= c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}.
\end{aligned}$$

These formulas, along with $\Gamma^1_{(2,3)} = 0$, simplify Eq. (45) to

$$\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(X_A)|_q X_{(\cdot)} = \Gamma^1_{(1,2)}E_2, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_q X_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_q X_{(\cdot)} = \lambda Z_A, \\
&\mathscr{L}_{\mathrm{R}}(X_A)|_q E_2 = -\Gamma^1_{(1,2)}X_A, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_q E_2 = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_q E_2 = -\Gamma^1_{(1,2)}Z_A, \\
&\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_q Z_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_q Z_{(\cdot)} = -\lambda X_A + \Gamma^1_{(1,2)}E_2, \quad (52)
\end{aligned}$$

at $q \in \tilde{O}_0$. We use these in the rest of the proof without further mention. Notice that, for any $q = (x, \hat{x}; A) \in (Q_1^+ \backslash S_1) \cap \tilde{O}$, and any sequence (which exist as $Q_1 \cap \tilde{O}$ is a nowhere dense subset of $\tilde{O}$) $q_n \in \tilde{O}_0$, $q_n \to q$, we have $\cos(\phi(q_n)) \to \cos(\phi(q)) = 1$, hence $0 \neq \sin(\phi(q_n)) \to 0$. Because

$$\lim_{n\to\infty} (c_\phi \Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)})(q_n) = (c_\phi \Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)})(q) = \Gamma^1_{(1,2)}(x) - \hat{\Gamma}^1_{(1,2)}(\hat{x}) \neq 0,$$

as $q \in Q_1^+ \backslash S_1$, we get

$$\lim_{n\to\infty} \big(\sin(\phi(q_n))\lambda(q_n)\big) \neq 0, \quad \lim_{n\to\infty} \sin(\phi(q_n)) = 0,$$

which implies that $\lim_{n\to\infty} \lambda(q_n) = \pm\infty$. In particular, we see that, even after shrinking $\tilde{O}$, one cannot extend the definition of θ in a smooth, or even C^1, way onto $\tilde{O}$, since if this were possible, the definition of λ above would imply that λ is continuous on $\tilde{O}$ and hence the above sequences $\lambda(q_n)$ would be bounded. This fact about the unboudedness of $\lambda(q)$ as q approaches $(Q_1^+ \backslash S_1) \cap \tilde{O}$ will be used later. To get around this problem, we will be working for a while uniquely on $\tilde{O}_0$.

Define on $\tilde{O}_0$ a 5-dimensional smooth distribution Δ spanned by

$$\mathscr{L}_{\rm R}(E_1)|_q, \mathscr{L}_{\rm R}(E_2)|_q, \mathscr{L}_{\rm R}(E_3)|_q, \nu(A \star E_2)|_q, \nu(A \star X_A)|_q, \quad q \in \tilde{O}_0.$$

We will proceed to show that the Lie algebra $\mathrm{Lie}(\Delta)$ spans at every point of $q \in \tilde{O}_0$ a 8-dimensional distribution $\mathrm{Lie}(\Delta)|_q$ which is then necessarily involutive. Notice that we consider VF^k_Δ, $k = 1, 2, \ldots$ and $\mathrm{Lie}(\Delta)$ as $C^\infty(\tilde{O}_0)$-modules. Since $\mathscr{L}_{\rm R}(X_{(\cdot)}), \mathscr{L}_{\rm R}(E_2), \mathscr{L}_{\rm R}(Z_{(\cdot)})$ span $\mathcal{D}_{\rm R}$ on $\tilde{O}_0$, they generate the module $\mathrm{VF}_{\mathcal{D}_{\rm R}|_{\tilde{O}_0}}$ and hence $\mathrm{Lie}(\mathcal{D}_{\rm R}|_{\tilde{O}_0})$. Moreover, the brackets

$$\begin{aligned}
[\mathscr{L}_{\rm R}(X_{(\cdot)}), \mathscr{L}_{\rm R}(E_2)]|_q =& -\Gamma^1_{(1,2)} \mathscr{L}_{\rm R}(X_A)|_q, \\
[\mathscr{L}_{\rm R}(E_2), \mathscr{L}_{\rm R}(Z_{(\cdot)})]|_q =& \Gamma^1_{(1,2)} \mathscr{L}_{\rm R}(Z_A)|_q - K_1^{\mathsf{Rol}} \nu(A \star X_A)|_q - \alpha \nu(A \star E_2)|_q, \\
[\mathscr{L}_{\rm R}(Z_{(\cdot)}), \mathscr{L}_{\rm R}(X_{(\cdot)})]|_q =& \lambda \mathscr{L}_{\rm R}(Z_A)|_q - \alpha \nu(A \star X_A)|_q - K_2^{\mathsf{Rol}} \nu(A \star E_2)|_q,
\end{aligned}$$

along with the definition of X_A, Z_A, show that $\mathrm{VF}^2_{\mathcal{D}_{\rm R}|_{\tilde{O}_0}} \subset \mathrm{VF}_\Delta$.

The first three Lie brackets in Proposition C.20 case (ii) show that VF^2_Δ contains vector fields L_1, L_3 given by $L_1|_q = \mathscr{L}_{\rm NS}(E_1)|_q - \Gamma^1_{(1,2)} \nu(A \star E_3)|_q$, $L_3|_q = \mathscr{L}_{\rm NS}(E_3)|_q + \Gamma^1_{(1,2)} \nu(A \star E_1)|_q$, and also $L_2|_q$, which in this setting is just the zero-vector field on $\tilde{O}_0$. We define $F_X|_q := c_\theta L_1|_q + s_\theta L_3|_q$ and $F_Z|_q := -s_\theta L_1|_q + c_\theta L_3|_q - \Gamma^1_{(1,2)} \nu(A \star X_A)|_q$, hence $F_X, F_Z \in \mathrm{VF}^2_\Delta$ and one easily sees that they simplify to

$$\begin{aligned}
F_X|_q =& \mathscr{L}_{\rm NS}(X_A)|_q - \Gamma^1_{(1,2)} \nu(A \star Z_A)|_q, \\
F_Z|_q =& \mathscr{L}_{\rm NS}(Z_A)|_q.
\end{aligned}$$

It is clear that the vector fields

$$\mathscr{L}_{\rm R}(X_{(\cdot)}), \mathscr{L}_{\rm R}(E_2), \mathscr{L}_{\rm R}(Z_{(\cdot)}), \nu((\cdot) \star E_2), \nu((\cdot) \star X_{(\cdot)}), F_X, F_Z,$$

span the same $C^\infty(\tilde{O}_0)$-submodule of VF^2_Δ as do

$$\mathscr{L}_{\rm R}(E_1), \mathscr{L}_{\rm R}(E_2), \mathscr{L}_{\rm R}(E_3), \nu((\cdot) \star E_2), \nu((\cdot) \star X_{(\cdot)}), L_1, L_3.$$

We now want to find generators of VF^2_Δ. By what we have already done and said, it remains us to compute need to prove that the Lie-brackets between the 4 vector fields

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(Z_A)|_q, \nu(A\star E_2)|_q,$$

and $\nu((\cdot)\star X_{(\cdot)})|_q$. Since we will have to derivate X_A, it follows that the derivatives of θ will also appear. That is why we first compute with respect to all the (pointwise linearly independent) vectors that appear above. As a first step, compute

$$\begin{aligned}
F_X|_q Z_{(\cdot)} =&(-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A,\\
F_Z|_q Z_{(\cdot)} =&\mathscr{L}_{\mathrm{NS}}(Z_A)|_q Z_{(\cdot)} = (-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A + \Gamma^1_{(1,2)}E_2.
\end{aligned}$$

Knowing already $\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta, \mathscr{L}_{\mathrm{R}}(Y)|_q\theta, \mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta$, we derivate the identity

$$\hat{g}(AZ_A, \hat{E}_2) = 0,$$

w.r.t. $\nu(A\star E_2)|_q, \nu(A\star X_A)|_q, F_X|_q, F_Z|_q$, which gives (notice that the derivative of $\hat{E}_2$ with respect to these vanishes)

$$\begin{aligned}
0 =&\hat{g}(A(\star E_2)Z_A - \nu(A\star E_2)|_q\theta AX_A, \hat{E}_2)\\
=&(1-\nu(A\star E_2))\hat{g}(AX_A,\hat{E}_2) = s_\phi(1-\nu(A\star E_2)),\\
0 =&\hat{g}(A(\star X_A)Z_A - \nu(A\star X_A)|_q\theta AX_A, \hat{E}_2)\\
=&-\hat{g}(AE_2,\hat{E}_2) - \nu(A\star X_A)|_q\theta\hat{g}(AX_A,\hat{E}_2)\\
=&-c_\phi - s_\phi\nu(A\star X_A)|_q\theta,\\
0 =&\hat{g}(-\Gamma^1_{(1,2)}A(\star Z_A)Z_A,\hat{E}_2) + (-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A,\hat{E}_2)\\
=&s_\phi(-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}),\\
0 =&(-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, E_2) + \Gamma^1_{(1,2)}\hat{g}(AE_2,\hat{E}_2)\\
=&s_\phi(-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)}) + c_\phi\Gamma^1_{(1,2)},
\end{aligned}$$

and since $s_\phi \neq 0$ on $\tilde{O}_0$,

$$\begin{aligned}
\nu(A\star E_2)|_q\theta =&1,\\
\nu(A\star X_A)|_q\theta =&-\cot(\phi),\\
F_X|_q\theta =&c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)},\\
F_Z|_q\theta =&-s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)} + \cot(\phi)\Gamma^1_{(1,2)}.
\end{aligned}$$

These simplify the above formulas to

$$\begin{aligned}
F_X|_q Z_{(\cdot)} =&0,\\
F_Z|_q Z_{(\cdot)} =&\mathscr{L}_{\mathrm{NS}}(Z_A)|_q Z_{(\cdot)} = -\cot(\phi)X_A + \Gamma^1_{(1,2)}E_2,
\end{aligned}$$

and moreover it is now easy to see that for $q\in\tilde{O}_0$,

$$\begin{aligned}
F_X|_q X_{(\cdot)} =&\Gamma^1_{(1,2)}E_2, \quad F_X|_q E_2 = -\Gamma^1_{(1,2)}X_A,\\
F_Z|_q X_{(\cdot)} =&\cot(\phi)\Gamma^1_{(1,2)}Z_A, \quad F_Z|_q E_2 = -\Gamma^1_{(1,2)}Z_A.
\end{aligned}$$

The Lie brackets

$$
\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q =&\cot(\phi)\mathscr{L}_{\mathrm{R}}(Z_A)|_q-\mathscr{L}_{\mathrm{NS}}(A\star(\star X_A)X_A)|_q+\Gamma^1_{(1,2)}\nu(A\star E_2)|_q\\
=&\cos(\phi)\mathscr{L}_{\mathrm{R}}(Z_A)|_q+\Gamma^1_{(1,2)}\nu(A\star E_2)|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(A(\star X_A)E_2)|_q+\nu(A\star 0)|_q=F_Z|_q-\mathscr{L}_{\mathrm{R}}(Z_A)|_q,\\
[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q =&-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q-\mathscr{L}_{\mathrm{NS}}(A\star(\star X_A)Z_A)|_q+\nu(A\star(\lambda Z_A)),\\
=&-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q+\mathscr{L}_{\mathrm{R}}(E_2)|_q-(\mathscr{L}_{\mathrm{NS}}(E_2)|_q-\lambda\nu(A\star Z_A)|_q),\\
[\nu(A\star E_2),\nu((\cdot)\star X_{(\cdot)})]|_q =&\nu(A[\star E_2,\star X_A]_{\mathfrak{so}})|_q+\nu(A\star Z_A)|_q=0,
\end{aligned}
$$

show that if one defines

$$F_Y|_q:=\mathscr{L}_{\mathrm{NS}}(E_2)|_q-\lambda\nu(A\star Z_A)|_q,$$

then one may write

$$[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q=-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q+\mathscr{L}_{\mathrm{R}}(E_2)|_q-F_Y|_q,$$

and hence we have shown that VF^2_Δ is generated by vector fields

$$\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}),\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star E_2),\nu((\cdot)\star X_{(\cdot)}),F_X,F_Y,F_Z,$$

which are all pointwise linearly independent on $\tilde{O}_0$.

Next we will proceed to show that the VF^2_Δ generated by the above 8 vector fields is in fact involutive, which then establishes that $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$. At first, the last 9 Lie brackets in Proposition C.20 (recall that we have $\Gamma^1_{(2,3)}=0$) show that $[F_Z,F_X]$ and the brackets of $\mathscr{L}_{\mathrm{R}}(X_{(\cdot)})$, $\mathscr{L}_{\mathrm{R}}(E_2)$, $\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)})$, $\nu((\cdot)\star E_2)$, with F_X and F_Z all belong to VF^2_Δ as well as do

$$
\begin{aligned}
[F_X,\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(-\cot(\phi)Z_A)|_q+\nu(A\star(\mathscr{L}_{\mathrm{NS}}(X_A)|_qX_{(\cdot)}))|_q\\
&-\Gamma^1_{(1,2)}\nu(A[\star Z_A,\star X_A]_{\mathfrak{so}}+\nu(A\star Z_A)|_qX_{(\cdot)}-\cot(\phi)A\star X_A)|_q\\
=&\cot(\phi)\mathscr{L}_{\mathrm{NS}}(Z_A)|_q+\nu(A\star F_X|_qX_{(\cdot)})|_q\\
&-\Gamma^1_{(1,2)}\nu(A\star E_2)|_q+\Gamma^1_{(1,2)}\cot(\phi)\nu(A\star X_A)|_q\\
=&\cot(\phi)F_Z|_q+\Gamma^1_{(1,2)}\cot(\phi)\nu(A\star X_A)|_q,\\
[F_Z,\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(\cot(\phi)X_A)|_q+\cot(\phi)\Gamma^1_{(1,2)}\nu(A\star Z_A)|_q\\
=&-\cot(\phi)F_X|_q.
\end{aligned}
$$

Therefore, it remains to us to prove that the brackets of F_Y with all the other 7 generators of VF^2_Δ, as listed above, also belong to VF^2_Δ. Since the expression of F_Y involves λ, which was defined earlier, we need to know its derivatives in all the possible directions (except in F_Y-direction) as well as the expression for $F_Y|_q\theta$. We begin by computing this latter derivative. As usual, the way to proceed is to derivate $0=\hat{g}(AZ_A,\hat{E}_2)$ w.r.t. $F_Y|_q$, for which, we first compute

$$F_Y|_qZ_{(\cdot)}=(-F_Y|_q\theta+\Gamma^2_{(3,1)})X_A,$$

and hence (notice that $F_Y|_q\hat{E}_2=0$)

$$0=\hat{g}(-\lambda A(\star Z_A)Z_A,\hat{E}_2)+(-F_Y|_q\theta+\Gamma^2_{(3,1)})\hat{g}(AX_A,\hat{E}_2)=s_\phi(-F_Y|_q\theta+\Gamma^2_{(3,1)}),$$

from where one deduces that $F_Y|_q\theta = \Gamma^2_{(3,1)}$. One then easily computes that on $\tilde{O}_0$,

$$F_Y|_q X_{(\cdot)} = 0, \quad F_Y|_q E_2 = 0, \quad F_Y|_q Z_{(\cdot)} = 0.$$

To compute the derivatives of λ, we differentiate the identity $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$ proved above. Obviously, this will require the knowledge of derivatives of ϕ, so we begin there. To do that, one will differentiate the identity $c_\phi = \hat{g}(AE_2, \hat{E}_2)$ in different directions. One has,

$$\begin{aligned}
\hat{\nabla}_{AX_A}\hat{E}_2 &= -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A, \\
\hat{\nabla}_{AE_2}\hat{E}_2 &= s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A, \\
\hat{\nabla}_{AZ_A}\hat{E}_2 &= -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A,
\end{aligned}$$

and hence

$$\begin{aligned}
-s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AX_A, \hat{E}_2) + \hat{g}(AE_2, \hat{\nabla}_{AX_A}\hat{E}_2), \\
=& - s_\phi\Gamma^1_{(1,2)} + \hat{g}(AE_2, -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A), \\
=& - s_\phi\Gamma^1_{(1,2)} + s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}, \\
-s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\phi =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(E_2)|_q E_2, \hat{E}_2) + \hat{g}(AE_2, \hat{\nabla}_{AE_2}\hat{E}_2), \\
=& 0 + \hat{g}(AE_2, s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) = -s_\phi^2\hat{\Gamma}^1_{(1,2)}, \\
-s_\phi\mathscr{L}_{\mathrm{R}}(Z_A)|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AZ_A, \hat{E}_2) + \hat{g}(AE_2, -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A) = 0, \\
-s_\phi\nu(A\star E_2)|_q\phi =& \hat{g}(A(\star E_2)E_2, \hat{E}_2) = 0, \\
-s_\phi\nu(A\star X_A)|_q\phi =& \hat{g}(A(\star X_A)E_2, \hat{E}_2) = \hat{g}(AZ_A, \hat{E}_2) = 0, \\
-s_\phi F_X|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}A(\star Z_A)E_2 - \Gamma^1_{(1,2)}AX_A, \hat{E}_2) = 0, \\
-s_\phi F_Z|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AZ_A, \hat{E}_2) = 0, \\
-s_\phi F_Y|_q\phi =& \hat{g}(-\lambda A(\star Z_A)E_2 + 0, \hat{E}_2) = \lambda\hat{g}(AX_A, \hat{E}_2) = s_\phi\lambda.
\end{aligned}$$

Because $s_\phi \neq 0$ on $\tilde{O}_0$, one also gets

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\phi =& \Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\phi =& s_\phi\hat{\Gamma}^1_{(1,2)}, \\
F_Y|_q\phi =& - \lambda, \\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\phi =& \nu(A\star E_2)|_q\phi = \nu(A\star X_A)|_q\phi = F_X|_q\phi = F_Z|_q\phi = 0.
\end{aligned}$$

Next notice that

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\Gamma^1_{(1,2)} &= F_X|_q\Gamma^1_{(1,2)} = X_A(\Gamma^1_{(1,2)}) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\Gamma^1_{(1,2)} &= F_Y|_q\Gamma^1_{(1,2)} = E_2(\Gamma^1_{(1,2)}), \\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\Gamma^1_{(1,2)} &= F_Z|_q\Gamma^1_{(1,2)} = Z_A(\Gamma^1_{(1,2)}) = 0,
\end{aligned}$$

because $X_A, Z_A \in E_2^\perp$ and similarly, since $\hat{X}_A, \hat{Z}_A \in \hat{E}_2^\perp$,

$$\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(X_A)|_q\hat{\Gamma}^1_{(1,2)} = AX_A(\hat{\Gamma}^1_{(1,2)}) = s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}), \\
&\mathscr{L}_{\mathrm{R}}(E_2)|_q\hat{\Gamma}^1_{(1,2)} = AE_2(\hat{\Gamma}^1_{(1,2)}) = c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}), \\
&\mathscr{L}_{\mathrm{R}}(Z_A)|_q\hat{\Gamma}^1_{(1,2)} = AZ_A(\hat{\Gamma}^1_{(1,2)}) = 0, \\
&F_X|_q\hat{\Gamma}^1_{(1,2)} = F_Y|_q\hat{\Gamma}^1_{(1,2)} = F_Z|_q\hat{\Gamma}^1_{(1,2)} = 0.
\end{aligned}$$

Finally, derivating the identity $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$ and using the previously derived rules,

$$\begin{aligned}
&c_\phi(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)})\lambda + s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda = -s_\phi\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}) - s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
&s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}\lambda + s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = -s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
&s_\phi\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda = 0,\\
&s_\phi\nu(A\star E_2)|_q\lambda = 0,\\
&s_\phi\nu(A\star X_A)|_q\lambda = 0,\\
&s_\phi F_X|_q\lambda = 0,\\
&-c_\phi\lambda^2 + s_\phi F_Y|_q\lambda = s_\phi\Gamma^1_{(1,2)}\lambda + c_\phi E_2(\Gamma^1_{(1,2)}),\\
&s_\phi F_Z|_q\lambda = 0,
\end{aligned}$$

from which the last 6 simplify immediately to

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda =&\nu(A\star E_2)|_q\lambda = \nu(A\star X_A)|_q\lambda = F_X|_q\lambda = F_Z|_q\lambda = 0,\\
F_Y|_q\lambda =&\cot(\phi)(E_2(\Gamma^1_{(1,2)}) + \lambda^2) + \Gamma^1_{(1,2)}\lambda.
\end{aligned}$$

Next simplify $\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda$ by using first $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$, and obtain

$$\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda =& - s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}\lambda - s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - c_\phi\hat{\Gamma}^1_{(1,2)}(c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}) - s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - \Gamma^1_{(1,2)}\hat{\Gamma}^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) + c_\phi(-\hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2)
\end{aligned}$$

and then using $-K = -\hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2$, to deduce

$$s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = - \Gamma^1_{(1,2)}\hat{\Gamma}^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi K,$$

once more $\hat{\Gamma}^1_{(1,2)} = c_\phi\Gamma^1_{(1,2)} - s_\phi\lambda$,

$$\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda =& - \Gamma^1_{(1,2)}(c_\phi\Gamma^1_{(1,2)} - s_\phi\lambda) + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi K,\\
=&c_\phi(-K - (\Gamma^1_{(1,2)})^2 + E_2(\Gamma^1_{(1,2)})) + s_\phi\Gamma^1_{(1,2)}\lambda,
\end{aligned}$$

which finally simplifies, thanks to $-K = -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2$ and $s_\phi \neq 0$, to

$$\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = \lambda\Gamma^1_{(1,2)}.$$

Next we simplify $\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda$ by using the same identities as above when simplifying $\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda$ yields

$$\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda =& - c_\phi(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)})\lambda - s_\phi\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}) - s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - \lambda(s_\phi\lambda + \hat{\Gamma}^1_{(1,2)}) + c_\phi^2\hat{\Gamma}^1_{(1,2)}\lambda,\\
&- s_\phi(\Gamma^1_{(1,2)})^2 + s_\phi\hat{\Gamma}^1_{(1,2)}(s_\phi\lambda + \hat{\Gamma}^1_{(1,2)}) - s_\phi(K + (\hat{\Gamma}^1_{(1,2)})^2),\\
=& - s_\phi(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K) - \lambda\hat{\Gamma}^1_{(1,2)} + c_\phi^2\lambda\hat{\Gamma}^1_{(1,2)} + s_\phi^2\hat{\Gamma}^1_{(1,2)}\lambda,\\
=& - s_\phi(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K),
\end{aligned}$$

which implies, at last, that $\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda = -(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K)$.

Finally, on $\tilde{O}_0$, we compute Lie the brackets

$$
\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_A), F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}X_A)|_q - \mathscr{L}_{\mathrm{R}}(\mathscr{L}_{\mathrm{NS}}(E_2)|_q X_{(\cdot)})|_q \\
&+ \nu(AR(X_A\wedge E_2) - \hat{R}(AX_A\wedge 0)A)|_q - \mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda\nu(A\star Z_A)|_q \\
&- \lambda\big(- \mathscr{L}_{\mathrm{NS}}(A(\star Z_A)X_A) - \mathscr{L}_{\mathrm{R}}(\nu(A\star Z_A)|_q X_{(\cdot)}) + \nu(A\star 0)|_q\big) \\
=& -\Gamma^1_{(1,2)}F_X|_q - \mathscr{L}_{\mathrm{R}}(F_Y|_q X_{(\cdot)})|_q + \lambda\mathscr{L}_{\mathrm{R}}(E_2)|_q - \lambda F_Y|_q \\
&+ \underbrace{\left(-(\Gamma^1_{(1,2)})^2 - K - \mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda - \lambda^2\right)}_{=0}\nu(A\star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), F_Y]|_q =& -\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda\nu(A\star Z_A)|_q - \lambda(-\mathscr{L}_{\mathrm{NS}}(A(\star Z_A)E_2)|_q + \nu(A\star 0)|_q) \\
=& -\lambda\mathscr{L}_{\mathrm{R}}(X_A)|_q + \lambda F_X|_q + \underbrace{(\lambda\Gamma^1_{(1,2)} - \mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda)}_{=0}\nu(A\star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(Z_A), F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}Z_A)|_q + \mathscr{L}_{\mathrm{R}}(\mathscr{L}_{\mathrm{NS}}(E_2)|_q Z_{(\cdot)}) \\
&+ \nu(AR(Z_A\wedge E_2) - \hat{R}(AZ_A\wedge 0)A)|_q - \mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda\nu(A\star Z_A)|_q \\
&- \lambda\big(- \mathscr{L}_{\mathrm{NS}}(A(\star Z_A)Z_A)|_q + \mathscr{L}_{\mathrm{R}}(\nu(A\star Z_A)|_q Z_{(\cdot)})|_q\big) \\
&- \lambda\nu(A\star(-\lambda X_A + \Gamma^1_{(1,2)}E_2)|_q \\
=& -\Gamma^1_{(1,2)}F_Z|_q + \mathscr{L}_{\mathrm{R}}(F_Y|_q Z_{(\cdot)})|_q + K\nu(A\star X_A)|_q \\
&- \underbrace{\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda}_{=0}\nu(A\star Z_A)|_q - \lambda\nu(A\star(-\lambda X_A + \Gamma^1_{(1,2)}E_2)|_q,
\end{aligned}
$$

$$
\begin{aligned}
[\nu((\cdot)\star E_2), F_Y]|_q =& -\nu(A\star E_2)|_q\lambda\nu(A\star Z_A)|_q \\
&- \lambda\nu(A[\star E_2, \star Z_A]_{\mathfrak{so}} - \nu(A\star E_2)|_q\theta A\star X_A)|_q \\
=& -\nu(A\star E_2)|_q\lambda\nu(A\star Z_A)|_q = 0, \\
[\nu((\cdot)\star X_{(\cdot)}, F_Y]|_q =& -\nu(A\star\mathscr{L}_{\mathrm{NS}}(E_2)|_q X_{(\cdot)})|_q - \nu(A\star X_A)|_q\lambda\nu(A\star Z_A)|_q \\
&- \lambda\nu(A[\star X_A, \star Z_A]_{\mathfrak{so}} - \nu(A\star X_A)|_q\theta A\star X_A)|_q \\
&- \lambda\nu(-A\star\nu(A\star Z_A)|_q X_{(\cdot)}))|_q \\
=& -\nu(A\star\underbrace{F_Y|_q X_{(\cdot)}}_{=0})|_q - \underbrace{\nu(A\star X_A)|_q\lambda}_{=0}\nu(A\star Z_A)|_q \\
&- \lambda\nu(A\star(-E_2 + \cot(\phi)X_A))|_q, \\
[F_Z, F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}Z_A - \mathscr{L}_{\mathrm{NS}}(E_2)|_q Z_{(\cdot)})|_q + \nu(AR(Z_A\wedge E_2))|_q \\
&- F_Z|_q\lambda\nu(A\star Z_A)|_q - \lambda(-\mathscr{L}_{\mathrm{NS}}(\nu(A\star Z_A)|_q Z_{(\cdot)}) + \nu(A\star F_Z|_q Z_{(\cdot)})|_q) \\
=& -\Gamma^1_{(1,2)}F_Z|_q - \mathscr{L}_{\mathrm{NS}}(\underbrace{F_Y|_q Z_{(\cdot)}}_{=0})|_q + K\nu(A\star X_A)|_q \\
&- \underbrace{F_Z|_q\lambda}_{=0}\nu(A\star Z_A)|_q - \lambda\nu(A\star(-\cot(\phi)X_A + \Gamma^1_{(1,2)}E_2))|_q,
\end{aligned}
$$

and finally, noticing that $-\lambda F_X|_q + \Gamma^1_{(1,2)} F_Y|_q = -\lambda \mathscr{L}_{\mathrm{NS}}(X_A)|_q + \Gamma^1_{(1,2)} \mathscr{L}_{\mathrm{NS}}(E_2)|_q$,

$$
\begin{aligned}
[F_X, F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(\mathscr{L}_{\mathrm{NS}}(X_A)|_q E_2 - \mathscr{L}_{\mathrm{NS}}(E_2)|_q X_{(\cdot)})|_q + \nu(AR(X_A \wedge E_2))|_q \\
& - \mathscr{L}_{\mathrm{R}}(X_A)|_q \lambda \nu(A \star Z_A)|_q + E_2(\Gamma^1_{(1,2)}) \nu(A \star Z_A)|_q \\
& - \lambda(-\mathscr{L}_{\mathrm{NS}}(\nu(A \star Z_A)|_q X_{(\cdot)}) + \nu(A \star \mathscr{L}_{\mathrm{NS}}(X_A)|_q Z_{(\cdot)})|_q) \\
& + \Gamma^1_{(1,2)} \nu(A \star \mathscr{L}_{\mathrm{NS}}(E_2)|_q Z_{(\cdot)})|_q + \Gamma^1_{(1,2)} \nu(A \star Z_A)|_q \lambda \nu(A \star Z_A)|_q \\
=& - \Gamma^1_{(1,2)} \mathscr{L}_{\mathrm{NS}}(X_A)|_q - \mathscr{L}_{\mathrm{NS}}(\underbrace{F_Y|_q X_{(\cdot)}}_{=0})|_q \\
& + \nu(A \star \underbrace{(-\lambda F_X|_q + \Gamma^1_{(1,2)} F_Y|_q) Z_{(\cdot)}}_{=0})|_q \\
& + (-K - F_X|_q \lambda + E_2(\Gamma^1_{(1,2)})) \nu(A \star Z_A)|_q \\
=& - \Gamma^1_{(1,2)} F_X|_q + (-K - F_X|_q \lambda + E_2(\Gamma^1_{(1,2)}) - (\Gamma^1_{(1,2)})^2) \nu(A \star Z_A)|_q,
\end{aligned}
$$

which, after using $F_X|_q \lambda = 0$ and Eq. (57), simplifies to $[F_X, F_Y]|_q = -\Gamma^1_{(1,2)} F_X|_q$. Since all these Lie brackets also belong to VF^2_Δ, we conclude that VF^2_Δ is involutive and therefore $\mathrm{Lie}(\Delta) = \mathrm{VF}^2_\Delta$. Therefore the span of $\mathrm{Lie}(\Delta)$ at each point $\tilde{O}_0$ is 8-dimensional subspace of $T|_q Q$, since VF^2_Δ is generated by 8 pointwise linearly independent vector fields. Since $q_0 \in Q \backslash S_1$ was arbitrary and since the choice of X_A, E_2, Z_A in $\tilde{O}_0$ are unique up to multiplication by -1, we have shown that on Q_0 there is a smooth 5-dimensional distribution Δ containing $\mathcal{D}_{\mathrm{R}}|_{Q_0}$ such that $\mathrm{Lie}(\Delta) = \mathrm{VF}^2_\Delta$ spans an 8-dimensional distribution $\mathcal{D}$ and which is then, by construction, involutive. We already know from the beginning of the proof that $q \in S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \subset S_1$ so, equivalently, $q \in Q \backslash S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \subset Q \backslash S_1$. Hence we are interested to see how $\mathcal{D}$ can be extended on all over $Q \backslash S_1$ i.e. we have to see how to define it on $Q_1 \backslash S_1$. For this purpose, we define the Sasaki metric G on Q by

$$
\begin{aligned}
&\mathcal{X} = \mathscr{L}_{\mathrm{NS}}(X, \hat{X})|_q + \nu(A \star Z)|_q, \quad \mathcal{Y} = \mathscr{L}_{\mathrm{NS}}(Y, \hat{Y})|_q + \nu(A \star W)|_q, \\
&G(\mathcal{X}, \mathcal{Y}) = g(X, Y) + \hat{g}(\hat{X}, \hat{Y}) + g(Z, W),
\end{aligned}
$$

for $q = (x, \hat{x}; A) \in Q$, $X, Y, Z, W \in T|_x M$, $\hat{X}, \hat{Y} \in T|_{\hat{x}} \hat{M}$. Notice that any vector $\mathcal{X} \in T|_q Q$ can be written in the form $\mathscr{L}_{\mathrm{NS}}(X, \hat{X})|_q + \nu(A \star Z)|_q$ for some $X, \hat{X}, Z$ as above. Since $\mathcal{D}$ is a smooth codimension 1 distribution on Q_0, it has a smooth normal line bundle $\mathcal{D}^\perp$ w.r.t. G defined on Q_0 which uniquely determines $\mathcal{D}$. We will use the Sasaki metric G to determine a smooth vector field $\mathcal{N}$ near a point $q_0 \in Q_1 \backslash S_1$ spanning $\mathcal{D}^\perp$. So let $q_0 \in Q_1 \backslash S_1$ and assume, as before, that $q_0 \in Q_1^+ \backslash S_1$ the case of $Q_1^- \backslash S_1$ being handled similarly. Take the frames E_1, E_2, E_3, $\hat{E}_1, \hat{E}_2, \hat{E}_3$ and $\tilde{O}$, $\tilde{O}_0$, X_A, Z_A as done above (the case (b)). Because $\cos(\phi(q_0)) \Gamma^1_{(1,2)}(x_0) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_0) \neq 0$, one assumes, after shrinking $\tilde{O}$ around q_0, that we have $\cos(\phi(q)) \Gamma^1_{(1,2)}(x) - \hat{\Gamma}^1_{(1,2)}(\hat{x}) \neq 0$ for all $q = (x, \hat{x}; A) \in \tilde{O}$, which then implies that $\lambda(q) \neq 0$ on $\tilde{O}_0$. Here to say what is the value of $\cos(\phi(q))$ even at $q \in Q_1 \backslash S_1$, we use the fact that $\cos(\phi(q)) = g(AE_2, \hat{E}_2)$ for all $q \in \tilde{O}$ (though $\phi(q)$ is not *a priori* defined). To determine a smooth vector field $\mathcal{N} \in \mathcal{D}^\perp$ on $\tilde{O}_0$, we write

$$
\begin{aligned}
\mathcal{N}|_q =& a_1 \mathscr{L}_{\mathrm{NS}}(X_A)|_q + a_2 \mathscr{L}_{\mathrm{NS}}(E_2)|_q + a_3 \mathscr{L}_{\mathrm{NS}}(Z_A)|_q \\
& + b_1 \mathscr{L}_{\mathrm{NS}}(AX_A)|_q + b_2 \mathscr{L}_{\mathrm{NS}}(AE_2)|_q + b_3 \mathscr{L}_{\mathrm{NS}}(AZ_A)|_q \\
& + v_1 \nu(A \star X_A)|_q + v_2 \nu(A \star E_2)|_q + v_3 \nu(A \star Z_A)|_q,
\end{aligned}
$$

and since this must be G-orthogonal to $\mathcal{D}$, we get

$$
\begin{aligned}
0 =&G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(X_A)) = a_1 + b_1, \quad 0 = G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(E_2)) = a_2 + b_2, \\
0 =&G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(Z_A)) = a_3 + b_3, \ 0 = G(\mathcal{N}, \nu(A \star X_A)) = v_1, \ 0 = G(\mathcal{N}, \nu(A \star E_2)) = v_2, \\
0 =&G(\mathcal{N}, F_X) = a_1 - \Gamma^1_{(1,2)} v_3, \quad 0 = G(\mathcal{N}, F_Y) = a_2 - \lambda v_3, \quad 0 = G(\mathcal{N}, F_Z) = a_3.
\end{aligned}
$$

So if we set $v_3 = \frac{1}{\lambda}$ and introduce the notation

$$
\mathscr{L}^{\perp}_{\mathrm{R}}(X)|_q := \mathscr{L}_{\mathrm{NS}}(X, -AX) \in \mathcal{D}_{\mathrm{NS}}|_q, \quad q = (x, \hat{x}; A) \in Q, \ X \in T|_x M,
$$

we get a smooth vector field $\mathcal{N}$ on $\tilde{O}_0$ which is G-perpendicular to $\mathcal{D}$ and given by

$$
\begin{aligned}
\mathcal{N}|_q =&\frac{1}{\lambda(q)} \Gamma^1_{(1,2)}(x) \mathscr{L}^{\perp}_{\mathrm{R}}(X_A)|_q + \mathscr{L}^{\perp}_{\mathrm{R}}(E_2) + \frac{1}{\lambda(q)} \nu(A \star Z_A)|_q, \quad q \in \tilde{O}_0 \\
=&\frac{c_\theta}{\lambda(q)} (\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_1)|_q + \nu(A \star E_3)|_q) + \mathscr{L}^{\perp}_{\mathrm{R}}(E_2)|_q \\
&+ \frac{s_\theta}{\lambda(q)} (\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_3)|_q - \nu(A \star E_1)|_q).
\end{aligned}
$$

i.e.,

$$
\mathcal{N}|_q = H_1(q) \mathcal{X}_1|_q + \mathcal{X}_2|_q + H_3(q) \mathcal{X}_3|_q,
$$

where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are pointwise linearly independent smooth vector fields on $\tilde{O}$ (and not only $\tilde{O}_0$) given by

$$
\begin{aligned}
\mathcal{X}_1|_q =&\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_1)|_q + \nu(A \star E_3)|_q, \\
\mathcal{X}_2|_q =&\mathscr{L}^{\perp}_{\mathrm{R}}(E_2)|_q, \\
\mathcal{X}_3|_q =&\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_3)|_q - \nu(A \star E_1)|_q,
\end{aligned}
$$

while H_1, H_3 are smooth functions on $\tilde{O}_0$ defined by

$$
H_1 = \frac{\cos(\theta)}{\lambda}, \quad H_3 = \frac{\sin(\theta)}{\lambda}.
$$

Notice that θ and λ cannot be extended in a smooth or even C^1-way from $\tilde{O}_0$ to $\tilde{O}$, but as we will show, one can extend H_1, H_3 in at least C^1-way onto $\tilde{O}$. First, since $\lambda(q) \to \pm\infty$ while $\cos(\theta(q))$, $\sin(\theta(q))$ stay bounded, it follows that H_1, H_3 extend uniquely to $\tilde{O} \cap Q_1$ by declaring $H_1(q) = H_3(q) = 0$ for all $q \in \tilde{O} \cap Q_1$. Of course, these extensions, which we still denote by H_1, H_3, are continuous functions on $\tilde{O}$.

The next objective consists of showing that H_1, H_3 are at least C^1 on $\tilde{O}$. For this purpose, let $\mathcal{X} \in \mathrm{VF}(\tilde{O})$ and decompose it uniquely as

$$
\mathcal{X} = \sum_{i=1}^{3} a_i \mathscr{L}_{\mathrm{R}}(E_i) + \sum_{i=1}^{3} b_i \mathscr{L}_{\mathrm{NS}}(E_i) + \sum_{i=1}^{3} v_i \nu((\cdot) \star E_i),
$$

with $a_i, b_i, v_i \in C^\infty(\tilde{O})$. We will need to know the derivatives of θ and λ in all the directions on $\tilde{O}_0$. These have been computed above by using the frame X_A, E_2, Z_A

instead of E_1, E_2, E_3 except in the direction of $\nu(A\star Z_A)|_q$. As before, one computes (using that $s_\phi\neq 0$ on $\tilde{O}_0$ as usual),

$$\nu(A\star Z_A)|_q\theta=0,\quad \nu(A\star Z_A)|_q\phi=1,$$
$$\nu(A\star Z_A)|_q\lambda=-\Gamma^1_{(1,2)}(x)-\lambda(q)\cot(\phi(q)).$$

One now easily computes that on $\tilde{O}_0$,

$$\begin{aligned}\mathcal{X}(\theta)=&(-a_1s_\theta+a_3c_\theta)\lambda+(-b_1s_\theta\Gamma^1_{(1,2)}+b_3c_\theta\Gamma^1_{(1,2)}-v_1c_\theta-v_3s_\theta)\cot(\phi)+B_1(q),\\ \mathcal{X}(\lambda)=&(-a_1c_\theta-a_3s_\theta)\lambda^2+(-b_1c_\theta\Gamma^1_{(1,2)}-b_3s_\theta\Gamma^1_{(1,2)}+v_1s_\theta-v_3c_\theta)\lambda\cot(\phi),\\ &+a_2\Gamma^1_{(1,2)}\lambda+b_2\cot(\phi)E_2(\Gamma^1_{(1,2)})+B_2(q),\end{aligned}$$

where

$$\begin{aligned}B_1(q)&=(a_1+b_1)\Gamma^1_{(3,1)}+(a_2+b_2)\Gamma^2_{(3,1)}+(a_3+b_3)\Gamma^3_{(3,1)}+v_2,\\ B_2(q)&=(-a_1c_\theta-a_3s_\theta)((\Gamma^1_{(1,2)})^2+K)+(-b_1c_\theta-b_3s_\theta)(\Gamma^1_{(1,2)})^2+(v_1s_\theta-v_3c_\theta)\Gamma^1_{(1,2)}.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{X}(H_1)=&-s_\theta\frac{\mathcal{X}(\theta)}{\lambda}-c_\theta\frac{\mathcal{X}(\lambda)}{\lambda^2}\\ =&a_1+(b_1\Gamma^1_{(1,2)}+v_3)\frac{\cot(\phi)}{\lambda}-\frac{a_2c_\theta\Gamma^1_{(1,2)}}{\lambda}-\frac{b_2c_\theta E_2(\Gamma^1_{(1,2)})}{\lambda}\frac{\cot(\phi)}{\lambda}-\frac{s_\theta B_1}{\lambda}-\frac{c_\theta B_2}{\lambda^2},\\ \mathcal{X}(H_3)=&c_\theta\frac{\mathcal{X}(\theta)}{\lambda}-s_\theta\frac{\mathcal{X}(\lambda)}{\lambda^2}\\ =&a_3+(b_3\Gamma^1_{(1,2)}-v_1)\frac{\cot(\phi)}{\lambda}-\frac{a_2s_\theta\Gamma^1_{(1,2)}}{\lambda}-\frac{b_3s_\theta E_2(\Gamma^1_{(1,2)})}{\lambda}\frac{\cot(\phi)}{\lambda}+\frac{c_\theta B_1}{\lambda}-\frac{s_\theta B_2}{\lambda^2}.\end{aligned}$$

Since $s_\phi\lambda=c_\phi\Gamma^1_{(1,2)}-\hat{\Gamma}^1_{(1,2)}$, one has

$$\frac{\cot(\phi)}{\lambda}=\frac{c_\phi}{c_\phi\Gamma^1_{(1,2)}-\hat{\Gamma}^1_{(1,2)}},$$

and therefore as q tends to a point q_1 of $Q_1^+\cap\tilde{O}$, we have

$$\lim_{q\to q_1}\frac{\cot(\phi)}{\lambda}=\frac{1}{\Gamma^1_{(1,2)}-\hat{\Gamma}^1_{(1,2)}}.$$

Since B_1, B_2 stay bounded as q approaches a point of $Q_1^+\cap\tilde{O}$, we get for every $q_1=(x_1,\hat{x}_1;A_1)\in Q_1^+\cap\tilde{O}$ that

$$\begin{aligned}\lim_{q\to q_1}\mathcal{X}(H_1)=a_1(q_1)+\frac{b_1(q_1)\Gamma^1_{(1,2)}(x_1)+v_3(q_1)}{\Gamma^1_{(1,2)}(x_1)-\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)}=:D_{\mathcal{X}}H_1(q_1),\\ \lim_{q\to q_1}\mathcal{X}(H_3)=a_3(q_1)+\frac{b_3(q_1)\Gamma^1_{(1,2)}(x_1)-v_1(q_1)}{\Gamma^1_{(1,2)}(x_1)-\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)}=:D_{\mathcal{X}}H_3(q_1).\end{aligned}$$

From these, it is now readily seen that H_1, H_3 are differentiable on $\tilde{O}\cap Q_1^+$ with $\mathcal{X}|_{q_1}(H_1)=D_{\mathcal{X}}H_1(q_1)$, $\mathcal{X}|_{q_1}(H_3)=D_{\mathcal{X}}H_3(q_1)$ and that H_1, H_3 are C^1-functions on

$\tilde{O}$. We therefore have that $\mathcal{N}$ is a well-defined C^1 vector field on $\tilde{O}$ and since $\mathcal{D}=\mathcal{N}^\perp$ w.r.t. G on $\tilde{O}_0$, it follows that $\mathcal{D}$ extends in C^1-sense on $\tilde{O}$. Since $q_0\in Q_1^+\backslash S_1$ was arbitrary and because the case $q_0\in Q_1^-\backslash S_1$ is handled similarly, we see that $\mathcal{D}$ can be extended onto the open subset $Q\backslash S_1$ of Q as a (at least) C^1-distribution, which is C^∞ on Q_0. Since $\mathcal{D}_{\mathrm{R}}|_{Q\backslash S_1}\subset\mathcal{D}$ and because $q\in Q\backslash S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)\subset Q\backslash S_1$ as we have seen, it follows that for every $q_0\in Q\backslash S_1$ we have $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset\mathcal{O}_{\mathcal{D}}(q_0)$ where the orbit on the right is *a priori* an immersed C^1-submanifold of $Q\backslash S_1$. However, since $\mathcal{D}$ is involutive and $\dim\mathcal{D}=8$ on $Q\backslash S_1$, we get by the C^1-version of the Frobenius theorem that $\dim\mathcal{O}_{\mathcal{D}}(q_0)=8$ and hence

$$\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\le\dim\mathcal{O}_{\mathcal{D}}(q_0)=8,$$

for every $q_0\in Q\backslash S_1$.

We will now investigate when the equality holds here. Define

$$\begin{aligned}
M_0&=\{x\in M\mid K_2(x)\neq K\},\\
M_1&=\{x\in M\mid \exists\text{ open }V\ni x\text{ s.t. }K_2(x')=K\ \forall x'\in V\},\\
\hat{M}_0&=\{\hat{x}\in\hat{M}\mid \hat{K}_2(\hat{x})\neq K\},\\
\hat{M}_1&=\{\hat{x}\in\hat{M}\mid \exists\text{ open }\hat{V}\ni\hat{x}\text{ s.t. }K_2(\hat{x}')=K\ \forall\hat{x}'\in\hat{V}\},
\end{aligned}$$

and notice that $M_0\cup M_1$ (resp. $\hat{M}_0\cup\hat{M}_1$) is a dense subset of M (resp. $\hat{M}$). Here we also fix the choice of $q_0=(x_0,\hat{x}_0;A_0)\in Q\backslash S_1$ and define $M^\circ=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ as in the statement. Write also $Q^\circ:=\pi_Q^{-1}(M^\circ\times\hat{M}^\circ)$ and notice that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q^\circ$. We define on Q two 2-dimensional distributions D and $\hat{D}$. For every $q_1=(x_1,\hat{x}_1;A_1)\in Q$, take orthonormal frames E_1,E_2,E_3, $\hat{E}_1,\hat{E}_2,\hat{E}_3$ of $M,\hat{M}$ defined on open neighbourhoods $U,\hat{U}$ of $x_1,\hat{x}_1$ with $E_2=\frac{\partial}{\partial r}$, $\hat{E}_2=\frac{\partial}{\partial r}$. Then, for $q\in\pi_Q^{-1}(U\times\hat{U})\cap Q$, the 2-dimensional plane $D|_q$ is spanned by

$$\begin{aligned}
K_1|_q&=\mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_1)|_q-\hat{\Gamma}^1_{(1,2)}(x)\nu((\hat{\star}\hat{E}_3)A)|_q,\\
K_3|_q&=\mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_3)|_q+\hat{\Gamma}^1_{(1,2)}(x)\nu((\hat{\star}\hat{E}_1)A)|_q,
\end{aligned}$$

and $\hat{D}|_q$ is spanned by

$$\begin{aligned}
\hat{K}_1|_q&=\mathscr{L}_{\mathrm{NS}}(AE_1)|_q+\Gamma^1_{(1,2)}(x)\nu(A\star E_3)|_q,\\
\hat{K}_3|_q&=\mathscr{L}_{\mathrm{NS}}(AE_3)|_q-\Gamma^1_{(1,2)}(x)\nu(A\star E_1)|_q.
\end{aligned}$$

Obviously, different choices of frames $E_i,\hat{E}_i$, $i=1,2,3$, give $K_1,K_3,\hat{K}_1,\hat{K}_3$ that span the same planes $D,\hat{D}$, since we have fixed the choice of $E_2=\frac{\partial}{\partial r}$, $\hat{E}_2=\frac{\partial}{\partial r}$. Exactly as in proof of Proposition 5.30, one can show that for every $q_1=((r_1,y_1),(\hat{r}_1,\hat{y}_1);A_1)\in Q$ and smooth paths $\gamma:[0,1]\to N$, $\hat{\gamma}:[0,1]\to\hat{N}$ with $\gamma(0)=y_1$, $\hat{\gamma}(0)=\hat{y}_1$ there are unique smooth paths $\Gamma,\hat{\Gamma}:[0,1]\to Q$ such that for all $t\in[0,1]$,

$$\begin{aligned}
&\dot{\Gamma}(t)\in D|_{\Gamma(t)},\quad \Gamma(0)=q_1,\quad (\pi_{Q,M}\circ\Gamma)(t)=(r_1,\gamma(t)),\\
&\dot{\hat{\Gamma}}(t)\in\hat{D}|_{\hat{\Gamma}(t)},\quad \hat{\Gamma}(0)=q_1,\quad (\pi_{Q,\hat{M}}\circ\hat{\Gamma})(t)=(\hat{r}_1,\hat{\gamma}(t)).
\end{aligned}$$

Since $(\pi_{Q,\hat{M}})_*D=0$ (resp. $(\pi_{Q,M})_*\hat{D}=0$), one has $\pi_{Q,\hat{M}}(\Gamma(t))=\hat{x}_1$ (resp. $\pi_{Q,M}(\hat{\Gamma}(t))=x_1$) for all $t\in[0,1]$. We write these as $\Gamma=\Gamma(\gamma,q_1)$, $\hat{\Gamma}=\hat{\Gamma}(\hat{\gamma},q_1)$. If

$E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$, then by exactly the same arguments as in the proof of Proposition 5.30 we have

$$\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}),$$
$$\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

We next show how one can replace Q_0 by $Q\backslash S_1$. Take frames E_i, $\hat{E}_i$, $i = 1, 2, 3$, as above when defining $D, \hat{D}$ for some $q_1 \in Q_1\backslash S_1$. We assume here without loss of generality that $q_1 \in Q_1^+\backslash S_1$ since the case $q_1 \in Q_1^-\backslash S_1$ can be dealt with in a similar way. If $h_1, h_2 : \pi_Q^{-1}(U \times \hat{U}) \to \mathbb{R}$ are defined as $h_1(q) = \hat{g}(AE_1, \hat{E}_2)$, $h_2(q) = \hat{g}(AE_3, \hat{E}_2)$, we have $Q_1 \cap \pi_Q^{-1}(U \times \hat{U}) = (h_1, h_2)^{-1}(0)$ and $(h_1, h_2) : \pi_Q^{-1}(U \times \hat{U}) \to \mathbb{R}^2$ is a regular map at the points of Q_1 (see e.g. Remark 5.34 or the proof of Proposition 5.28). Since $q_1 \in Q_1^+\backslash S_1$, then $\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1} h_1 = \Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1) \neq 0$ and $\mathscr{L}_{\mathrm{R}}(E_3)|_{q_1} h_2 = \Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1) \neq 0$, which shows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ intersects Q_1^+ transversally at q_1 (hence at every point $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$), by dimensional reasons (because $\dim Q_1 = 7$, $\dim Q = 9$). From this, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1) \cap Q_1$ is a smooth closed submanifold of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ and that there is a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1) \cap Q_0$ such that $q'_n \to q_1$. If now $q_1 \in \pi_Q^{-1}(M_0 \times \hat{M}) \cap Q_1\backslash S_1$, then we know that for n large enough, $q'_n \in \pi_Q^{-1}(M_0 \times \hat{M}) \cap Q_0$ and hence $\nu(A \star E_2)|_{q'_n} \in T|_{q'_n}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q'_n) = T|_{q'_n}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Taking the limit implies that $\nu(A \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Similarly, if $q_1 \in \pi_Q^{-1}(M \times \hat{M}_0) \cap Q_1\backslash S_1$, one has $\nu((\hat{\star}\hat{E}_2)A)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Hence we have that if $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$, then

$$\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q\backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}),$$
$$\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q\backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

For every $q \in (Q\backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M})$, which is an open subset of Q, one has $\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ with $E_2 = \frac{\partial}{\partial r}$ and hence by Proposition C.20, case (i), it follows that

$$L_1|_q = \mathscr{L}_{\mathrm{NS}}(E_1)|_q - \Gamma^1_{(1,2)}(x)\nu(A \star E_3)|_q,$$
$$L_3|_q = \mathscr{L}_{\mathrm{NS}}(E_3)|_q + \Gamma^1_{(1,2)}(x)\nu(A \star E_1)|_q,$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, where $E_1, E_2 = \frac{\partial}{\partial r}, E_3$ is an orthonormal frame in an open neighbourhood of x_1. But because $\hat{K}_1|_q = \mathscr{L}_{\mathrm{R}}(E_1)|_q - L_1|_q$, $\hat{K}_3|_q = \mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q$, we get that

$$\hat{D}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q\backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

Moreover, if $q = (x, (\hat{r}, \hat{y}); A) \in (Q\backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M})$ and if $\hat{\gamma} : [0, 1] \to \hat{N}$ is any curve with $\hat{\gamma}(0) = \hat{y}$, then one shows with exactly the same argument as in the proof of Proposition 5.30 that

$$\hat{\Gamma}(\hat{\gamma}, q)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_Q^{-1}(M_0 \times \hat{M}), \quad \forall t \in [0, 1].$$

In particular,

$$\exists q = (x, (\hat{r}, \hat{y}); A) \in (Q\backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}) \implies \{x\} \times (\{\hat{r}\} \times \hat{N}) \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)).$$

A similar argument shows that

$$D|_q \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0),$$

and that for all $q = ((r,y), \hat{x}; A) \in (Q\backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0)$ and $\gamma : [0,1] \to N$ with $\gamma(0) = y$,

$$\Gamma(\gamma, q)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_Q^{-1}(M \times \hat{M}_0), \quad \forall t \in [0,1].$$

In particular,

$$\exists q = ((r,y), \hat{x}; A) \in (Q\backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0) \implies (\{r\} \times N) \times \{\hat{x}\} \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)).$$

Everything so far is similar to the proof of Proposition 5.30 and continues to be so, with few minor changes (notably, here $\dim D = \dim \hat{D} = 2$ instead of 3). Suppose that $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. Take $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_1 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, with $x_1 = (r_1, y_1)$. If $\sigma(y)$ is the unique sectional curvature of N at y, we have

$$K_2(r_1, y_1) = \frac{\sigma(y_1) - (f'(r_1))^2}{f(r_1)^2} = K.$$

We go from here case by case.

(I) If N does not have constant curvature, there exists $y_2 \in N$ with $\sigma(y_2) \neq \sigma(y_1)$ and hence

$$K_2(r_1, y_2) = \frac{\sigma(y_2) - (f'(r_1))^2}{f(r_1)^2} \neq K,$$

i.e., $(r_1, y_2) \in M_0$. Since $q_1 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q\backslash S_1$, we have by the above that

$$((r_1, y_2), \hat{x}_1) \in (\{r_1\} \times N) \times \{\hat{x}_1\} \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)) = \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)),$$

and since $((r_1, y_2), \hat{x}_1) \in M_0 \times \hat{M}_0$, we get that which implies that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$.

(II) Suppose that (N, h) has constant curvature C i.e. $\sigma(y) = C$ for all $y \in N$. We write $K_2(r, y) = K_2(r)$ on M since its value only depends on $r \in I$ and notice that for all $r \in I$,

$$\frac{\mathrm{d}K_2}{\mathrm{d}r} = -2\frac{f'(r)}{f(r)}(K_2(r) - K).$$

But $K_2(r_1) = K$, so by the uniqueness of solutions of ODEs, we get $K_2(r) = K$ for all $r \in I$ and hence (M, g) has constant curvature K.

Of course, regarding case (II), it is clear that if (M, g) has constant curvature K, then (N, h) has a constant curvature. Hence we have proved that if (M, g) does not have a constant curvature and if $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$, then also $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. The argument being symmetric in (M, g), $(\hat{M}, \hat{g})$, we also have that if $(\hat{M}, \hat{g})$ does not have a constant curvature and if $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$, then also $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. Notice that (M°, g) and $(\hat{M}^\circ, \hat{g})$ cannot both have constant curvature, since this violates the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$ (see Corollary 4.16 and Remark 4.17). We can now finish the proof by considering, again, different cases.

a) Assume that $(\hat{M}^\circ,\hat{g})$ has constant curvature equal then to K. We have $\hat{M}_0\cap \hat{M}^\circ=\emptyset$. If $E_2=\frac{\partial}{\partial r}$, then Hence, $\widetilde{\mathsf{Rol}}_q(\star X)=0$ for all $q\in Q^\circ=\pi_Q^{-1}(M^\circ\times\hat{M}^\circ)$, $X\in E_2^\perp$ while $\widetilde{\mathsf{Rol}}_q(\star E_2)=(-K_2(x)+K)\star E_2$. At $q_1=(x_1,\hat{x}_1;A_1)\in Q^\circ$, take an open neighbourhood U of x_1 and an ortonormal basis E_1,E_2,E_3 with $E_2=\frac{\partial}{\partial r}$ and let D_1 be a distribution on $\pi_{Q,M}^{-1}(U)$ spanned by

$$\mathscr{L}_{\mathrm{R}}(E_1),\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star E_2),L_1,L_3,$$

where L_1,L_3 are as in Proposition C.20. Obviously, one defines in this way a 6-dimensional smooth distribution D_1 on the whole Q° and the above from of $\widetilde{\mathsf{Rol}}_q$, $q\in Q^\circ$, along with Proposition C.20, case (ii), reveal that it is involutive (recall that $\Gamma^1_{(2,3)}=0$ there). Clearly, $\mathcal{D}_{\mathrm{R}}\subset D_1$ on Q° and since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q^\circ$, we have $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset\mathcal{O}_{D_1}(q_0)$ and hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\le 6$. Because (M°,g) does not have constant curvature (as noticed previously), we have $M_0\cap M^\circ\neq\emptyset$ and thus $O:=\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\cap\pi_{Q,M}^{-1}(M_0)$ is a non-empty open subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. For every $q=(x,\hat{x};A)\in O$, one has $\widetilde{\mathsf{Rol}}_q(\star E_2)=(-K_2(x)+K)\star E_2\neq 0$ and hence that $\nu(A\star E_2)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Therefore, Proposition C.20, case (i), implies that $D_1|_O$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This gives $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\ge 6$ and hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$.

b) If (M°,g) has constant curvature, then the argument of case a) with the roles of (M,g), $(\hat{M},\hat{g})$ interchanged, shows that $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$.

Hence we have proven (ii). For the rest of the cases, we may assume that neither (M°,g) nor $(\hat{M}^\circ,\hat{g})$ has constant curvature i.e. $M^\circ\cap M_0\neq\emptyset$, $\hat{M}^\circ\cap\hat{M}_0\neq\emptyset$.

c) Suppose $(M_0\times\hat{M}_0)\cap\pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))\neq\emptyset$ and let $q_1=(x_1,\hat{x}_1;A_1)\in\pi_Q^{-1}(M_0\times \hat{M}_0)\cap\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. We already know that $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ contains vectors

$$\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1},\mathscr{L}_{\mathrm{R}}(E_2)|_{q_1},\mathscr{L}_{\mathrm{R}}(E_3)|_{q_1},\nu(A\star E_2)|_{q_1},\nu((\hat{\star}\hat{E}_2)A)|_{q_1},L_1|_{q_1},L_3|_{q_1},\hat{L}_1|_{q_1},\hat{L}_3|_{q_1},$$

where

$$\begin{aligned}\hat{L}_1|_{q_1}&=\mathscr{L}_{\mathrm{NS}}(\hat{E}_1)|_{q_1}+\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)\nu((\hat{\star}\hat{E}_3)A_1)|_{q_1},\\ \hat{L}_3|_{q_1}&=\mathscr{L}_{\mathrm{NS}}(\hat{E}_3)|_{q_1}-\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)\nu((\hat{\star}\hat{E}_1)A_1)|_{q_1}.\end{aligned}$$

If $q_1\in Q_0$. these vectors span an 8-dimensional subspace of $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, Indeed, by considering $X_{A_1},Z_{A_1},\hat{X}_{A_1},\hat{Z}_{A_1}$ and angles $\phi,\theta,\hat{\theta}$ as before, one has $\sin(\phi(q_1))\neq 0$ and

$$\begin{aligned}\nu((\hat{\star}\hat{E}_2)A_1)|_{q_1}=&\nu(A_1\star(A_1^{\overline{T}}\hat{E}_2))|_{q_1}\\ =&\sin(\phi(q_1))\nu(A_1\star X_{A_1})|_{q_1}+\cos(\phi(q_1))\nu(A_1\star E_2)|_{q_1},\\ c_\theta L_1|_{q_1}+s_\theta L_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(X_{A_1})|_{q_1}-\Gamma^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1},\\ -s_\theta L_1|_{q_1}+c_\theta L_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(Z_{A_1})|_{q_1}+\Gamma^1_{(1,2)}(x_1)\nu(A_1\star X_{A_1})|_{q_1},\\ c_{\hat{\theta}}\hat{L}_1|_{q_1}+s_{\hat{\theta}}\hat{L}_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(\hat{X}_{A_1})|_{q_1}+\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1}\\ =&c_\phi\mathscr{L}_{\mathrm{NS}}(A_1X_{A_1})|_{q_1}-s_\phi\mathscr{L}_{\mathrm{NS}}(A_1E_2)|_{q_1}+\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1},\\ -s_{\hat{\theta}}\hat{L}_1|_{q_1}+c_{\hat{\theta}}\hat{L}_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(A_1Z_{A_1})|_{q_1}-\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star(A^{\overline{T}}\hat{X}_{A_1}))|_{q_1}\\ =&\mathscr{L}_{\mathrm{NS}}(A_1Z_{A_1})|_{q_1}-\hat{\Gamma}^1_{(1,2)}(x_1)\big(c_\phi\nu(A_1\star X_{A_1})|_{q_1}-s_\phi\nu(A_1\star E_2)|_{q_1}\big).\end{aligned}$$

On the other hand, if $q_1 \in Q_1$, then since Q_1 is transversal to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at q_1, we can replace q_1 by a nearby $q_1' \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap Q_0$ and the above holds at q_1'. Therefore $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 8$ and since we have also shown that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$, we have the equality.

d) Since $M^\circ \cap M_0 \neq \emptyset$, there is a $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $x_1 \in M_0$. If $\hat{x}_1 \in \hat{M}_0$, one has that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and hence case c) implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$. If $\hat{x}_1 \notin \hat{M}_0$, then $\hat{x}_1 \in \overline{\hat{M}_1}$. Therefore, we may find a sequence $q_n' = (x_n', \hat{x}_n'; A_n') \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $q_n' \to q_1$ and $\hat{x}_n' \in \hat{M}_1$. So for n large enough, we have $(x_n', \hat{x}_n') \in (M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Thus $(\hat{M}, \hat{g})$ does not have constant curvature and $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ which we have shown to imply that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ from which the above case c) implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$.

The cases c) and d) above give (iii) and therefore the proof is complete. □

Remark 5.36 It is not difficult to see that Proposition 5.32 generalizes to higher dimension as follows. Keeping the same notations as before, let $(M, g) = (I, s_1) \times_f (N, h)$ and $(\hat{M}, \hat{g}) = (\hat{I}, s_1) \times_{\hat{f}} (\hat{N}, \hat{h})$, $I, \hat{I} \subset \mathbb{R}$, be warped products where (N, h) and $(\hat{N}, \hat{h})$ are now connected, oriented $(n-1)$-dimensional Riemannian manifolds. As before, let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ be such that if we write $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, then (50) and (51) hold true. Then, the exact argument of Proposition 5.32 yields that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has dimension at most equal to $n(n+1)/2$. One can even have equality, if the $(n-1)$-dimensional manifolds (N, h) and $(\hat{N}, \hat{h})$ are such that that the corresponding $\widetilde{\mathsf{Rol}}_{q_0'}$ operator (in $(n-1)$-dimensional setting) is invertible at $q_0' = (y_0, \hat{y}_0; A_0') \in Q(N, \hat{N})$, where $A_0' : \frac{\partial}{\partial r}\big|_{x_0}^{\perp} \to \frac{\partial}{\partial r}\big|_{\hat{x}_0}^{\perp}$ is the restriction of A_0 and if we also assume that $f(r_0) = 1$, $\hat{f}(r_0) = 1$, an assumption that can always be satisfied after rescaling the metrics of (N, h) and $(\hat{N}, \hat{h})$.

A Fiber Coordinates and Control Theoretic Points of View

In this section we describe equations of the control system $(\Sigma)_R$ in terms of moving orthonormal frames. Assume that Let $F = (F_1, \dots, F_n)$, $\hat{F} = (\hat{F}_1, \dots, \hat{F}_n)$ be oriented orthonormal local frames of M and $\hat{M}$ defined on U and $\hat{U}$ respectively. We assume moreover that $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0, 1]$, is an a.c. curve in Q sucht that $\gamma([0, 1]) \subset U$ and $\hat{\gamma}([0, 1]) \subset \hat{U}$.

Define for every $x \in U$ and $\hat{x} \in \hat{U}$ the linear maps

$$\Gamma : T|_x M \to \mathfrak{so}(n), \quad \Gamma(X)_i^j = g(\nabla_X F_i, F_j),$$
$$\hat{\Gamma} : T|_{\hat{x}} \hat{M} \to \mathfrak{so}(n), \quad \hat{\Gamma}(\hat{X})_i^j = \hat{g}(\hat{\nabla}_{\hat{X}} \hat{F}_i, \hat{F}_j).$$

Let $\mathcal{A} : [0, 1] \to \mathrm{SO}(n)$ be given by $\mathcal{A}(t) = \mathcal{M}_{F, \hat{F}}(A(t)) = [A_j^i(t)]$ i.e.,

$$(A(t)F_1|_{\gamma(t)}, \dots, A(t)F_n|_{\gamma(t)}) = (\hat{F}_1|_{\hat{\gamma}(t)}, \dots, \hat{F}_n|_{\hat{\gamma}(t)})\mathcal{A}(t).$$

Taking $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}$ of this gives

$$
\begin{aligned}
&(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_1|_{\gamma(t)},\dots,\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_n|_{\gamma(t)})+(A(t)\nabla_{\dot{\gamma}(t)}F_1,\dots,A(t)\nabla_{\dot{\gamma}(t)}F_n)\\
=&(\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{F}_1,\dots,\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{F}_n)\mathcal{A}(t)+(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\dot{\mathcal{A}}(t),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\big(-\mathcal{A}(t)\Gamma(\dot{\gamma}(t))+\hat{\Gamma}(\dot{\hat{\gamma}}(t))\mathcal{A}(t)+\dot{\mathcal{A}}(t)\big)\\
=&(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_1|_{\gamma(t)},\dots,\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_n|_{\gamma(t)}).
\end{aligned}
$$

Hence one sees that

$$
q(t)=(\gamma(t),\hat{\gamma}(t);A(t))\ \text{ satisfies Eq. (9)} \iff \dot{\mathcal{A}}(t)=\mathcal{A}(t)\Gamma(\dot{\gamma}(t))-\hat{\Gamma}(\dot{\hat{\gamma}}(t))\mathcal{A}(t).
$$

We newt show how to interpret $(\Sigma)_R$ as an affine driftless control system in $\pi_Q^{-1}(U\times\hat{U})$. Fix $q_0=(x_0,\hat{x}_0;A_0)\in\pi_Q^{-1}(U\times\hat{U})$. Note that there is an open subset $\mathcal{U}\subset L^1([0,1],\mathbb{R}^n)$ and a one-to-one correspondence between a.c. curves $\gamma:[0,1]\to U$ with $\gamma(0)=x_0$ and $\mathcal{U}$ given by

$$
\dot{\gamma}(t)=(F_1|_{\gamma(t)},\dots,F_n|_{\gamma(t)})\begin{pmatrix}u^1(t)\\ \vdots\\ u^n(t)\end{pmatrix},\quad (u^1,\dots,u^n)\in\mathcal{U}. \tag{53}
$$

The no-slip condition, Eq. (11) now becomes

$$
\dot{\hat{\gamma}}(t)=(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\mathcal{A}(t)\begin{pmatrix}u^1(t)\\ \vdots\\ u^n(t)\end{pmatrix}, \tag{54}
$$

and, by the above, the no-spin condition, Eq. (9), becomes

$$
\dot{\mathcal{A}}(t)=\sum_{i=1}^n u^i(t)\Big(\mathcal{A}(t)\Gamma(F_i|_{\gamma(t)})-\sum_{j=1}^n A_i^j(t)\hat{\Gamma}(\hat{F}_j|_{\hat{\gamma}(t)})\mathcal{A}(t)\Big). \tag{55}
$$

Hence, the problem $(\Sigma)_R$ is equivalent on $\pi_Q^{-1}(U\times\hat{U})$ to the control system defined by Eqs. (53), (54), (55) where the controls $(u^1,\dots,u^n)$ belong to $\mathcal{U}\subset L^1([0,1],\mathbb{R}^n)$ and $\mathcal{A}(t)=\mathcal{M}_{F,\hat{F}}(A(t))=[A_j^i(t)]$. If $v=(v_1,\dots,v_n)\in\mathbb{R}^n$ we write $\langle F,v\rangle=\sum_{i=1}^n v_iF_i$ and $\Big\langle \hat{F},v\Big\rangle=\sum_{i=1}^n v_i\hat{F}_i$. With this notation, if we write $u=(u^1,\dots,u^n)$, we write the system (53), (54), (55) more compactly as

$$
\begin{cases}
\dot{\gamma}(t)=\left\langle F|_{\gamma(t)},u(t)\right\rangle,\\
\dot{\hat{\gamma}}(t)=\Big\langle \hat{F}|_{\hat{\gamma}(t)}\mathcal{A}(t),u(t)\Big\rangle,\\
\dot{\mathcal{A}}(t)=\mathcal{A}(t)\Gamma\big(\left\langle F|_{\gamma(t)},u(t)\right\rangle\big)-\hat{\Gamma}\big(\Big\langle \hat{F}|_{\hat{\gamma}(t)}\mathcal{A}(t),u(t)\Big\rangle\big)\mathcal{A}(t).
\end{cases}
$$

B The Rolling Problem Embedded in $\mathbb{R}^N$

In this section, we compare the rolling model defined by the state space $Q = Q(M, \hat{M})$, whose dynamics is governed by the conditions (10)-(11) (or, equivalently, by $\mathcal{D}_{\mathrm{R}}$), with the rolling model of two n-dimensional manifolds embedded in $\mathbb{R}^N$ as given in [41] (Appendix B). See also [30], [20].

Let us first fix $N \in \mathbb{N}$ and introduce some notations. The special Euclidean group of $\mathbb{R}^N$ is the set $\mathrm{SE}(N) := \mathbb{R}^N \times \mathrm{SO}(N)$ equipped with the group operation $\star$ given by

$$(p, A) \star (q, B) = (Aq + p, AB), \quad (p, A), (q, B) \in \mathrm{SE}(N).$$

We identify $\mathrm{SO}(N)$ with the subgroup $\{0\} \times \mathrm{SO}(N)$ of $\mathrm{SE}(N)$, while $\mathbb{R}^N$ is identified with the normal subgroup $\mathbb{R}^N \times \{\mathrm{id}_{\mathbb{R}^N}\}$ of $\mathrm{SE}(N)$. With these identifications, the action $\star$ of the subgroup $\mathrm{SO}(N)$ on the normal subgroup $\mathbb{R}^N$ is given by

$$(p, A) \star q = Aq + p, \quad (p, A) \in \mathrm{SE}(N),\ p \in \mathbb{R}^N.$$

Let $\mathcal{M}$ and $\hat{\mathcal{M}} \subset \mathbb{R}^N$ be two (embedded) submanifolds of dimension n. For every $z \in \mathcal{M}$, we identify $T|_z\mathcal{M}$ with a subspace of $\mathbb{R}^N$ (the same holding in the case of $\hat{\mathcal{M}}$) i.e., elements of $T|_z\mathcal{M}$ are derivatives $\dot{\sigma}(0)$ of curves $\sigma : I \to M$ with $\sigma(0) = z$ ($I \ni 0$ a nontrivial real interval).

The *rolling of $\mathcal{M}$ against $\hat{\mathcal{M}}$ without slipping or twisting* in the sense of [41] is realized by a smooth curves $G : I \to \mathrm{SE}(N)$; $G(t) = (p(t), U(t))$ (I a nontrivial real interval) called the *rolling map* and $\sigma : I \to \mathcal{M}$ called the *development* curve such that the following conditions (1)-(3) hold for every $t \in I$:

(1) (a) $\hat{\sigma}(t) := G(t) \star \sigma(t) \in \hat{\mathcal{M}}$ and
(b) $T|_{\hat{\sigma}(t)}(G(t) \star \mathcal{M}) = T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$.

(2) No-slip: $\dot{G}(t) \star \sigma(t) = 0$.

(3) No-twist: (a) $\dot{U}(t)U(t)^{-1}T|_{\hat{\sigma}(t)}\hat{\mathcal{M}} \subset (T|_{\hat{\sigma}(t)}\hat{\mathcal{M}})^\perp$ (tangential no-twist),
(b) $\dot{U}(t)U(t)^{-1}(T|_{\hat{\sigma}(t)}\hat{\mathcal{M}})^\perp \subset T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$ (normal no-twist).

The orthogonal complements are taken w.r.t. the Euclidean inner product of $\mathbb{R}^N$. In condition (2) we define the action '$\star$' of $\dot{G}(t) = (\dot{U}(t), \dot{p}(t))$ on $\mathbb{R}^N$ by the same formula as for the action '$\star$' of $\mathrm{SE}(N)$ on $\mathbb{R}^N$.

We next consider two classical cases of rolling and interpret the no-twist conditions in these cases.

Example B.1 (i) Suppose $N = 3$, $n = 2$ i.e., $\mathcal{M}, \hat{\mathcal{M}}$ are surfaces of $\mathbb{R}^3$. Assuming that they are oriented, there exist smooth normal vector fields N, $\hat{N}$ of $\mathcal{M}$ and $\hat{\mathcal{M}}$ respectively. For a given t, choose oriented orthonormal frame $\hat{X}, \hat{Y} \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$ and $\hat{a}, \hat{b}, \hat{c} \in \mathbb{R}$ such that $\dot{U}(t)U(t)^{-1} \in \mathfrak{so}(3)$ can be written as

$$\dot{U}(t)U(t)^{-1} = \hat{a}(\hat{N}|_{\hat{\sigma}(t)}\times) + \hat{b}(\hat{X}\times) + \hat{c}(\hat{Y}\times),$$

where $\times$ denotes the cross product in $\mathbb{R}^3$ and for a vector $V \in \mathbb{R}^3$ we denote by $(V\times)$ the element of $\mathfrak{so}(3)$ given by $W \in \mathbb{R}^3 \mapsto V \times W \in \mathbb{R}^3$. It is now easy to see, by applying $\dot{U}(t)U(t)^{-1}$ to $\hat{X}, \hat{Y}$, that the tangential no-twist condition (3)-(a) is equivalent to the fact that $\hat{a} = 0$ i.e.,

$$\dot{U}(t)U(t)^{-1} \quad \text{does not contain } (\hat{N}|_{\hat{\sigma}(t)}\times)\text{-component.}$$

This is what is intuitively understood by "no spinning" since it is the $(\hat{N}|_{\hat{\sigma}(t)}\times)$ component $\hat{a}$ of $\dot{U}(t)U(t)^{-1}$ that mesures the instantaneous speed of rotation of $\mathcal{M}$ about the axis $\hat{N}|_{\hat{\sigma}(t)}$ at the corresponding point of contact. Notice also that

$$\dot{U}(t)U(t)^{-1}\hat{N}|_{\hat{\sigma}(t)} = -\hat{b}\hat{Y} + \hat{c}\hat{X} \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}},$$

so the normal no-twist condition (3)-(b) is automatically satisfied. This example can be easily generalized to any case of oriented hypersurfaces i.e. when $N = n+1$.

(ii) Suppose now that $N = 3$ and $n = 1$ i.e. $\mathcal{M}, \hat{\mathcal{M}}$ are regular curves in $\mathbb{R}^3$. Without loss of generality, we assume that $\|\dot{\sigma}(t)\| = 1$, hence also $\left\|\dot{\hat{\sigma}}(t)\right\| = 1$. Let $\hat{X}, \hat{Y} \in \dot{\hat{\sigma}}(t)^\perp$ such that $\hat{X}, \hat{Y}, \dot{\hat{\sigma}}(t)$ is an oriented orthonormal frame in $\mathbb{R}^3$. One may write $\dot{U}(t)U(t)^{-1} \in \mathfrak{so}(3)$ as

$$\dot{U}(t)U(t)^{-1} = \hat{a}(\dot{\hat{\sigma}}(t)\times) + \hat{b}(\hat{X}\times) + \hat{c}(\hat{Y}\times).$$

Since then

$$\dot{U}(t)U(t)^{-1}\dot{\hat{\sigma}} = -\hat{b}\hat{Y} + \hat{c}\hat{X},$$

the tangential no-twist condition (3)-(a) is trivially satisfied. As for the normal no-twist condition (3)-(b), one sees that it is equivalent to $\hat{a} = 0$ i.e.,

$$\dot{U}(t)U(t)^{-1} \quad \text{does not contain } (\dot{\hat{\sigma}}(t)\times)\text{-component.}$$

Intuitively this means that the instantaneous speed of rotation $\hat{a}$ of $\mathcal{M}$ about the axis $\dot{\hat{\sigma}}(t)$ is zero at the point of contact, so $\mathcal{M}$ does not turn around $\hat{\mathcal{M}}$.

The two manifolds M and $\hat{M}$ are embedded inside $\mathbb{R}^N$ by embeddings $\iota : M \to \mathbb{R}^N$ and $\hat{\iota} : \hat{M} \to \mathbb{R}^N$ and their metrics g and $\hat{g}$ are induced from the Euclidean metric s_N of $\mathbb{R}^N$ i.e., $g = \iota^* s_N$ and $\hat{g} = \hat{\iota}^* s_N$. In the above setting, we take now $\mathcal{M} = \iota(M)$, $\hat{\mathcal{M}} = \hat{\iota}(\hat{M})$. For $z \in \mathcal{M}$ and $\hat{z} \in \hat{\mathcal{M}}$, consider the linear orthogonal projections

$$P^T : T|_z\mathbb{R}^N \to T|_z\mathcal{M} \text{ and } P^\perp : T|_z\mathbb{R}^N \to T|_z\mathcal{M}^\perp,$$

and

$$\hat{P}^T : T|_{\hat{z}}\mathbb{R}^N \to T|_{\hat{z}}\hat{\mathcal{M}} \text{ and } \hat{P}^\perp : T|_{\hat{z}}\mathbb{R}^N \to T|_{\hat{z}}\hat{\mathcal{M}}^\perp,$$

respectively. For $X \in T|_z\mathbb{R}^N$ and $Y \in \Gamma(\pi_{T\mathbb{R}^N}|_{\mathcal{M}})$ (here $\pi_{T\mathbb{R}^N}|_{\mathcal{M}}$ is the pull-back bundle of $T\mathbb{R}^N$ over $\mathcal{M}$), we use $\nabla^\perp_X Y$ to denote $P^\perp(\nabla^{s_N}_X Y)$ and one writes similarly $\hat{\nabla}^\perp_{\hat{X}}\hat{Y} = \hat{P}^\perp(\nabla^{s_N}_{\hat{X}}\hat{Y})$ for $\hat{X} \in T|_{\hat{z}}\mathbb{R}^N$ and $Y \in \Gamma(\pi_{T\mathbb{R}^N}|_{\hat{\mathcal{M}}})$. We notice that, for any $z \in \mathcal{M}$, $X \in T|_z\mathcal{M}$ and $Y \in \mathrm{VF}(\mathcal{M})$, we have

$$\nabla^{s_N}_X Y = \iota_*(\nabla_{\iota_*^{-1}(X)}\iota_*^{-1}(Y)) + \nabla^\perp_X Y,$$

and similarly on $\hat{\mathcal{M}}$. Notice that $\nabla^\perp$ and $\hat{\nabla}^\perp$ determine (by restriction) connections of vector bundles $\pi_{T\mathcal{M}^\perp} : T\mathcal{M}^\perp \to \mathcal{M}$ and $\pi_{T\hat{\mathcal{M}}^\perp} : T\hat{\mathcal{M}}^\perp \to \hat{\mathcal{M}}$. These connections can then be used in an obvious way to determine a connection $\overline{\nabla}^{\perp}$ on the vector bundle

$$\pi_{(T\mathcal{M}^{\perp})^*\otimes T\mathcal{M}^{\perp}} : (T\mathcal{M}^{\perp})^* \otimes T\mathcal{M}^{\perp} \to \mathcal{M} \times \hat{\mathcal{M}}.$$

Let us take any rolling map $G : I \to \mathrm{SE}(N)$, $G(t) = (p(t), U(t))$ and development curve $\sigma : I \to \mathcal{M}$ and define $x = \iota^{-1} \circ \sigma$. We will go throught the meaning of each of the above conditions (1)-(3).

(1) (a) Since $\hat{\sigma}(t) \in \hat{\mathcal{M}}$, we may define a smooth curve $\hat{x} := \hat{\iota}^{-1} \circ \hat{\sigma}$ in $\hat{M}$.
(b) One easily sees that

$$U(t)T|_{\hat{\sigma}(t)}\mathcal{M} = T|_{\hat{\sigma}(t)}(G(t) \star \mathcal{M}) = T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}.$$

Thus $A(t) := \hat{\iota}_*^{-1} \circ U(t) \circ \iota_*|_{T|_{x(t)}M}$ defines a map $T|_{x(t)}M \to T|_{\hat{x}(t)}\hat{M}$, which is also orthogonal i.e., $A(t) \in Q|_{(x(t),\hat{x}(t))}$ for all t. Moreover, if $B(t) := U(t)|_{T|_{\sigma(t)}\mathcal{M}^{\perp}}$, then $B(t)$ is a map $T|_{\sigma(t)}\mathcal{M}^{\perp} \to T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}^{\perp}$ and, by a slight abuse of notation, we can write $U(t) = A(t) \oplus B(t)$. Thus Condition (1) just determines a smooth curve $t \mapsto (x(t), \hat{x}(t); A(t))$ inside the state space $Q = Q(M, \hat{M})$.

(2) We compute

$$\begin{aligned}0 =& \dot{G}(t) \star \sigma(t) = \dot{U}(t)\sigma(t) + \dot{p}(t)\\ =& \frac{\mathrm{d}}{\mathrm{d}t}(G(t) \star \sigma(t)) - U(t)\dot{\sigma}(t) = \dot{\hat{\sigma}}(t) - U(t) \circ \iota_* \circ \iota_*^{-1} \circ \dot{\sigma}(t),\end{aligned}$$

which, once composed with $\hat{\iota}_*^{-1}$ from the left, gives $0 = \dot{\hat{x}}(t) - A(t)\dot{x}(t)$. This is exactly the no-slip condition, Eq. (11).

(3) Notice that, on $\mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$, the sum metric $s_N \oplus s_N$ is just s_{2N}. Moreover, if $\gamma : I \to \mathbb{R}^N$ is a smooth curve, then smooth vector fields $X : I \to T(\mathbb{R}^N)$ along γ can be identified with smooth maps $X : I \to \mathbb{R}^N$ and with this observation one has: $\dot{X}(t) = \nabla^{s_N}_{\dot{\gamma}(t)} X$.

(a) Since $U(t) = A(t) \oplus B(t)$, we get, for $t \mapsto \hat{X}(t) \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$, that

$$\begin{aligned}&\dot{U}(t)U(t)^{-1}\hat{X}(t) = \nabla^{s_{2N}}_{(\dot{\sigma},\dot{\hat{\sigma}})(t)}\hat{X}(\cdot) - U(t)\nabla^{s_{2N}}_{(\dot{\sigma},\dot{\hat{\sigma}})(t)}(U(\cdot)^{-1}\hat{X}(\cdot))\\ =& P^T\big(\hat{\nabla}^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot)\big) + \hat{\nabla}^{\perp}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot)\\ &- U(t)\big(P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big) + \nabla^{\perp}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big)\\ =& \big(\overline{\nabla}_{(\dot{x},\dot{\hat{x}})(t)}A(\cdot)\big)A(t)^{-1}(\hat{\iota}_*^{-1}\hat{X}(t)) + \big(\hat{\nabla}^{\perp}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot) - B(t)\nabla^{\perp}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big),\end{aligned}$$

from which it is clear that the tangential no-twist condition corresponds to the condition that $\overline{\nabla}_{(\dot{x}(t),\dot{\hat{x}}(t))}A(\cdot) = 0$. This means that $t \mapsto (x(t), \hat{x}(t); A(t))$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ for all $t \in I$. Thus, the tangential no-twist condition (3)-(a) is equivalent to the no-spinning condition, Eq. (9).

(b) Choose $t \mapsto \hat{X}^{\perp}(t) \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}^{\perp}$ and calculate as above

$$\begin{aligned}\dot{U}(t)U(t)^{-1}\hat{X}^{\perp}(t) =& P^T(\nabla^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}^{\perp}(\cdot)) + \hat{\nabla}^{\perp}\dot{\hat{\sigma}}(t)\\ &- U(t)\big(P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big) + \nabla^{\perp}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big)\\ =& \Big(P^T(\nabla^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}^{\perp}(\cdot) - A(t)P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big)\Big)\\ &+ \big(\overline{\nabla}^{\perp}_{(\dot{\sigma}(t),\dot{\hat{\sigma}}(t))}B(\cdot)\big)B(t)^{-1}\hat{X}(t),\end{aligned}$$

and hence we see that the normal no-twist condition (3)-(b) corresponds to the condition that

$$\overline{\nabla}^{\perp}_{(\sigma(t),\hat{\sigma}(t))} B(\cdot) = 0, \quad \forall t.$$

In a similar spirit to how Definition 3.5 was given, one easily sees that this condition just amounts to say that B maps parallel translated normal vectors to $\mathcal{M}$ to parallel translated normal vectors to $\hat{\mathcal{M}}$. More precisely, if $X_0 \in T\mathcal{M}^{\perp}$ and $X(t) = (P^{\nabla^{\perp}})_0^t(\sigma)X_0$ is a parallel translate of X_0 along σ w.r.t. to the connection $\nabla^{\perp}$ (notice that $X(t) \in T|_{\sigma(t)}\mathcal{M}^{\perp}$ for all t), then the normal no-twist condition (3)-(b) requires that $t \mapsto B(t)X(t)$ (which is the same as $U(t)X(t)$) is parallel to $t \mapsto \hat{\sigma}(t)$ w.r.t the connection $\hat{\nabla}^{\perp}$ i.e., for all t,

$$B(t)((P^{\nabla^{\perp}})_0^t(\sigma)X_0) = (P^{\hat{\nabla}^{\perp}})_0^t(\hat{\sigma})(B(0)X_0).$$

We formulate the preceding remarks to a proposition.

Proposition B.2 Let $\iota : M \to \mathbb{R}^N$ and $\hat{\iota} : \hat{M} \to \mathbb{R}^N$ be smooth embeddings and let $g = \iota^*(s_N)$ and $\hat{g} = \hat{\iota}^*(s_N)$. Fix points $x_0 \in M$, $\hat{x}_0 \in \hat{M}$ and an element $B_0 \in \mathrm{SO}(T|_{\iota(x_0)}\mathcal{M}^{\perp}, T|_{\hat{\iota}(\hat{x}_0)}\hat{\mathcal{M}}^{\perp})$. Then, there is a bijective correspondence between the smooth curves $t \mapsto (x(t), \hat{x}(t); A(t))$ of Q tangent to $\mathcal{D}_{\mathrm{NS}}$ (resp. $\mathcal{D}_{\mathrm{R}}$), satisfying $(x(0), \hat{x}(0)) = (x_0, \hat{x}_0)$ and the pairs of smooth curves $t \mapsto G(t) = (p(t), U(t))$ of $\mathrm{SE}(N)$ and $t \mapsto \sigma(t)$ of $\mathcal{M}$ which satisfy the conditions (1), (3) (resp. (1),(2),(3) i.e., rolling maps) and $U(0)|_{T|_{\sigma(0)}\mathcal{M}^{\perp}} = B_0$.

Proof. Let $t \mapsto q(t) = (x(t), \hat{x}(t); A(t))$ to be a smooth curve in Q such that $(x(0), \hat{x}(0)) = (x_0, \hat{x}_0)$. Denote $\sigma = \iota \circ x$, $\hat{\sigma} = \hat{\iota} \circ \hat{x}$ and let $B(t) = (P^{\overline{\nabla}^{\perp}})_0^t((\sigma, \hat{\sigma}))B_0$ be the parallel translate of B_0 along $t \mapsto (\sigma(t), \hat{\sigma}(t))$ w.r.t the connection $\overline{\nabla}^{\perp}$. We define

$$U(t) := (\hat{\iota}_* \circ A(t) \circ \iota_*^{-1}) \oplus B(t) : T|_{\sigma(t)}\mathcal{M} \to T|_{\hat{\sigma}(t)}\hat{\mathcal{M}},$$

and $p(t) = \hat{\sigma}(t) - U(t)\sigma(t)$. Then, by the above remarks, the smooth curve $t \mapsto G(t) = (p(t), U(t))$ satisfies Conditions (1),(3) (resp. (1),(2),(3)) if $t \mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ (resp. $\mathcal{D}_{\mathrm{R}}$). This clearly gives the claimed bijective correspondence. □

C Special Manifolds in 3D Riemanniann Geometry

C.1 Preliminaries

On an oriented Riemannian manifold (M, g), the Hodge-dual $\star_M$ is defined as the linear map uniquely given by

$$\star_M : \wedge^k T|_x M \to \wedge^{n-k} T|_x M; \quad \star_M(X_1 \wedge \cdots \wedge X_k) = X_{k+1} \wedge \cdots \wedge X_n,$$

with $x \in M$, $k = 0, \ldots, n = \dim M$ and $X_1, \ldots, X_n \in T|_x M$ any oriented basis. For an oriented Riemannian manifold (M, g) and $x \in M$, let $\mathfrak{so}(T|_x M)$ be the set of g-antisymmetric linear maps $T|_x M \to T|_x M$. and writes $\mathfrak{so}(M)$ as the disjoint union of $\mathfrak{so}(T|_x M)$, $x \in M$. If $A, B \in \mathfrak{so}(T|_x M)$, we define

$$[A, B]_{\mathfrak{so}} := A \circ B - B \circ A \in \mathfrak{so}(T|_x M).$$

Also, we define the following natural isomorphism ϕ by

$$\phi : \wedge^2 TM \to \mathfrak{so}(M); \quad \phi(X \wedge Y) := g(\cdot, X)Y - g(\cdot, Y)X.$$

Using this isomorphism, the curvature tensor R of (M, g) at $x \in M$, is the linear map given by

$$\mathcal{R} : \wedge^2 T|_x M \to \wedge^2 T|_x M; \quad \mathcal{R}(X \wedge Y) := \phi^{-1}(R(X, Y)),$$

where $X, Y \in T|_x M$. Here of course $R(X, Y)$, as an element of $T^*|_x M \otimes T|_x M$, belongs to $\mathfrak{so}(T|_x M)$. It is a standard fact that $\mathcal{R}$ is a symmetric map when $\wedge^2 T|_x M$ is endowed with the inner product, also written as g,

$$g(X \wedge Y, Z \wedge W) := g(X, Z)g(Y, W) - g(X, W)g(Y, Z).$$

For $A, B \in \mathfrak{so}(T|_x M)$, $\operatorname{tr}(AB) = g(\phi^{-1}(A), \phi^{-1}(B))$. The map $\mathcal{R}$ is the curvature operator and we will, with a slight abuse of notation, write it as R.

If $\dim M = 3$, then $\star_M^2 = \mathrm{id}$ when $\star_M$ is the map $\wedge^2 TM \to TM$ and $TM \to \wedge^2 TM$. Let $X, Y, Z \in T|_x M$ be an orthonormal positively oriented basis. Then

$$\star_M(X \wedge Y) = Z, \quad \star_M(Y \wedge Z) = X, \quad \star_M(Z \wedge X) = Y.$$

In terms of this basis X, Y, Z one has

$$\star_M \phi^{-1} \begin{pmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ \beta \\ \alpha \end{pmatrix}.$$

Lemma C.1 If (M, g) is a 3-dimensional oriented Riemannian manifold and $x \in M$.

(i) Then each 2-vector $\xi \in \wedge^2 T|_x M$ is pure i.e. there exist $X, Y \in T|_x M$ such that $\xi = X \wedge Y$.

(ii) For every $X, Y \in T|_x M$ one has

$$[\phi(\star_M X), \phi(\star_M Y)]_{\mathfrak{so}} = \phi(X \wedge Y).$$

C.2 Manifolds of class M_β

In this subsection, we define and investigate some properties of special type of 3-dimensional manifolds. Following the paper [2] we make the following definition.

Definition C.2 A 3-dimensional manifold M is called a *contact manifold of type* $(\kappa, 0)$ where $\kappa \in C^\infty(M)$ if there are everywhere linearly independent vector fields $F_1, F_2, F_3 \in \mathrm{VF}(M)$ and smooth functions $c, \gamma_1, \gamma_3 \in C^\infty(M)$ such that

$$\begin{aligned}
[F_1, F_2] =& cF_3, \\
[F_2, F_3] =& cF_1, \\
[F_3, F_1] =& -\gamma_1 F_1 + F_2 - \gamma_3 F_3,
\end{aligned}$$

and

$$-\kappa = F_3(\gamma_1) - F_1(\gamma_3) + (\gamma_1)^2 + (\gamma_3)^2 - c.$$

The frame F_1, F_2, F_3 is said to be an *(normalized) adapted frame of M* and c, γ_1, γ_3 the corresponding structure functions.

Remark C.3 More generally, one could say that a 3-dimensional contact manifold M is of class (κ, χ) if its so-called *first and second invariants* are κ and χ, respectively, where $\kappa, \chi \in C^\infty(M)$, cf. [2].

One may define on such a manifold a Riemannian metric g in a natural way by declaring F_1, F_2, F_3 to be orthogonal. In order to see that a manifold defined in the above definition C.2 is indeed a contact manifold, it suffices to define a 1-form α by $\alpha(X) = g(X, F_2)$, $X \in TM$, and observe that $\mathrm{d}\alpha(F_3, F_1) = -g([F_3, F_1], F_2) = -1$, from which one can conclude that $\alpha \wedge d\alpha \neq 0$ (cf. [15], Definition 1.1.3 and Lemma 1.1.7).

The structure of the connection table (see section C.4) of the Levi-Civita connection and the eigenvalues of the corresponding curvature tensor are given in the following lemma, which is a direct consequence of Koszul's formula.

Lemma C.4 Let M be a contact manifold of type $(\kappa, 0)$ with adapted frame F_1, F_2, F_3 and structure functions c, γ_1, γ_2. If g is the unique Riemannian metric which makes F_1, F_2, F_3 orthonormal, then the connection table w.r.t. F_1, F_2, F_3 is

$$\Gamma = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \gamma_1 & c - \frac{1}{2} & \gamma_3 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},$$

Moreover, at each point, $\star F_1, \star F_2, \star F_3$ (with $\star$ the Hodge dual) are eigenvectors of the curvature tensor R with eigenvalues $-K, -K_2(\cdot), -K$, respectively, where

$$\begin{aligned} K =& \frac{1}{4}, \quad \text{(constant)}, \\ K_2(x) =& \kappa(x) - \frac{3}{4}, \quad x \in M. \end{aligned}$$

To justify somewhat our next definition, we make the following remark.

Remark C.5 Notice that if $\beta \in \mathbb{R}$, $\beta \neq 0$ and $g_\beta := \beta^{-2} g$ then the Koszul-formula gives,

$$2g_\beta(\nabla^{g_\beta}_{F_i} F_j, F_k) = \beta^{-2} g([F_i, F_j], F_k) - \beta^{-2} g([F_i, F_k], F_j) - \beta^{-2} g([F_j, F_k], F_i) = 2\beta^{-2}\Gamma^i_{(j,k)},$$

because $g_\beta(F_i, F_j) = \beta^{-2}\delta_{ij}$. Then, $E_i := \beta F_i$, $i = 1, 2, 3$, is a g_β-orthonormal basis and if $(\Gamma_\beta)^i_{(j,k)} = g_\beta(\nabla_{E_i} E_j, E_k)$, then for every i, j, k.

$$\beta^{-3}(\Gamma_\beta)^i_{(j,k)} = \beta^{-3} g_\beta(\nabla^{g_\beta}_{E_i} E_j, E_k) = g_\beta(\nabla^{g_\beta}_{F_i} F_j, F_k) = \beta^{-2}\Gamma^i_{(j,k)},$$

i.e. $(\Gamma_\beta)^i_{(j,k)} = \beta \Gamma^i_{(j,k)}$.

Definition C.6 A 3-dimensional Riemannian manifold (M, g) is said to belong to *class* $\mathcal{M}_\beta$, for $\beta \in \mathbb{R}$, if there exists an orthonormal frame $E_1, E_2, E_3 \in \mathrm{VF}(M)$ w. r. t. which the connection table is

$$\Gamma = \begin{pmatrix} \beta & 0 & 0 \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ 0 & 0 & \beta \end{pmatrix}.$$

In this case the frame E_1, E_2, E_3 is called an *adapted frame of* (M, g).

Remark C.7 For a given $\beta \in \mathbb{R}$, one can say that a Riemannian space (M, g) is *locally of class* $\mathcal{M}_\beta$, if every $x \in M$ has an open neighbourhood U such that $(U, g|_U)$ is of class $\mathcal{M}_\beta$. Since we are interested in local results, we usually speak of manifolds of (globally) class $\mathcal{M}_\beta$.

Lemma C.8 If $\beta \neq 0$ and (M, g) is of class $\mathcal{M}_\beta$ with an adapted frame, then $\star E_1, \star E_2, \star E_3$ are eigenvectors of R with eigenvalues $-\beta^2, -K_2(\cdot), -\beta^2$, where

$$-K_2(x) = \beta^2 + E_3(\Gamma^1_{(3,1)}) - E_1(\Gamma^3_{(3,1)}) + (\Gamma^1_{(3,1)})^2 + (\Gamma^3_{(3,1)})^2 - 2\beta\Gamma^2_{(3,1)}, \quad x \in M.$$

Proof. Immediate from Proposition C.17, Eq. (58). □

Next lemma is the converse of what has been done before the above definition.

Lemma C.9 Let (M, g) be of class $\mathcal{M}_\beta$, $\beta \neq 0$, with an adapted frame E_1, E_2, E_3. Then M is a contact manifold of type $(\kappa, 0)$ with (normalized) adapted frame $F_i := \frac{1}{2\beta} E_i$, $i = 1, 2, 3$. Moreover, for $x \in M$, κ and the structure functions c, γ_1, γ_3 are given by

$$c = \frac{\beta + \Gamma^2_{(3,1)}}{2\beta}, \quad \gamma_1 = \frac{\Gamma^1_{(3,1)}}{2\beta}, \quad \gamma_3 = -\frac{\Gamma^3_{(3,1)}}{2\beta}, \quad \kappa = \frac{K_2}{4\beta^2} + \frac{3}{4}.$$

Proof. From the torsion freeness of the Levi-Civita connection on (M, g) and from the connection table w.r.t. E_1, E_2, E_3, we get

$$\begin{aligned} [E_1, E_2] &= (\beta + \Gamma^2_{(3,1)}) E_3, \\ [E_2, E_3] &= (\beta + \Gamma^2_{(3,1)}) E_1, \\ [E_3, E_1] &= -\Gamma^1_{(3,1)} E_1 + 2\beta E_2 - \Gamma^3_{(3,1)} E_3. \end{aligned}$$

From this and the fact that $\beta \neq 0$, the claims are immediate. □

Remark C.10 (i) Note that the classes $\mathcal{M}_\beta$ and $\mathcal{M}_{-\beta}$ are the same. Indeed, if (M, g) is of class $\mathcal{M}_\beta$ and E_1, E_2, E_3 is an adapted orthonormal frame, then (M, g) is of class $\mathcal{M}_{-\beta}$ with a adapted frame F_1, F_2, F_3 where $F_1 = E_3$, $F_3 = E_1$ (i.e., the change of orientation of E_1, E_3 plane moves from $\mathcal{M}_\beta$ to $\mathcal{M}_{-\beta}$). It would then be better to speak of Riemannian manifolds of class $\mathcal{M}_\beta$ with $\beta \geq 0$ or of class $\mathcal{M}_{|\beta|}$.

(ii) If one has a Riemannian manifold (M, g) of class $\mathcal{M}_\beta$, then scaling the metric by $\lambda \neq 0$ one gets a Riemannian manifold $(M, \lambda^2 g)$ of class $\mathcal{M}_{\beta/\lambda}$. This follows from Remark C.5 above.

Remark C.11 If (M, g) is of class $\mathcal{M}_0$, then since $\beta = 0$ and $\Gamma^1_{(1,2)} = 0$, one deduces e.g. from Theorem C.14 that (M, g) is locally a warped product. Conversely, a Riemannian product manifold is locally of class $\mathcal{M}_0$. Hence there are many non-isometric spaces of class $\mathcal{M}_0$.

To conclude this subsection, we will show that for every $\beta \in \mathbb{R}$ there exist 3-dimensional Riemannian manifolds of class $\mathcal{M}_\beta$ which are not isometric. See also [2].

Example C.12 (i) Let M be $\mathrm{SO}(3)$. There exists left-invariant vector fields E_1, E_2, E_3 such that

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

Hence with the metric g rendering E_1, E_2, E_3 orthonormal, we get a space (M, g) of class $\mathcal{M}_{1/2}$. By the definition of κ and Lemma C.9 we have $\kappa = 1$ and $K_2 = \frac{1}{4}$.

(ii) Let M be the Heisenberg group H_3. There exists left-invariant vector fields E_1, E_2, E_3 such that

$$[E_1, E_2] = 0, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = E_2.$$

Then, M endowed with the metric for which E_1, E_2, E_3 are orthonormal, is of class $\mathcal{M}_{1/2}$ and $\kappa = 0$, $K_2 = -\frac{3}{4}$.

(iii) Let M be $\mathrm{SL}(2)$. There exists left-invariant vector fields such that

$$[E_1, E_2] = -E_3, \quad [E_2, E_3] = -E_1, \quad [E_3, E_1] = E_2.$$

If g is a metric with respect to which E_1, E_2, E_3 are orthonormal, then M is of class $\mathcal{M}_{1/2}$, with $\kappa = -1$ and $K_2 = -\frac{7}{4}$.

Note that if one takes the "usual" basis of $\mathfrak{sl}(2)$ as a, b, c satisfying,

$$[c, a] = 2a, \quad [c, b] = -2b, \quad [a, b] = c,$$

then one may define $e_1 = \frac{a+b}{2}$, $e_2 = \frac{a-b}{2}$, $e_3 = \frac{c}{2}$ to obtain

$$[e_1, e_2] = -e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2.$$

None of the examples in (i)-(iii) of Riemannian manifolds of class $\mathcal{M}_\beta$ with $\beta = \frac{1}{2}$ are (locally) isometric one to the other. This fact is immediately read from the different values of K_2 (constant). Hence by Remarks C.10 and C.11, we see that for every $\beta \in \mathbb{R}$ there are non-isometric Riemannian manifolds of the same class $\mathcal{M}_\beta$.

C.3 Warped Products

Definition C.13 Let (M, g), (N, h) be Riemannian manifolds and $f \in C^\infty(M)$. Define a metric h_f on $M \times N$

$$h_f = \mathrm{pr}_1^*(g) + (f \circ \mathrm{pr}_1)^2 \mathrm{pr}_2^*(h),$$

where $\mathrm{pr}_1, \mathrm{pr}_2$ are projections onto the first and second factor of $M \times N$, respectively. Then the Riemannian manifold $(M \times N, h_f)$ is called a *warped product of* (M, g) *and* (N, h) with the *warping function* f. One may write $(M \times N, h_f)$ as $(M, g) \times_f (N, h)$ and h_f as $g \oplus_f h$ if there is a risk of ambiguity.

We are mainly interested in the case where $(M,g)=(I,s_1)$, where $I\subset\mathbb{R}$ is an open non-empty interval and s_1 is the standard Euclidean metric on $\mathbb{R}$. By convention, we write $\frac{\partial}{\partial r}$ for the natural positively directed unit (w.r.t. s_1) vector field on $\mathbb{R}$ and identify it in the canonical way as a vector field on the product $I\times N$ and notice that it is also a unit vector field w.r.t. h_f.

Since needed in section 5, we state (a local version of) the main result of [19] in 3-dimensional case. The general result allows one to detect Riemannian spaces which are locally warped products. In our setting we use it (in the below form) to detect when a 3-dimensional Riemannian manifold (M,g) is, around a given point, a warped product of the form $(I\times N,h_f)$, with $I\subset\mathbb{R}$, $f\in C^\infty(I)$, and (N,h) a 2-dimensional Riemannian manifold.

Theorem C.14 ([19]) Let (M,g) be a Riemannian manifold of dimension 3. Suppose that at every point $x_0\in M$ there is an orthonormal frame E_1,E_2,E_3 defined in a neighbourhood of x_0 such that the connection table w.r.t. E_1,E_2,E_3 on this neighbourhood is of the form

$$\Gamma=\begin{pmatrix} 0 & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & 0\end{pmatrix},$$

and moreover

$$X(\Gamma^1_{(1,2)})=0,\quad \forall X\in E_2^\perp.$$

Then there is a neighbourhood U of x, an interval $I\subset\mathbb{R}$, $f\in C^\infty(I)$ and a 2-dimensional Riemannian manifold (N,h) such that $(U,g|_U)$ is isometric to the warped product $(I\times N,h_f)$. If $F:(I\times N,h_f)\to(U,g|_U)$ is the isometry in question, then for all $(r,y)\in I\times N$,

$$\frac{f'(r)}{f(r)}=-\Gamma^1_{(1,2)}(F(r,y)),\quad F_*\frac{\partial}{\partial r}\big|_{(r,y)}=E_2|_{\phi(r,y)}.$$

C.4 Technical propositions

Since we will be dealing frequently with orthonormal frames and connection coefficients, it is convenient to define the following concept.

Definition C.15 Let (M,g) be a 3-dimensional Riemannian manifold. If E_1,E_2,E_3 is an orthonormal frame of M defined on an open set U, then $\Gamma^j_{(i,k)}=g(\nabla_{E_j}E_i,E_k)$, we call the matrix

$$\Gamma=\begin{pmatrix} \Gamma^1_{(2,3)} & \Gamma^2_{(2,3)} & \Gamma^3_{(2,3)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & \Gamma^2_{(1,2)} & \Gamma^3_{(1,2)}\end{pmatrix},$$

the *connection table w.r.t.* E_1,E_2,E_3. To emphasize the frame, we may write $\Gamma=\Gamma_{(E_1,E_2,E_3)}$.

Remark C.16 (i) Since E_1,E_2,E_3 is orhonormal, one has $\Gamma^i_{(j,k)}=-\Gamma^i_{(k,j)}$ for all i,j,k. These relations mean that to know all the connection coefficients (of an orthonormal frame), it is enough to know exactly 9 of them. It is these 9 coefficients, that appear in the connection table.

(ii) Here it is important that the frame E_1, E_2, E_3 is ordered and hence one should speak of the connection table w.r.t. (E_1, E_2, E_3) (as in the notation $\Gamma = \Gamma_{(E_1,E_2,E_3)}$), but since we always list the frame in the correct order, there will be no room for confusion.

(iii) Notice that the above connection table could be written as $\Gamma = [(\Gamma^j_{\star i})^i_j]$, if one writes $\star 1 = (2,3)$, $\star 2 = (3,1)$ and $\star 3 = (1,2)$ i.e.

$$\Gamma = \begin{pmatrix} \Gamma^1_{\star 1} & \Gamma^2_{\star 1} & \Gamma^3_{\star 1} \\ \Gamma^1_{\star 2} & \Gamma^2_{\star 2} & \Gamma^3_{\star 2} \\ \Gamma^1_{\star 3} & \Gamma^2_{\star 3} & \Gamma^3_{\star 3} \end{pmatrix}.$$

(iv) One should notice that usually the Christoffel symbols Γ^k_{ji} of a frame E_1, E_2, E_3 are defined by

$$\nabla_{E_j} E_i = \sum_{k=1}^{3} \Gamma^k_{ji} E_k.$$

The notation $\Gamma^j_{(i,k)}$ introduced above differs from this by

$$\Gamma^j_{(i,k)} = \Gamma^k_{ji}.$$

We find this notation convenient and we only use it when dealing with 3-dimensional manifolds.

Proposition C.17 Suppose (M, g) is a 3-dimensional Riemannian manifold and in some neighbourhood U of $x \in M$ there is an orthonormal frame E_1, E_2, E_3 defined on an open set U with respect to which the connection table is of the form

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)} \end{pmatrix},$$

and $V(\Gamma^1_{(2,3)}) = 0$, $V(\Gamma^1_{(1,2)}) = 0$, for all $V \in E_2|_y^{\perp}$, $\quad y \in U$. Then the following are true:

(i) For every $y \in U$, $\star E_1|_y, \star E_2|_y, \star E_3|_y$ are eigenvectors of R with eigenvalues $-K(y), -K_2(y), -K(y)$, respectively (i.e. the eigenvalues of $\star E_1|_y$ and $\star E_3|_y$ coincide).

(ii) If $\Gamma^1_{(2,3)} \neq 0$ on U and if U is connected, it follows that on U the coefficient $\Gamma^1_{(2,3)}$ is constant, $\Gamma^1_{(1,2)} = 0$ and $K(y) = (\Gamma^1_{(2,3)})^2$ (constant). Hence $(U, g|_U)$ is of class $\mathcal{M}_\beta$, for $\beta = \Gamma^1_{(2,3)}$.

(iii) If $\Gamma^1_{(2,3)} = 0$ in the open set U, then every $y \in U$ has a neighbourhood $U' \subset U$ such that $(U', g|_{U'})$ is isometric to a warped product $(I \times N, h_f)$ where $I \subset \mathbb{R}$ is an open interval. Moreover, if $F : (I \times N, h_f) \to (U', g|_{U'})$ is the isometry in question, then, for every $(r, y) \in I \times N$,

$$\frac{f'(r)}{f(r)} = -\,\Gamma^1_{(1,2)}(F(r,y)), \quad F_* \frac{\partial}{\partial r}\Big|_{(r,y)} = E_2|_{F(r,y)}.$$

Moreover, one has

$$
\begin{aligned}
0 =& - E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)}, && (56)\\
-K =& - E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2, && (57)\\
-K_2 =& E_3(\Gamma^1_{(3,1)}) - E_1(\Gamma^3_{(3,1)}) + (\Gamma^1_{(3,1)})^2 + (\Gamma^3_{(3,1)})^2 - 2\Gamma^1_{(2,3)}\Gamma^2_{(3,1)} + (\Gamma^1_{(1,2)})^2 + (\Gamma^1_{(2,3)})^2. && (58)
\end{aligned}
$$

Proof. (i) We begin by computing in the basis $\star E_1, \star E_2, \star E_3$ that

$$
\begin{aligned}
R(E_3 \wedge E_1) =& \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} \wedge \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} + \begin{pmatrix} E_3(\Gamma^1_{(2,3)}) \\ E_3(\Gamma^1_{(3,1)}) \\ E_3(\Gamma^1_{(1,2)}) \end{pmatrix} - \begin{pmatrix} -E_1(\Gamma^1_{(1,2)}) \\ E_1(\Gamma^3_{(3,1)}) \\ E_1(\Gamma^1_{(2,3)}) \end{pmatrix} \\
&+ \Gamma^1_{(3,1)} \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - 2\Gamma^1_{(2,3)} \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} + \Gamma^3_{(3,1)} \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} = \begin{pmatrix} 0 \\ -K_2 \\ 0 \end{pmatrix},
\end{aligned}
$$

where we omitted the further computation of row 2 and wrote it simply as $-K_2$ and use the fact that $E_i(\Gamma^1_{(1,2)}) = 0$, $E_i(\Gamma^1_{(2,3)}) = 0$ for $i \in \{1, 3\}$. Thus $\star E_2|_y$ is an eigenvector of $R|_y$ for all $y \in U$. Since $R|_y$ is a symmetric linear map $\wedge^2 T|_y M$ to itself and $\star E_2|_y$ is an eigenvector for $R|_y$, we know that the other eigenvectors lie in $\star E_2|_y$, which is spanned by $\star E_1|_y, \star E_3|_y$. By rotating E_1, E_3 among themselves by a constant matrix, we may well assume that $\star E_1|_y, \star E_3|_y$ are eigenvectors of $R|_y$ corresponding to eigenvalues, say, $-K_1(y), -K_3(y)$. We want to show that $K_1(y) = K_3(y)$. Computing $R|_y(E_1 \wedge E_2)$ in the basis $\star E_1|_y, \star E_2|_y, \star E_3|_y$ gives (we write simply $\Gamma^i_{(j,k)}$ for $\Gamma^i_{(j,k)}(y)$ etc.)

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ -K_3(y) \end{pmatrix} =& R|_y(\star E_3) = R|_y(E_1 \wedge E_2) \\
=& \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ E_1(\Gamma^2_{(3,1)}) \\ 0 \end{pmatrix} - \begin{pmatrix} E_2(\Gamma^1_{(2,3)}) \\ E_2(\Gamma^1_{(3,1)}) \\ E_2(\Gamma^1_{(1,2)}) \end{pmatrix} \\
&+ \Gamma^1_{(1,2)} \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)}) \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} \\
=& \begin{pmatrix} -E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)} \\ E_1(\Gamma^2_{(3,1)}) - E_2(\Gamma^1_{(3,1)}) + \Gamma^1_{(1,2)}\Gamma^1_{(3,1)} - (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)})\Gamma^3_{(3,1)} \\ -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2 \end{pmatrix},
\end{aligned}
$$

from where $-K_3(y) = -E_2|_y(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)}(y))^2 - (\Gamma^1_{(2,3)}(y))^2$. Similarly, computing

$R|_y(E_2\wedge E_3)$ in the basis $\star E_1|_y, \star E_2|_y, \star E_3|_y$,

$$\begin{aligned}
\begin{pmatrix} -K_1(y) \\ 0 \\ 0 \end{pmatrix} =& R|_y(\star E_1) = R|_y(E_2\wedge E_3) \\
=& \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} + \begin{pmatrix} -E_2(\Gamma^1_{(1,2)}) \\ E_2(\Gamma^3_{(3,1)}) \\ E_2(\Gamma^1_{(2,3)}) \end{pmatrix} - \begin{pmatrix} 0 \\ E_3(\Gamma^2_{(3,1)}) \\ 0 \end{pmatrix} \\
& - (\Gamma^2_{(3,1)} + \Gamma^1_{(2,3)}) \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - \Gamma^1_{(1,2)} \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} \\
=& \begin{pmatrix} -E_2(\Gamma^1_{(1,2)}) - (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2 \\ E_2(\Gamma^3_{(3,1)}) - E_3(\Gamma^2_{(3,1)}) - (\Gamma^2_{(3,1)} + \Gamma^1_{(2,3)})\Gamma^1_{(3,1)} - \Gamma^1_{(1,2)}\Gamma^3_{(3,1)} \\ E_2(\Gamma^1_{(2,3)}) - 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)}, \end{pmatrix}
\end{aligned}$$

leads us to $-K_1(y) = -E_2|_y(\Gamma^1_{(1,2)}) - (\Gamma^1_{(2,3)}(y))^2 + (\Gamma^1_{(1,2)}(y))^2$. By comparing to the result of the computations of $R|_y(E_1\wedge E_2)$ and $R|_y(E_2\wedge E_3)$ implies that $K_1(y) = K_3(y)$. In other words, if one writes $K(y)$ for this common value $K_1(y) = K_3(y)$, one sees that $E_2|_y^{\perp}$ is contained in the eigenspace of $R|_y$ corresponding to the eigenvalue $-K(y)$. This finishes the proof of (i).

(ii) Suppose now that $\Gamma^1_{(2,3)} \neq 0$ on an open connected subset U of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$. Then since $E_1(\Gamma^1_{(2,3)}) = 0$, $E_3(\Gamma^1_{(2,3)}) = 0$ on U, one has, on U,

$$[E_3, E_1](\Gamma^1_{(2,3)}) = E_3(E_1(\Gamma^1_{(2,3)})) - E_1(E_3(\Gamma^1_{(2,3)})) = 0.$$

On the other hand, $[E_3, E_1] = -\Gamma^1_{(3,1)}E_1 + 2\Gamma^1_{(2,3)}E_2 - \Gamma^3_{(3,1)}E_3$, so

$$\begin{aligned}
0 =& [E_3, E_1](\Gamma^1_{(2,3)}) \\
=& -\Gamma^1_{(3,1)}E_1(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(2,3)}E_2(\Gamma^1_{(2,3)}) - \Gamma^3_{(3,1)}E_3(\Gamma^1_{(2,3)}) = 2\Gamma^1_{(2,3)}E_2(\Gamma^1_{(2,3)}).
\end{aligned}$$

Since $\Gamma^1_{(2,3)} \neq 0$ everywhere on U, one has $E_2(\Gamma^1_{(2,3)}) = 0$ on U. Because E_1, E_2, E_3 span TM on U, we have that all the derivatives of $\Gamma^1_{(2,3)}$ vanish on U and thus it is constant. From the first row of the computation of $R(E_1\wedge E_2)$ in the case (ii) above, one gets

$$0 = -E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)} = 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)},$$

which implies $\Gamma^1_{(1,2)} = 0$ on U. Finally from the last row computation of $R(E_1\wedge E_2)$ (recall that $K_1(y) = K_3(y) =: K(y)$), one gets

$$-K(y) = -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2 = -(\Gamma^1_{(2,3)})^2.$$

This concludes the proof of (ii).

(iii) This case follows from Theorem C.14. □

We next provide two technical propositions which are needed to conclude the proof of Theorem 5.1 and Theorem 5.3.

Proposition C.18 Let (M,g), $(\hat{M},\hat{g})$ be two Riemannian manifolds of dimension 3, $q_0=(x_0,\hat{x}_0;A_0)\in Q$ and suppose there is an open subset O of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and a smooth unit vector field $E_2\in\mathrm{VF}(\pi_{Q,M}(O))$ such that $\nu(A\star E_2)|_q$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q\in O$. If the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q, then for any $x\in\pi_{Q,M}(O)$ and any unit vector fields E_1,E_3 such that E_1,E_2,E_3 is an orthonormal frame in some neighbourhood U of x in M, the connection table associated to E_1,E_2,E_3 is given by

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},$$

and

$$V(\Gamma^1_{(2,3)})=0,\quad V(\Gamma^1_{(1,2)})=0,\quad \forall V\in E_2|_y^{\perp},\quad y\in U,$$

where $\Gamma=[(\Gamma^j_{\star i})^i_j]$, $\Gamma^j_{(i,k)}=g(\nabla_{E_j}E_i,E_k)$ and $\star 1=(2,3)$, $\star 2=(3,1)$ and $\star 3=(1,2)$.

Remark C.19 In particular, this means that the assumptions of the previous proposition imply that the assumptions of Proposition C.17 are fulfilled.

Proof. Notice that $\pi_{Q,M}(O)$ is open in M since $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}=\pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$ is a submersion. Without loss of generality, we may assume that there exist $E_1,E_3\in\mathrm{VF}(\pi_{Q,M}(O))$ such that E_1,E_2,E_3 form an orthonormal basis.

We begin by computing in O the following Lie bracket,

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star E_2)]|_q=&-\mathscr{L}_{\mathrm{NS}}(A(\star E_2)E_2)|_q+\nu(A\star\nabla_{E_2}E_2)|_q\\ =&\nu(A\star(-\Gamma^2_{(1,2)}E_1+\Gamma^2_{(2,3)}E_3)|_q=:V_2|_q,\end{aligned}$$

whence V_2 is a vector field in O and furthermore

$$\begin{aligned}[V_2,\nu((\cdot)\star E_2)]|_q=&\nu(A[\star(-\Gamma^2_{(1,2)}E_1+\Gamma^2_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\ =&\nu(A\star(-\Gamma^2_{(1,2)}E_3-\Gamma^2_{(2,3)}E_1))|_q=:M_2|_q,\end{aligned}$$

where M_2 is a vector field in O as well. Now if there were an open subset O' of O the $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$-vertical vector fields where $\nu(A\star E_2)|_q,V_2|_q,M_2|_q$ were linearly independent for all $q\in O'$, it would follow that they form a basis of $V|_q(\pi_Q)$ for $q\in O'$ and hence $V|_q(\pi_Q)\subset T|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ for $q\in O'$. Then Corollary 4.18 would imply that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open, which is a contradiction. Hence in a dense subset O_d of O one has that $\nu(A\star E_2)|_q,V_2|_q,M_2|_q$ are linearly dependent which implies

$$0=\det\begin{pmatrix}0 & 1 & 0\\ -\Gamma^2_{(1,2)} & 0 & \Gamma^2_{(2,3)}\\ -\Gamma^2_{(2,3)} & 0 & -\Gamma^2_{(1,2)}\end{pmatrix}=-((\Gamma^2_{(1,2)})^2+(\Gamma^2_{(2,3)})^2),$$

i.e.,

$$\Gamma^2_{(1,2)}=0,\quad \Gamma^2_{(2,3)}=0,$$

on $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O_d)$. It is clear that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O_d)$ is dense in $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$ so the above relation holds on the open subset $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$ of M.

Next compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star E_2)]|_q =&\mathscr{L}_{\mathrm{NS}}(AE_3)|_q+\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q=\mathscr{L}_{\mathrm{R}}(E_3)|_q-L_3|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star E_2)]|_q =&-\mathscr{L}_{\mathrm{NS}}(AE_1)|_q-\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q\\
=&-\mathscr{L}_{\mathrm{R}}(E_1)|_q+L_1|_q,
\end{aligned}$$

where $L_1,L_3\in\mathrm{VF}(O')$ such that

$$\begin{aligned}
L_1|_q :=&\mathscr{L}_{\mathrm{NS}}(E_1)|_q+\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q,\\
L_3|_q :=&\mathscr{L}_{\mathrm{NS}}(E_3)|_q-\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q.
\end{aligned}$$

Continuing by taking brackets of these against $\nu(A\star E_2)|_q$ gives

$$\begin{aligned}
[L_1,\nu((\cdot)\star E_2)]|_q =&\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q+\nu(A[\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\
=&\nu(A\star(-(\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)})E_1+(\Gamma^1_{(2,3)}-\Gamma^3_{(1,2)})E_3)|_q=:M_3,\\
[L_3,\nu((\cdot)\star E_2)]|_q =&\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q-\nu(A[\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\
=&\nu(A\star((-\Gamma^3_{(1,2)}+\Gamma^1_{(2,3)})E_1+(\Gamma^3_{(2,3)}+\Gamma^1_{(1,2)})E_3)|_q=:M_1.
\end{aligned}$$

Since $\nu(A\star E_2)|_q, M_1|_q, M_3|_q$ are smooth $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$-vertical vector fields defined on O', we may again resort to Corollary 4.18 to deduce that

$$0=\det\begin{pmatrix} 0 & 1 & 0\\ -(\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)}) & 0 & \Gamma^1_{(2,3)}-\Gamma^3_{(1,2)}\\ -\Gamma^3_{(1,2)}+\Gamma^1_{(2,3)} & 0 & \Gamma^3_{(2,3)}+\Gamma^1_{(1,2)}\end{pmatrix}=-((\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)})^2+(\Gamma^1_{(2,3)}-\Gamma^3_{(1,2)})^2),$$

i.e., $\Gamma^3_{(2,3)}=-\Gamma^1_{(1,2)}$, $\Gamma^3_{(1,2)}=\Gamma^1_{(2,3)}$ on $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$. We will now prove that derivatives of $\Gamma^1_{(2,3)}$ and $\Gamma^1_{(1,2)}$ in the $E_2^{\perp}$-directions vanish on $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$. To reach this we first notice that

$$L_1|_q=\mathscr{L}_{\mathrm{NS}}(E_1)|_q-\nu(A\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))|_q,$$

and then compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(E_1),L_1]|_q =&\mathscr{L}_{\mathrm{NS}}(\Gamma^1_{(1,2)}E_2-\Gamma^1_{(3,1)}E_3)|_q-\mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_1)|_q\\
&+\nu(AR(E_1\wedge E_1)-\hat{R}(AE_1\wedge 0)A)|_q\\
&+\Gamma^1_{(1,2)}\mathscr{L}_{\mathrm{NS}}(AE_2)|_q-\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q\\
&-\nu\big(A\star(\Gamma^1_{(2,3)}(\Gamma^1_{(1,2)}E_2-\Gamma^1_{(3,1)}E_3)+\Gamma^1_{(1,2)}(\Gamma^1_{(3,1)}E_1-\Gamma^1_{(2,3)}E_2))\big)|_q\\
=&\Gamma^1_{(1,2)}\mathscr{L}_{\mathrm{R}}(E_2)|_q-\Gamma^1_{(3,1)}L_3|_q-\mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_1)|_q\\
&-\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q.
\end{aligned}$$

So if one define $J_1|_q:=\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q$, then J_1 is a smooth vector field in O (tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$) and

$$[J_1,\nu((\cdot)\star E_2]|_q=\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_3-E_1(\Gamma^1_{(1,2)})E_1)\big)|_q.$$

Since $\nu(A\star E_1)|_q, J_1|_q$ and $[J_1,\nu((\cdot)\star E_2]|_q$ are $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)}$ vertical vector fields in O and $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ is not open, we again deduce that

$$E_1(\Gamma^1_{(2,3)})=0,\quad E_1(\Gamma^1_{(1,2)})=0.$$

In a similar way,

$$\begin{aligned}
[\mathscr{L}_{\rm R}(E_3),L_3]|_q =&\mathscr{L}_{\rm NS}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2)|_q-\mathscr{L}_{\rm R}(\nabla_{E_3}E_3)|_q\\
&+\nu(AR(E_3\wedge E_3)-\hat{R}(AE_3\wedge 0)A)|_q\\
&+\Gamma^1_{(1,2)}\mathscr{L}_{\rm NS}(AE_2)|_q-\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q\\
&-\nu\big(A\star(-\Gamma^1_{(1,2)}(\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)+\Gamma^1_{(2,3)}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2)\big)|_q\\
=&\Gamma^3_{(3,1)}L_1|_q+\Gamma^1_{(1,2)}\mathscr{L}_{\rm R}(E_2)|_q-\mathscr{L}_{\rm R}(\nabla_{E_3}E_3)|_q\\
&-\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q,
\end{aligned}$$

so $J_3|_q:=\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q$ defines a smooth vector field on O and

$$[J_3,\nu((\cdot)\star E_2)]|_q=\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_3-E_3(\Gamma^1_{(2,3)})E_1)\big)|_q.$$

The same argument as before implies that $E_3(\Gamma^1_{(1,2)})=0$, $E_3(\Gamma^1_{(2,3)})=0$. Since $E_2^\perp$ is spanned by E_1,E_3, the claim follows. This completes the proof. □

We next provide a completary result to Proposition C.18 which will be fundamental for the proof of Theorem 5.3.

Proposition C.20 Let (M,g), $(\hat{M},\hat{g})$ be two Riemannian manifolds of dimension 3, $q_0=(x_0,\hat{x}_0;A_0)\in Q$. Assume that there is an open subset O of $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ and a smooth orthonormal local frame $E_1,E_2,E_3\in{\rm VF}(U)$ defined on the open subset $U:=\pi_{Q,M}(O)$ of M with respect to which the connection table has the form

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},$$

and that moreover

$$V(\Gamma^1_{(2,3)})=0,\quad V(\Gamma^1_{(1,2)})=0,\quad \forall V\in E_2|_y^\perp,\quad y\in U.$$

Define smooth vector fields L_1,L_2,L_3 on the open subset $\tilde{O}:=\pi_{Q,M}^{-1}(U)$ of Q by

$$\begin{aligned}
L_1|_q=&\mathscr{L}_{\rm NS}(E_1)|_q-\nu(A\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))|_q\\
L_2|_q=&\Gamma^1_{(2,3)}(x)\mathscr{L}_{\rm NS}(E_2)|_q\\
L_3|_q=&\mathscr{L}_{\rm NS}(E_3)|_q-\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q.
\end{aligned}$$

Then we have the following:

(i) If $\nu(A\star E_2)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ at every point $q\in O$, then the vectors

$$\mathscr{L}_{\rm R}(E_1)|_q,\ \mathscr{L}_{\rm R}(E_2)|_q,\ \mathscr{L}_{\rm R}(E_3)|_q,\ \nu(A\star E_2)|_q,\ L_1|_q,\ L_2|_q,\ L_3|_q,$$

are all tangent to $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ for every $q\in O$.

(ii) On $\tilde{O}$ we have the following Lie-bracket formulas

$$
\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star E_2)]|_q &= \mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star E_2)]|_q &= 0,\\
[\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star E_2)]|_q &= -\mathscr{L}_{\mathrm{R}}(E_1)|_q + L_1|_q,\\
[L_1,\nu((\cdot)\star E_2)]|_q &= 0,\\
[L_3,\nu((\cdot)\star E_2)]|_q &= 0,\\
[\mathscr{L}_{\mathrm{R}}(E_1),L_1]|_q &= -\Gamma^1_{(3,1)}L_3|_q + \Gamma^1_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_3)|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_3),L_3]|_q &= \Gamma^3_{(3,1)}L_1|_q - \Gamma^3_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_1)|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_2),L_1]|_q &=\Gamma^1_{(1,2)}L_1|_q - (\Gamma^1_{(2,3)}+\Gamma^2_{(3,1)})L_3|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_2),L_3]|_q &=(\Gamma^1_{(2,3)}+\Gamma^2_{(3,1)})L_1|_q + \Gamma^1_{(1,2)}L_3|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_3),L_1]|_q &=2L_2|_q - \Gamma^3_{(3,1)}L_3|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_3)|_q - \Gamma^1_{(2,3)}\mathscr{L}_{\mathrm{R}}(E_2)|_q,\\
&\quad -(K_2+(\Gamma^1_{(2,3)})^2+(\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_1),L_3]|_q &=-2L_2|_q + \Gamma^1_{(3,1)}L_1|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_3}E_1)|_q + \Gamma^1_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_2)|_q,\\
&\quad +(K_2+(\Gamma^1_{(1,2)})^2+(\Gamma^1_{(2,3)})^2)\nu(A\star E_2)|_q,\\
[L_3,L_1]|_q &=2L_2|_q - \Gamma^1_{(3,1)}L_1|_q - \Gamma^3_{(3,1)}L_3|_q,\\
&\quad -(K_2+(\Gamma^1_{(2,3)})^2+(\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q.
\end{aligned}
$$

Proof. It has been already shown in the course of the proof of Proposition C.18 that the vectors $\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,L_1|_q,L_3|_q$ are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for $q\in O$. Moreover, the first 7 brackets appearing in the statement of this corollary are immediately established from the computations done explicitly in the proof of Proposition C.18. We compute,

$$
\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(E_2),L_1]|_q\\
=&-\mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_2)|_q + \mathscr{L}_{\mathrm{NS}}(-\Gamma^2_{(3,1)}E_3)|_q + \nu(AR(E_2\wedge E_1) - \hat{R}(AE_2\wedge 0)A)\\
&+\mathscr{L}_{\mathrm{NS}}(A(\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))E_2)|_q\\
&-\nu\big(A\star(\Gamma^1_{(2,3)}(-\Gamma^2_{(3,1)}E_3)+\Gamma^1_{(1,2)}(\Gamma^2_{(3,1)}E_1))\big)\big|_q\\
&-\nu\big(A\star(E_2(\Gamma^1_{(2,3)})E_1+E_2(\Gamma^1_{(1,2)})E_3)\big)|_q\\
=&-\mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_2)|_q - \Gamma^2_{(3,1)}L_3|_q + K\nu(A\star E_3)|_q + \mathscr{L}_{\mathrm{NS}}(A(\Gamma^1_{(2,3)}E_3-\Gamma^1_{(1,2)}E_1))|_q\\
&-\nu\big(A\star(E_2(\Gamma^1_{(2,3)})E_1+E_2(\Gamma^1_{(1,2)})E_3)\big)|_q\\
=&-\mathscr{L}_{\mathrm{R}}(\nabla_{E_1}E_2)|_q - \Gamma^2_{(3,1)}L_3|_q + \mathscr{L}_{\mathrm{R}}(\Gamma^1_{(2,3)}E_3-\Gamma^1_{(1,2)}E_1)|_q - \Gamma^1_{(2,3)}L_3 + \Gamma^1_{(1,2)}L_1\\
&+(2\Gamma^1_{(2,3)}\Gamma^1_{(1,2)} - E_2(\Gamma^1_{(2,3)}))\nu(A\star E_1)|_q\\
&+(-E_2(\Gamma^1_{(1,2)})+K-(\Gamma^1_{(2,3)})^2+(\Gamma^1_{(1,2)})^2)\nu(A\star E_3)|_q.
\end{aligned}
$$

One knows from Eq. (57) that $-K=-E_2(\Gamma^1_{(1,2)})+(\Gamma^1_{(1,2)})^2-(\Gamma^1_{(2,3)})^2$ and $-E_2(\Gamma^1_{(2,3)})+2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)}=0$ and since also $\nabla_{E_1}E_2=-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3$, this simplifies to

$$[\mathscr{L}_{\mathrm{R}}(E_2),L_1]_q = -\Gamma^2_{(3,1)}L_3|_q - \Gamma^1_{(2,3)}L_3 + \Gamma^1_{(1,2)}L_1.$$

The Lie bracket $[\mathscr{L}_{\mathrm{R}}(E_2),L_3]_q$ can be found by similar computations. We compute

$[\mathscr{L}_{\rm R}(E_3),L_1]|_q$. We have, recalling that $E_i(\Gamma^1_{(2,3)})=0,\ E_i(\Gamma^1_{(2,3)})=0$ for $i=1,3$,

$$\begin{aligned}
&[\mathscr{L}_{\rm R}(E_3),L_1]|_q\\
=&-\mathscr{L}_{\rm R}(\nabla_{E_1}E_3)|_q+\mathscr{L}_{\rm NS}(\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)|_q\\
&+\nu(AR(E_3\wedge E_1)|_q-\hat{R}(AE_3\wedge 0)|_q\\
&+\mathscr{L}_{\rm NS}(A(\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))E_3)|_q\\
&-\nu\big(A\star(\Gamma^1_{(2,3)}(\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)+\Gamma^1_{(1,2)}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2)\big)\big|_q\\
=&-\mathscr{L}_{\rm R}(\nabla_{E_1}E_3)|_q+(-K_2-(\Gamma^1_{(2,3)})^2-(\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q\\
&-\Gamma^3_{(3,1)}L_3|_q-\Gamma^1_{(2,3)}\mathscr{L}_{\rm R}(E_2)|_q+2L_2|_q.
\end{aligned}$$

The computation of $[\mathscr{L}_{\rm R}(E_1),L_3]|_q$ is similar. We compute $[L_3,L_1]$ with the following 4 steps:

$$\begin{aligned}
[\mathscr{L}_{\rm NS}(E_3),\mathscr{L}_{\rm NS}(E_1)]|_q=&\mathscr{L}_{\rm NS}(-\Gamma^1_{(3,1)}E_1+2\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)|_q\\
&+\nu(AR(E_3\wedge E_1)-\hat{R}(0\wedge 0)A)|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\mathscr{L}_{\rm NS}(E_3),\nu\big((\cdot)\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3)\big)\big]|_q\\
=&\nu\big(A\star(\Gamma^1_{(2,3)}(\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)+\Gamma^1_{(1,2)}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2))\big)\big|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\nu\big((\cdot)\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3)\big),\mathscr{L}_{\rm NS}(E_1)\big]|_q\\
=&-\nu\big(A\star(-\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)}E_2-\Gamma^1_{(3,1)}E_3)+\Gamma^1_{(2,3)}(\Gamma^1_{(3,1)}E_1-\Gamma^1_{(2,3)}E_2))\big)\big|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\nu\big((\cdot)\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3)\big),\nu\big((\cdot)\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3)\big)\big]|_q\\
=&\nu\big(A\big[\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3),\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3)\big]_{\mathfrak{so}}\big)\big|_q\\
=&((\Gamma^1_{(1,2)})^2+(\Gamma^1_{(2,3)})^2)\nu(A\star E_2)|_q.
\end{aligned}$$

Collecting these gives,

$$\begin{aligned}
[L_3,L_1]|_q=&-\Gamma^1_{(3,1)}L_1|_q-\Gamma^3_{(3,1)}L_3|_q+2\Gamma^1_{(2,3)}\mathscr{L}_{\rm NS}(E_2)|_q\\
&-(K_2+(\Gamma^1_{(2,3)})^2+(\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q.
\end{aligned}$$

□

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