WEISS-WEINSTEIN BOUND FOR AN ABRUPT UNKNOWN FREQUENCY CHANGE

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ABSTRACT

In this paper, we derive analytical expressions of the Weiss-Weinstein bound (WWB) in the context of observations whose frequency abruptly changes at an unknown time instant. Since both frequencies before and after the change are assumed to be unknown as well, it is appropriate to consider the multiparameter version of the WWB. Furthermore, numerical simulations are provided in order to illustrate the tightness of the proposed bound expressions regarding to the estimates errors.

***Index Terms*—** Weiss-Weinstein bound, change-point, frequency estimation, MSE bayesian lower bound

1. INTRODUCTION

In many practical applications, abrupt changes, refered to as change-points, may occur in the distribution of the signal of interest, e.g., in quality control, in speech processing, as well as in navigation system monitoring [1]. Such problem is also encountered in radar signal processing, and more specifically in SAR image edge detection [2], in clutter map segmentation [3] or in the detection of target signals in clutter [4]. Most often, the time instants at which the changes occur are unknown and must be estimated jointly with the parameters of interest that change from one change-point to another.

In such scenario, change-point estimation has been widely studied in the last five decades. A plethora of algorithms have been proposed, among which the maximum likelihood based schemes received a particular attention [5, 6, 7]. Its optimality properties led to important studies of its asymptotic distribution. First, in [5], Hinlkey derived the exact asymptotic distribution of the maximum likelihood estimate (MLE), and proposed in the same paper a computationally tractable approximation of the latter. Fotopoulos and Jandhyala later worked on deriving exact computable forms of the asymptotic distribution of the MLE, in the cases of exponentially distributed and Gaussian random processes [6, 7]. The knowledge of the distribution of the MLE, which is highly informative, seems unfortunately impossible to obtain in non asymptotic context. At least, as far as we know, such result does not exist in the literature. To overcome this difficulty, an alternative is to study the so-called lower bounds on the variance of the change-point estimation problem. One of the most commonly used lower bound on the mean square error (MSE), is the Cramér-Rao bound (CRB) due to its easy derivation and tightness in the asymptotic region [8, 9]. Nevertheless, the CRB is not tight in the non-asymptotic region and more importantly, it requires strong regularity conditions as the smoothness of the log likelihood function which is not satisfied in our context since the change-point parameter is discrete. A first alternative was to regularize the problem by approximating the signal by a smoother one [10, 11]. A second alternative was to consider a lower bound on the MSE that does not require the smoothness of the log-likelihood function. This has been first done using the Chapman-Robbins bound for a single change-point in [12], and then extended to multiple changes in [13], both with known parameters of interest before and after the change-point(s). The results in both cases are not satisfying since the resulting bounds are not tight. This drawback led us to consider the problem from a Bayesian point of view, for which we recently derived the Weiss-Weinstein bound (WWB) for a single change-point and known parameters of interest [14].

In this paper, we extend [14] to the case of complex observation vector with unknown (changing) parameters of interest. Specifically, we consider a widely used model in the literature, namely the complex sinusoidal signal, where the unknown parameters of interest (frequencies) are suject to a change at an unknown time instant. We chose cisoid model for sake of clarity, nevertheless, it is encountered in several radar area or can be easily extended to specific radar applications involving sophisticated model based on the cisoid. One can cite target detection in abruptly non-stationary Doppler-spread clutter [15, 4]. The aim is to analyze the estimation performance on both the frequencies before and after the change jointly with the time instant at which the change occurs. It is known that the problem of frequency estimation is non-linear and gives rise to a threshold phenomenon under which the estimation performance is highly degraded [16]. Consequently, the use of the WWB is interesting since the breakdown point is also rendered by this bound, unlike the CRB [17]. Finally, this study allows us to have a glance at

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how the estimation behavior is influenced by the joint estimation of these two types of parameters, namely the discrete change-point as well as the continuous frequencies parameters.

2. PROBLEM FORMULATION

Let us consider the complex sinusoidal model for frequency change-point scenario given by

$$\begin{cases} y(n) = \mathrm{e}^{\mathrm{i}2\pi f_1 n} + w(n) & \text{for } n = 1, \dots, \tau \\ y(n) = \mathrm{e}^{\mathrm{i}2\pi f_2 n} + w(n) & \text{for } n = \tau+1, \dots, N \end{cases} \quad (1)$$

where $y(n)$ denotes the observation at the time instant n, f_1 and f_2 are unknown (normalized) frequencies, τ is the unknown change-point, and $w(n)$ is a zero mean, complex, circular, Gaussian, i.i.d. noise with variance σ^2. Model (1) can be seen as a simplified model of radars for target detection in abruptly non-stationary Doppler-spread clutter [15, 4].

The whole observation can be gathered in the vector $\mathbf{y} \triangleq \left(y(1), \dots, y(N)\right)^T \in \Omega$, whose mean is denoted by $\mathbf{s}_\tau(f_1, f_2) \triangleq \left(\mathrm{e}^{\mathrm{i}2\pi f_1}, \dots, \mathrm{e}^{\mathrm{i}2\pi f_1 \tau}, \mathrm{e}^{\mathrm{i}2\pi f_2(\tau+1)}, \dots, \mathrm{e}^{\mathrm{i}2\pi f_2 N}\right)^T$ and the proper noise vector by $\mathbf{w} \triangleq \left(w(1), \dots, w(N)\right)^T \sim \mathbb{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}_N\right)$. Consequently, (1) can be rewritten as

$$\mathbf{y} = \mathbf{s}_\tau(f_1, f_2) + \mathbf{w}. \quad (2)$$

Following a Bayesian approach, we assume a uniform *a priori* for the normalized frequencies, i.e., $f_i \sim \mathcal{U}_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$, $i = 1, 2$, and that τ follows a discrete uniform distribution between 1 and $N-1$, i.e., $\tau \sim \mathcal{U}_{\{1,\dots,N-1\}}$. The unknown parameter vector is denoted by $\boldsymbol{\theta} = (f_1, f_2, \tau)^T \in \Theta$, in which $\Theta = \mathbb{R}^2 \times \mathbb{N}$. Since every component of $\boldsymbol{\theta}$ is independent one from another, the joint prior pdf can be written as

$$p(f_1, f_2, \tau = k) = I_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(f_1)\, I_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(f_2)\, \frac{I_{\{1,\dots,N-1\}}(k)}{N-1} \quad (3)$$

where $I_{\mathcal{A}}(x)$ denotes the indicator function w.r.t. the set $\mathcal{A}$, i.e., it equals 1 if $x \in \mathcal{A}$ and 0 otherwise.

3. WWB DEFINITION AND DERIVATION

3.1. Background on the WWB

The multiparameter form of the WWB [18] satisfies the following matrix inequality for any Bayesian estimator $\widehat{\boldsymbol{\theta}}(\mathbf{y})$ of $\boldsymbol{\theta}$:

$$\mathbb{E}_{\mathbf{y},\boldsymbol{\theta}}\left\{\left[\widehat{\boldsymbol{\theta}}(\mathbf{y}) - \boldsymbol{\theta}\right]\left[\widehat{\boldsymbol{\theta}}(\mathbf{y}) - \boldsymbol{\theta}\right]^T\right\} \succeq \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T \quad (4)$$

in which $\mathbf{A} \succeq \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is a nonnegative matrix, $\mathbf{H} = [\boldsymbol{\theta}_1 - \boldsymbol{\theta}, \boldsymbol{\theta}_2 - \boldsymbol{\theta}, \boldsymbol{\theta}_3 - \boldsymbol{\theta}]$ is a function of the so-called test-points $\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3\}$ which can be chosen by the user as long as each $\boldsymbol{\theta}_q \in \Theta$. We define $\mathbf{h}_q = \boldsymbol{\theta}_q - \boldsymbol{\theta}$ and $\mathbf{H}$ becomes $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$. The 3×3 matrix $\mathbf{G}$ is defined by its (i,j)-th element as

$$G_{i,j} = \frac{\mathbb{E}_{\mathbf{y},\boldsymbol{\theta}}\left\{\begin{array}{l}\left[L^{s_i}(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_i, \boldsymbol{\theta}) - L^{1-s_i}(\mathbf{y}, \boldsymbol{\theta}-\mathbf{h}_i, \boldsymbol{\theta})\right] \\ \times \left[L^{s_j}(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_j, \boldsymbol{\theta}) - L^{1-s_j}(\mathbf{y}, \boldsymbol{\theta}-\mathbf{h}_j, \boldsymbol{\theta})\right]\end{array}\right\}}{\mathbb{E}_{\mathbf{y},\boldsymbol{\theta}}\left\{L^{s_i}(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_i, \boldsymbol{\theta})\right\}\mathbb{E}_{\mathbf{y},\boldsymbol{\theta}}\left\{L^{s_j}(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_j, \boldsymbol{\theta})\right\}} \quad (5)$$

where $L(\mathbf{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{p(\mathbf{y}, \boldsymbol{\alpha})}{p(\mathbf{y}, \boldsymbol{\beta})}$. The inequality (4) holds for all combinations of $\mathbf{h}_q$ and $0 < s_q < 1$ such that $\mathbf{G}$ is invertible. Then, the WWB is obtained by maximizing the right side of (4) w.r.t. the degrees of freedom $\mathbf{h}_q$ and s_q. To reduce the computational cost of this maximization procedure, we first assume one test point per parameter, i.e., $\mathbf{h}_q$ has only one non-zero element h_q, leading to $\mathbf{H} = \mathrm{diag}(h_1, h_2, h_3)$. Second, it often has been noticed numerically that choosing $s_q = \frac{1}{2}\ \forall q$ leads to the maximum bound, thus we make this choice here and perform the maximization only with respect to h_1, h_2 and h_3. Consequently, (5) can be rewritten as

$$G_{i,j} = \frac{\zeta(\mathbf{h}_i, \mathbf{h}_j) + \zeta(-\mathbf{h}_i, -\mathbf{h}_j) - \zeta(\mathbf{h}_i, -\mathbf{h}_j) - \zeta(-\mathbf{h}_i, \mathbf{h}_j)}{\zeta(\mathbf{h}_i, \mathbf{0})\, \zeta(\mathbf{h}_j, \mathbf{0})} \quad (6)$$

where

$$\zeta(\mathbf{h}_i, \mathbf{h}_j) = \int_\Theta \int_\Omega \sqrt{p(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_i)\, p(\mathbf{y}, \boldsymbol{\theta}+\mathbf{h}_j)}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\boldsymbol{\theta} \quad (7)$$

in which the notation $\int_\Theta \cdot\, \mathrm{d}\boldsymbol{\theta}$ stands for $\int_\mathbb{R}\int_\mathbb{R} \sum_\mathbb{N} \cdot\, \mathrm{d}f_1 \mathrm{d}f_2$.

3.2. Derivation of the WWB

3.2.1. Derivation of the diagonal terms $G_{1,1}$ and $G_{2,2}$

We start by deriving $G_{1,1}$, relative to frequency f_1, from which we will deduce $G_{2,2}$ for frequency f_2. First,

$$\zeta(\mathbf{h}_1, \mathbf{0}) = \int_\Theta \left[\int_\Omega \sqrt{p(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1)\, p(\mathbf{y}|\boldsymbol{\theta})}\, \mathrm{d}\mathbf{y}\right] \sqrt{p(\boldsymbol{\theta}+\mathbf{h}_1)\, p(\boldsymbol{\theta})}\, \mathrm{d}\boldsymbol{\theta} \quad (8)$$

where $p(\boldsymbol{\theta})$ is given by (3) and $p(\boldsymbol{\theta}+\mathbf{h}_1)$ is determined accordingly:

$$p(\boldsymbol{\theta}+\mathbf{h}_1) = I_{\left[-\frac{1}{2}-h_1, \frac{1}{2}-h_1\right]}(f_1)\, I_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(f_2)\, \frac{I_{\{1,\dots,N-1\}}(k)}{N-1} \quad (9)$$

Consequently,

$$\sqrt{p(\boldsymbol{\theta}+\mathbf{h}_1)\, p(\boldsymbol{\theta})} = I_{\left[-\frac{1}{2}+(-h_1)^+, \frac{1}{2}-(h_1)^+\right]}(f_1)\, I_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(f_2) \times \frac{I_{\{1,\dots,N-1\}}(k)}{N-1} \quad (10)$$

where for any real parameter, $(x)^+ \triangleq \max(x, 0)$.

On the other hand, from the i.i.d. assumption we have

$$p(\mathbf{y}|\boldsymbol{\theta}) \triangleq p(\mathbf{y}|f_1, f_2, \tau = k) = \prod_{n=1}^{k} p(y(n)|f_1) \prod_{n=k+1}^{N} p(y(n)|f_2) \quad (11)$$

and $p\left(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1\right)$ is modified accordingly. This leads to:

$$\sqrt{p(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1)\,p(\mathbf{y}|\boldsymbol{\theta})} = \frac{1}{(\pi\sigma^2)^N}\exp\left\{-\frac{1}{\sigma^2}(\mathbf{y}-\mathbf{m_s})^H(\mathbf{y}-\mathbf{m_s})\right\} \times \exp\left\{-\frac{\left(N-\mathrm{Re}\left(\mathbf{s}_k(f_1+h_1,f_2)^H\mathbf{s}_k(f_1,f_2)\right)\right)}{2\sigma^2}\right\} \quad (12)$$

where $\mathbf{m_s} = \frac{1}{2}\left(\mathbf{s}_k\left(f_1+h_1,f_2\right)+\mathbf{s}_k\left(f_1,f_2\right)\right)$. Then, the integration over Ω with respect to $\mathbf{y}$ gives:

$$\int_\Omega\sqrt{p\left(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1\right)p\left(\mathbf{y}|\boldsymbol{\theta}\right)}\mathrm{d}\mathbf{y} = \exp\left\{-\frac{\left(N-\mathrm{Re}\left(\mathbf{s}_k(f_1+h_1,f_2)^H\mathbf{s}_k(f_1,f_2)\right)\right)}{2\sigma^2}\right\} \quad (13)$$

More explicitely, since

$$\mathbf{s}_k(f_1+h_1,f_2)^H\mathbf{s}_k(f_1,f_2) = \sum_{n=1}^{k}\mathrm{e}^{-\mathrm{i}2\pi h_1 n}+N-k = \mathrm{e}^{-\mathrm{i}\pi h_1(k+1)}\frac{\sin\left(\pi h_1 k\right)}{\sin\left(\pi h_1\right)}+N-k \quad (14)$$

the left side of (13) does not depend on f_1 and f_2. Finally, by substituting (13) and (10) into (8) and by integrating out f_1 and f_2 and summing over $\mathbb{N}$, we obtain

$$\zeta(\mathbf{h}_1,\mathbf{0}) = \frac{1-|h_1|}{N-1}\sum_{k=1}^{N-1}\exp\left\{-\frac{1}{4\sigma^2}\left(2k+1-\frac{\sin(\pi h_1(2k+1))}{\sin(\pi h_1)}\right)\right\} \quad (15)$$

which concludes the derivation of the denominator terms of (6). For the numerator, first, we notice that

$$\zeta\left(\mathbf{h}_1,\mathbf{h}_1\right) = \zeta\left(-\mathbf{h}_1,-\mathbf{h}_1\right) = \int_\Theta\int_\Omega p\left(\mathbf{y},\boldsymbol{\theta}\pm\mathbf{h}_1\right)\mathrm{d}\mathbf{y}\mathrm{d}\boldsymbol{\theta} = 1. \quad (16)$$

The other terms of the numerator of (6) can be derived as follows:

$$\zeta\left(\mathbf{h}_1,-\mathbf{h}_1\right) = \zeta\left(-\mathbf{h}_1,\mathbf{h}_1\right) = \zeta\left(2\mathbf{h}_1,\mathbf{0}\right) \quad (17)$$

using a proper variable change. Finally, by substituting (16) and (17) into (6), we obtain

$$G_{1,1} = \frac{2\left(1-\zeta\left(2\mathbf{h}_1,\mathbf{0}\right)\right)}{\zeta^2\left(\mathbf{h}_1,\mathbf{0}\right)} \quad (18)$$

where $\zeta\left(\mathbf{h}_1,\mathbf{0}\right)$ and $\zeta\left(2\mathbf{h}_1,\mathbf{0}\right)$ are given according to (15).

Following the same methodology, we obtain

$$G_{2,2} = \frac{2\left(1-\zeta\left(2\mathbf{h}_2,\mathbf{0}\right)\right)}{\zeta^2\left(\mathbf{h}_2,\mathbf{0}\right)} \quad (19)$$

in which

$$\zeta\left(\mathbf{h}_2,\mathbf{0}\right) = \frac{1-|h_2|}{N-1}\sum_{k=1}^{N-1}\exp\left\{-\frac{1}{2\sigma^2}\left(N-k-\cos\left(\pi h_2\left(N+k+1\right)\right)\frac{\sin\left(\pi h_2\left(N-k\right)\right)}{\sin\left(\pi h_2\right)}\right)\right\}. \quad (20)$$

3.2.2. *Derivation of the diagonal term $G_{3,3}$*

First, (10) becomes

$$\sqrt{p\left(\boldsymbol{\theta}+\mathbf{h}_3\right)p\left(\boldsymbol{\theta}\right)} = I_{\left[-\frac{1}{2},\frac{1}{2}\right]}\left(f_1\right)I_{\left[-\frac{1}{2},\frac{1}{2}\right]}\left(f_2\right) \times \frac{I_{\left\{1+(-h_3)^+,\ldots,N-1-(h_3)^+\right\}}(k)}{N-1}. \quad (21)$$

Furthermore, similar manipulations to those in (11), (12), and (14) lead to

$$\int_\Omega\sqrt{p(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_3)\,p(\mathbf{y}|\boldsymbol{\theta})}\mathrm{d}\mathbf{y} = \exp\left\{-\frac{1}{2\sigma^2}\left(|h_3|-\sum_{n=k+1}^{k+|h_3|}\cos\left(2\pi\left(f_2-f_1\right)n\right)\right)\right\} \quad (22)$$

The expression of $\zeta\left(\mathbf{h}_3,\mathbf{0}\right)$ is then obtained by integrating out f_1 and f_2 and summing over $\mathbb{N}$:

$$\zeta\left(\mathbf{h}_3,\mathbf{0}\right) = \frac{\exp\left\{-\frac{|h_3|}{2\sigma^2}\right\}}{(N-1)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{k=1}^{N-1-|h_3|}\exp\left\{\frac{1}{2\sigma^2}\times\sum_{n=k+1}^{k+|h_3|}\cos\left(2\pi\left(f_2-f_1\right)n\right)\right\}\mathrm{d}f_1\mathrm{d}f_2 \quad (23)$$

which can be efficiently numerically evaluated [19]. Finally, in the same manner as in (18) and (19), we obtain

$$G_{3,3} = \frac{2\left(1-\zeta\left(2\mathbf{h}_3,\mathbf{0}\right)\right)}{\zeta^2\left(\mathbf{h}_3,\mathbf{0}\right)} \quad (24)$$

whose explicit expression is deduced from (23).

3.2.3. *Cross-terms derivation*

Similar considerations to those in subsection 3.2.1 lead to

$$\zeta\left(\mathbf{h}_1,\mathbf{h}_2\right) = \frac{\left(1-|h_1|\right)\left(1-|h_2|\right)}{N-1} \times\sum_{k=1}^{N-1}\exp\left\{-\frac{1}{2\sigma^2}\left(N-\cos\left(\pi h_1\left(k+1\right)\right)\frac{\sin(\pi h_1 k)}{\sin(\pi h_1)}-\cos\left(\pi h_2\left(N+k+1\right)\right)\frac{\sin\left(\pi h_2\left(N-k\right)\right)}{\sin\left(\pi h_2\right)}\right)\right\} \quad (25)$$

which directly implies, by (6), that $G_{1,2}=G_{2,1}=0$.

So far, two terms remain to be derived: $G_{1,3}$ and $G_{2,3}$. They are actually very similar, so details are given only for the former. The denominator terms of $G_{1,3}$ are given by (15) and (23). We then derive $\zeta\left(\mathbf{h}_1,\mathbf{h}_3\right)$ and we obtain

$$\zeta\left(\mathbf{h}_1,\mathbf{h}_3\right) = \int_\Theta\left[\int_\Omega\sqrt{p\left(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1-\mathbf{h}_3\right)p\left(\mathbf{y}|\boldsymbol{\theta}\right)}\mathrm{d}\mathbf{y}\right] \times\sqrt{p\left(\boldsymbol{\theta}+\mathbf{h}_1-\mathbf{h}_3\right)p\left(\boldsymbol{\theta}\right)}\mathrm{d}\boldsymbol{\theta} \quad (26)$$

and similarly as in (10) and (21), we have

$$\sqrt{p\left(\boldsymbol{\theta}+\mathbf{h}_1-\mathbf{h}_3\right)p\left(\boldsymbol{\theta}\right)} = I_{\left[-\frac{1}{2}+(-h_1)^+,\frac{1}{2}-(h_1)^+\right]}\left(f_1\right) \times I_{\left[-\frac{1}{2},\frac{1}{2}\right]}\left(f_2\right)\frac{I_{\left\{1+(h_3)^+,\ldots,N-1-(-h_3)^+\right\}}(k)}{N-1}. \quad (27)$$

Finally, since

$$\int_{\Omega}\sqrt{p\left(\mathbf{y}|\boldsymbol{\theta}+\mathbf{h}_1-\mathbf{h}_3\right)p\left(\mathbf{y}|\boldsymbol{\theta}\right)}\mathrm{d}\mathbf{y} = \exp\left\{-\frac{1}{2\sigma^2}\left[k+|h_3|-\sum_{n=1}^{k}\cos\left(2\pi h_1 n\right)-\sum_{n=k+1}^{k+|h_3|}\cos\left(2\pi\left(f_2-f_1-h_1 I_{\mathbb{R}_-^*}\left(h_3\right)\right)n\right)\right]\right\} \quad (28)$$

where $\mathbb{R}_-^* \triangleq]-\infty,0[$, we are now able to write the expression of $\zeta\left(\mathbf{h}_1,\mathbf{h}_3\right)$ by substituting (27) and (28) into (26), which leads to

$$\zeta\left(\mathbf{h}_1,\mathbf{h}_3\right)=\frac{1}{N-1}\exp\left\{-\frac{|h_3|}{2\sigma^2}\right\}\times \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}+(-h_1)^+}^{\frac{1}{2}-(h_1)^+}\sum_{k=1}^{N-1-|h_3|}\exp\left\{-\frac{1}{2\sigma^2}\left[k-\sum_{n=1}^{k}\cos\left(2\pi h_1 n\right)-\sum_{n=k+1}^{k+|h_3|}\cos\left(2\pi\left(f_2-f_1-h_1 I_{\mathbb{R}_-^*}\left(h_3\right)\right)n\right)\right]\right\}\mathrm{d}f_1\mathrm{d}f_2 \quad (29)$$

and from which the expressions of $\zeta\left(-\mathbf{h}_1,-\mathbf{h}_3\right)$, $\zeta\left(\mathbf{h}_1,-\mathbf{h}_3\right)$, and $\zeta\left(-\mathbf{h}_1,\mathbf{h}_3\right)$ are obtained. Then one can deduce $G_{1,3}$ from (6). Notice that generally, $G_{1,3}\neq 0$.

Finally, the expression of $\zeta\left(\mathbf{h}_2,\mathbf{h}_3\right)$ is obtained in the same way as that of $\zeta\left(\mathbf{h}_1,\mathbf{h}_3\right)$ as

$$\zeta\left(\mathbf{h}_2,\mathbf{h}_3\right)=\frac{1}{N-1}\exp\left\{-\frac{N}{2\sigma^2}\right\}\times \int_{-\frac{1}{2}+(-h_2)^+}^{\frac{1}{2}-(h_2)^+}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{k=1}^{N-1-|h_3|}\exp\left\{\frac{1}{2\sigma^2}\left[k+\sum_{n=k+|h_3|+1}^{N}\cos\left(2\pi h_2 n\right)+\sum_{n=k+1}^{k+|h_3|}\cos\left(2\pi\left(f_2+h_2 I_{\mathbb{R}_+^*}\left(h_3\right)-f_1\right)n\right)\right]\right\}\mathrm{d}f_1\mathrm{d}f_2 \quad (30)$$

where $\mathbb{R}_+^* \triangleq]0,+\infty[$.

As for $G_{1,3}$, the expression of $G_{2,3}$ arises from (20), (23) and (30), that one properly substitutes into (6).

4. NUMERICAL RESULTS

In this section, we present the simulation results comparing the empirical global mean square errors (GMSE) of the maximum *a posteriori* (MAP) estimator of (f_1, f_2, τ) with the WWB derived above, for model (2) with $N = 20$. Notice that the maximization with respect to the h_q, $q = 1,2,3$ is performed by maximizing the trace of $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T$.

In Figs. 1 and 2, we perform 1000 Monte-Carlo simulations. These figures reveals that, as expected, the WWB in good agreement with the behavior of the MAP estimator and remains reasonably tight. Indeed, Fig. 1 which represents the GMSE of the MAP and WWB for the change-point τ shows a gap of approximately 1 sample at low and high SNR.

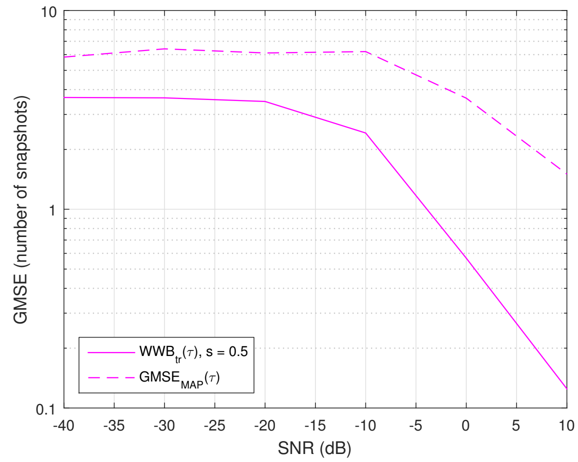

Fig. 1. GMSE of the MAP estimator (dashed line) and WWB (solid line) for the estimation of the change-point τ (in magenta), with $N = 20$ snapshots.

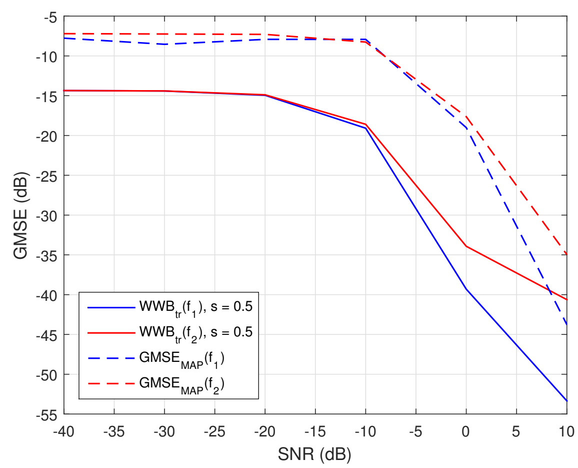

Fig. 2. GMSE of the MAP estimator (dashed lines) and WWB (solid lines) for the estimation of the frequencies f_1 (in blue) and f_2 (in red), with $N = 20$ snapshots.

5. CONCLUSION

In this communication, we derived the Weiss-Weinstein bound (WWB) in the case of an abruptly changing frequency. Numerical simulations reveal that the WWB with respect to the frequencies and the change-point remains reasonably tight regarding to the global mean square error of the maximum a posteriori estimator.

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