Inverse Problems: From Regularization Theory to Probabilistic and Bayesian Inference

Ali Mohammad-Djafari
Laboratoire des Signaux et Systèmes,
UMR8506 CNRS-SUPELEC-UNIV PARIS SUD 11
SUPELEC, 91192 Gif-sur-Yvette, France
http://lss.supelec.free.fr

Email: djafari@lss.supelec.fr
http://djafari.free.fr

Ali Mohammad-Djafari: Parcours en quelques mots

- 1975: Arrivé d'Iran avec un diplôme d'Ing.
- 1977: Diplôme de Supélec
- 1981: Docteur-Ing.
- 1988: Docteur d'état

- 1981-1982: Ing. Société Equipments Scientifiques
- 1982-1984: Assistant associé Univ Paris Sud
- 1984: CR CNRS
- 1997: Visiting Associate Professor, Univ of Notre Dame, Indiana, USA
- 1998: DR CNRS

- Plus de 50 articles, 200 communications, ...
- 20 thèses déj soutenues
- 4 thèses en cours
- Plus de 20 contrats, EDF, ONERA, Thales, ANR,
- Plusieurs organisation du congrès: SPIE, WIO, 5 **MaxEnt**, ...

Regularization and Bayesian approaches

- Invese problems: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$
- Regularization: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$$

- Bayesian

$$p(\boldsymbol{f}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})$$

- Infer on $\boldsymbol{f}$
  - MAP $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} \longrightarrow$ Optimization
  - EAP $\widehat{\boldsymbol{f}} = \int \boldsymbol{f}\, p(\boldsymbol{f}|\boldsymbol{g})\, \mathrm{d}\boldsymbol{f} \longrightarrow$ Integration
  - Full exploration using MCMC
  - Approximations (Laplace,VBA)

Bayesian approaches

$$p(\boldsymbol{f}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})$$

- Choix pour $p(\boldsymbol{g}|\boldsymbol{f}) = p_\epsilon(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$
  - Gaussian
  - Poisson
  - Generalized Gaussian
  - Cauchy or Student
- Choix pour $p(\boldsymbol{f})$
  - Gaussian / Gauss-Markov
  - Generalized Gaussian / GGM
  - Cauchy or Student
  - Gauss-Markov-Potts

Unsupervised Bayesian

Supervised:

$$p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f},\boldsymbol{\theta}_2), \quad \boldsymbol{\theta} = (\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$$

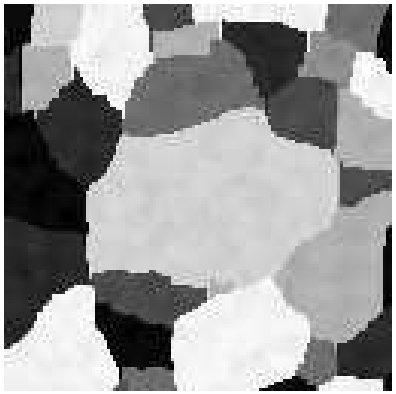
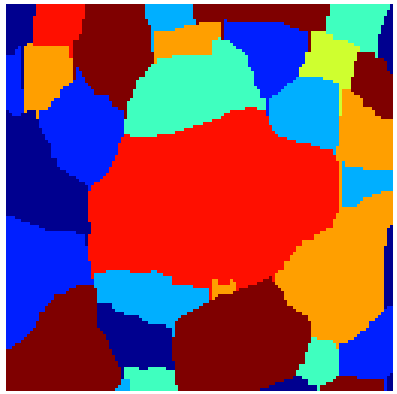
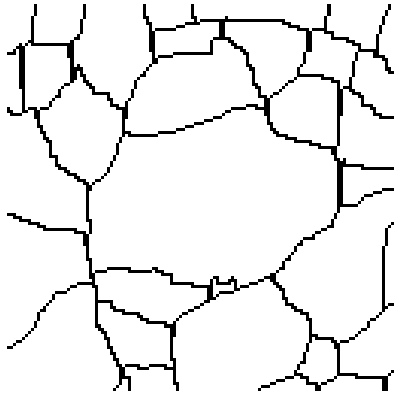
Unsupervised:

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f},\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

Infer on $\boldsymbol{f}$ and $\boldsymbol{\theta}$

- JMAP $(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\widehat{\boldsymbol{\theta}})} \{p(\boldsymbol{f}|\boldsymbol{g})\} \longrightarrow$ Alternate Optimization
- Full exploration using MCMC, Gibbs sampling
- Variational Bayesian Approximations (VBA)
  - Approximate $p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})$ by $q(\boldsymbol{f},\boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta})$
  - Use $KL(q|p) = \int q \ln q/p$ as a measure of proximity and obtain $q_1(\boldsymbol{f})$ and $q_2(\boldsymbol{\theta})$
  - Use them to infer $\boldsymbol{f}$ and $\boldsymbol{\theta}$

Gauss-Markov-Potts prior models for images

$f(\boldsymbol{r})$ $z(\boldsymbol{r})$ $c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables: $p(\boldsymbol{z})$ Potts-Markov:

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Four different cases

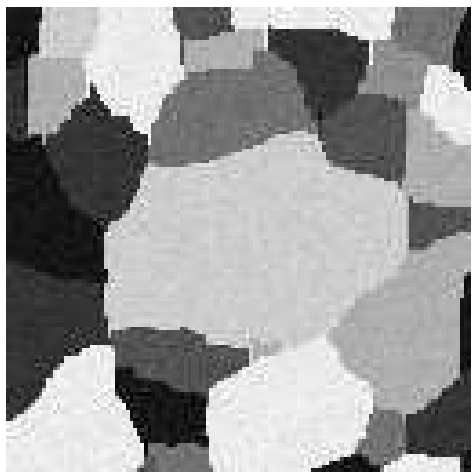
To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid,
- $\boldsymbol{z}$ iid :
  Mixture of Gaussians

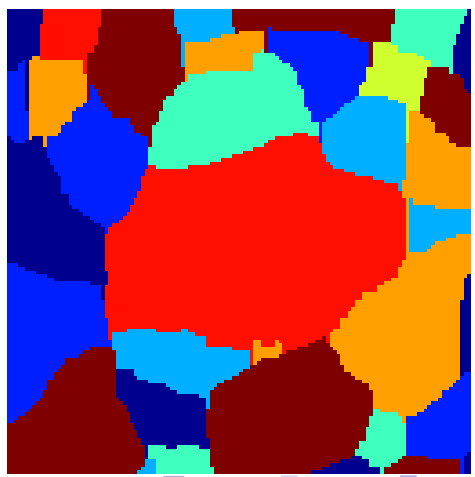
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid,
- $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)

- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)

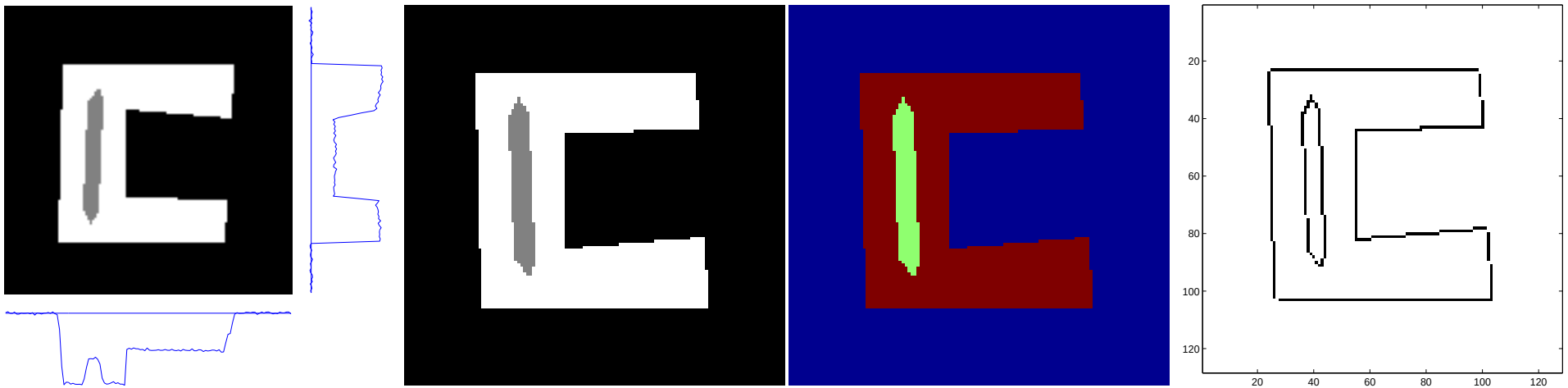


$f(\boldsymbol{r})$



$z(\boldsymbol{r})$

Application in CT



$\boldsymbol{g}\|\boldsymbol{f}$	$\boldsymbol{f}\|\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g}=\boldsymbol{H}\boldsymbol{f}+\boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\boldsymbol{r})\in\{0,1\}$
$\boldsymbol{g}\|\boldsymbol{f}\sim\mathcal{N}(\boldsymbol{H}\boldsymbol{f},\sigma_\epsilon^2\boldsymbol{I})$	or	or	$1-\delta(z(\boldsymbol{r})-z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})\propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\,p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta}_2)\,p(\boldsymbol{z}|\boldsymbol{\theta}_3)\,p(\boldsymbol{\theta})$$

JMAP, MCMC, VBA

Supervised PhD

1. **Denis Prémel (1992), CEA**
2. **Mila Nikolova (1994), DR CNRS**
3. **Stphane Gautier (1996), EDF**
4. **Hervé Carfantan (1998), Prof. Toulouse**
5. **Guillaume Montmont (2000), CEA LETI**
6. **Charles Soussen (2000) CRAN**
7. **Hichem Snoussi (2003), Professor at Technical Univ. of Troys UTT**
8. **Patrice Brault (2005), L2S**
9. **Mahieddine Ichir (2005), Industry**
10. **Fabrice Humblot (2005),Industry**
11. **Olivier Féron (2006), Research Dept. of EDF, France**

Supervised PhD

1. **Nadia Bali (2007), Associate Professor in Tunis**
2. **Sofia FEKIH-SALEM (2009), Industry**
3. **Hacheme AYASSO (2010), MC Grenoble**
4. **Doriano-Boris POUGAZA (2011), Research South Africa**
5. **Diarra FALL (2012) (E. Bara, CEA, C. Compta, SFHJ), MC Orléan**
6. **Sha ZHU (2012), (Cooperation with Changsha NUDT, China), Assist. Prof. China**
7. **Caifang CAI (2013) (Co-supervisers: Th. Rodet (Associate Professor, University of Paris & S. Le Goupil (CEA-LIST), 2013), Post-doc ENS Cachan**
8. **Rmi PRENON (2013) (Co-superviser: Pierre Grangeat, CEA-LETI, Grenoble), Post-doc, Industry**
9. **Thomas BOULAY (2013) (Co-Supervisers: N. Gac and J. Lagoutte, Thales Air System), Post-doc L2S, Projet SAFRAN**
10. **Ning CHU (2013) (Co-supervisers: N. Gac & J. Picheral), Post-doc EPFL**

Current PhD's and projects

PhD's:

1. **Microwave imaging: PhD Leila Gharsalli (co-supervising B. Duchne)**
2. **Multivariate and multicomponents biological data processing: PhD Mircea Dumitru (co-supervising F. Lévi), ERASYSBIO**
3. **ANR: HONTOMIN, PhD Safa AlAli, (CO2 stock supervising using electrical imaging) (B. Duchne & G. Perruson)**
4. **New methods for reducing dose in Computed Tomography, PhD Li Wang (N. Gac)**
5. **Information fusion for radar target recognition, starting PhD, May Abou Chahine, Thales Systèmes Aéroports**

Post-docs

1. **ANR: SURMITO (Optical imaging), S. Mehrab, (B. Duchne)**
2. **3D Tomography (SAFRAN), Th. Boulay, (N. Gac)**