Patterned Complex-Valued Matrix Derivatives

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Notation

- Scalars: Lowercase non-bold letters, function f, variable z
- Vectors: Lowercase bold letters, function $\boldsymbol{f}$, variable $\boldsymbol{z}$
- Matrices: Uppercase bold letters, function $\boldsymbol{F}$, variable $\boldsymbol{Z}$

In engineering, functions are often real-valued.
If $f \in \mathbb{R}$, $\boldsymbol{f} \in \mathbb{R}^{M \times 1}$, and $\boldsymbol{F} \in \mathbb{R}^{M \times P}$, then they have the forms:

$$f(\boldsymbol{Z}, \boldsymbol{Z}^*), \quad \boldsymbol{f}(\boldsymbol{Z}, \boldsymbol{Z}^*), \quad \text{or} \quad \boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*).$$

Function type	Matrix variables $\boldsymbol{Z}, \boldsymbol{Z}^* \in \mathbb{C}^{N \times Q}$
Scalar function $f \in \mathbb{C}$	$f(\boldsymbol{Z}, \boldsymbol{Z}^*)$ $f : \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \to \mathbb{C}$
Vector function $\boldsymbol{f} \in \mathbb{C}^{M \times 1}$	$\boldsymbol{f}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ $\boldsymbol{f} : \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \to \mathbb{C}^{M \times 1}$
Matrix function $\boldsymbol{F} \in \mathbb{C}^{M \times P}$	$\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ $\boldsymbol{F} : \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \to \mathbb{C}^{M \times P}$

Motivation

Common problems in signal processing for communications:

- Unitary precoder design: Minimize SER/PEP/capacity with respect to (wrt.) a unitary precoder $\boldsymbol{F}$
- Minimize/maximize $f(\boldsymbol{W}, \boldsymbol{W}^*) \in \mathbb{R}$ wrt. the patterned matrix $\boldsymbol{W} \in \mathbb{C}^{N \times Q}$
- The connection between Information theory and Estimation theory goes through derivatives and sometimes the derivative wrt. an autocorrelation matrix $\boldsymbol{Q}$ which is positive semi-definite is needed

Problem: How to find the derivatives of $\boldsymbol{G}(\boldsymbol{W}, \boldsymbol{W}^*)$, where $\boldsymbol{G} : \mathbb{C}^{M \times P} \times \mathbb{C}^{M \times P} \to \mathbb{C}^{R \times S}$, when the matrices $\boldsymbol{W}$ and $\boldsymbol{W}^*$ contain pattern (or no pattern)?

Existing Works (Among Others)

- Theory developed for finding derivatives wrt. *real-valued unpatterned* matrices [Magnus & Neudecker 1988]
- Theory for finding derivatives wrt. *unpatterned* complex-valued vectors [Brandwood 1983]
- Systematic and simple way to find derivatives wrt. *unpatterned* complex-valued matrices [Hjørungnes & Gesbert 2007a]
- Theory for finding derivatives of real-valued functions which depends on patterned real-valued matrices [Tracy & Jinadasa 1988]
- Theory for finding derivatives of functions which depends on complex-valued matrices [Hjørungnes & Palomar 2008a]
- Increased insight into the way of finding derivatives of functions which depends on *patterned* complex-valued matrices and connections to manifolds [Hjørungnes & Palomar 2008b]

$\mathrm{vec}(\cdot)$ Operator

Definition

Let $\boldsymbol{A} \in \mathbb{C}^{M \times N}$ *and denote the* i*th column of* $\boldsymbol{A}$ *by* $\boldsymbol{a}_i$*, where* $i \in \{0, 1, \ldots, N-1\}$*, such that* $\boldsymbol{A} = [\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}]$*. Then the* $\mathrm{vec}(\cdot)$ *operator is defined as the* $MN \times 1$ *vector given by:*

$$\mathrm{vec}\left(\boldsymbol{A}\right) = \begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{N-1} \end{bmatrix}.$$

Independence of $\mathcal{Z}$ and $\mathcal{Z}^*$

In $z = x + \jmath y$, the variables x and y are independent such that

$$\frac{\partial}{\partial x} y = 0, \quad \frac{\partial}{\partial y} x = 0$$

This can be translated into [Brandwood 1983]:

$$\frac{\partial}{\partial z^*} z = 0, \quad \frac{\partial}{\partial z} z^* = 0$$

$\Rightarrow z$ and z^* are independent

This can again be generalized to: $\mathcal{Z}$ and $\mathcal{Z}^*$ are independent

Unpatterned Case: Definition of Derivative of $\boldsymbol{F} \in \mathbb{C}^{M \times P}$

- Let $\boldsymbol{F} : \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \rightarrow \mathbb{C}^{M \times P}$. Assume that the elements of $\boldsymbol{Z}$ are linear independent (unpatterned).
- Derivative of $\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ wrt. $\boldsymbol{Z}$ is denoted $\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F}$.
- Derivative of $\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ wrt. $\boldsymbol{Z}^*$ is denoted $\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F}$.
- $\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F} \in \mathbb{C}^{MP \times NQ}$ and $\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F} \in \mathbb{C}^{MP \times NQ}$ are defined by:

$$d \operatorname{vec}(\boldsymbol{F}) = (\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F}) \, d \operatorname{vec}(\boldsymbol{Z}) + (\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F}) \, d \operatorname{vec}(\boldsymbol{Z}^*).$$

- $\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ and $\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*)$ are also called the *Jacobian* matrices of $\boldsymbol{F}$.

Chain Rule

Let $(\mathcal{S}_0, \mathcal{S}_1) \subseteq \mathbb{C}^{N\times Q} \times \mathbb{C}^{N\times Q}$, and let $\boldsymbol{F} : \mathcal{S}_0 \times \mathcal{S}_1 \to \mathbb{C}^{M\times P}$ be differentiable wrt. both its first and second argument at an interior point $(\boldsymbol{Z}, \boldsymbol{Z}^*)$ in the set $\mathcal{S}_0 \times \mathcal{S}_1$. Let $\mathcal{T}_0 \times \mathcal{T}_1 \subseteq \mathbb{C}^{M\times P} \times \mathbb{C}^{M\times P}$ be such that $(\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*), \boldsymbol{F}^*(\boldsymbol{Z}, \boldsymbol{Z}^*)) \in \mathcal{T}_0 \times \mathcal{T}_1$ for all $(\boldsymbol{Z}, \boldsymbol{Z}^*) \in \mathcal{S}_0 \times \mathcal{S}_1$. Assume that $\boldsymbol{G} : \mathcal{T}_0 \times \mathcal{T}_1 \to \mathbb{C}^{R\times S}$ is differentiable at an interior point $(\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*), \boldsymbol{F}^*(\boldsymbol{Z}, \boldsymbol{Z}^*)) \in \mathcal{T}_0 \times \mathcal{T}_1$. Define the composite function $\boldsymbol{H} : \mathcal{S}_0 \times \mathcal{S}_1 \to \mathbb{C}^{R\times S}$ by $\boldsymbol{H}(\boldsymbol{Z}, \boldsymbol{Z}^*) \triangleq \boldsymbol{G}(\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*), \boldsymbol{F}^*(\boldsymbol{Z}, \boldsymbol{Z}^*))$. The derivatives $\mathcal{D}_{\boldsymbol{Z}}\boldsymbol{H}$ and $\mathcal{D}_{\boldsymbol{Z}^*}\boldsymbol{H}$ are obtained through:

$$\mathcal{D}_{\boldsymbol{Z}}\boldsymbol{H} = (\mathcal{D}_{\boldsymbol{F}}\boldsymbol{G})\,\mathcal{D}_{\boldsymbol{Z}}\boldsymbol{F} + (\mathcal{D}_{\boldsymbol{F}^*}\boldsymbol{G})\,\mathcal{D}_{\boldsymbol{Z}}\boldsymbol{F}^*,$$

$$\mathcal{D}_{\boldsymbol{Z}^*}\boldsymbol{H} = (\mathcal{D}_{\boldsymbol{F}}\boldsymbol{G})\,\mathcal{D}_{\boldsymbol{Z}^*}\boldsymbol{F} + (\mathcal{D}_{\boldsymbol{F}^*}\boldsymbol{G})\,\mathcal{D}_{\boldsymbol{Z}^*}\boldsymbol{F}^*.$$

Chain rule can be used to find complicated derivatives

Definitions from the Theory of Manifolds

Definition (Smooth Function)

A function is called *smooth* if it has continuous partial derivatives of all orders with respect to all its input variables.

Definition (Diffeomorphism)

A smooth bijective function is called a *diffeomorphism* if the inverse function is also smooth.

Definition (Manifold)

Let $\mathcal{X}$ be a subset of a big ambient Euclidean space $\mathbb{R}^{N\times 1}$. Then $\mathcal{X}$ is a k-dimensional manifold, if it is locally diffeomorphic to $\mathbb{R}^{k\times 1}$, where $k \leq N$.

Patterned Matrix

Definition (Patterned Matrix Set)

A set of matrices $\mathcal{W} \subseteq \mathbb{C}^{M \times P}$ is said to be a *patterned matrix set* if there exist a *diffeomorphic function* $\boldsymbol{F} : \mathbb{R}^{K \times L} \times \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \to \mathcal{W}$, where $\mathcal{W} \subseteq \mathbb{C}^{M \times P}$, i.e., if $\mathcal{W}$ is a manifold.
The function $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$ is then named the *pattern producing function*.

Examples of Patterned Matrices

Example (Symmetric Matrix)

Let $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$ be a symmetric matrix such that $\boldsymbol{Z}^T = \boldsymbol{Z}$. This is manifold of $\frac{N(N+1)}{2}$ complex dimensions.

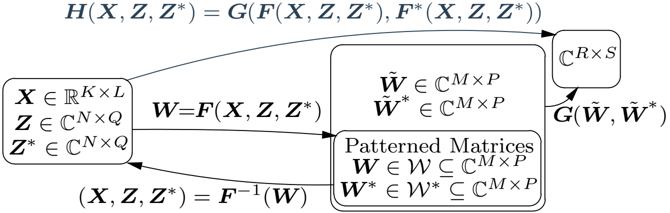
Example (Hermitian Matrix)

Let $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$ be a Hermitian matrix parameterized by

- N *real-valued* diagonal elements
- $\frac{(N-1)N}{2}$ complex-valued parameters strictly below the main diagonal
- Above diagonal: The complex conjugated of the elements strictly below the main diagonal

Manifold of $N + \frac{N(N-1)}{2} \times 2 = N^2$ real dimensions.

Problem Setup Including Function Definitions



- Let $\boldsymbol{W} \in \mathcal{W}$ be a patterned matrix within the set of patterned matrices $\mathcal{W}$, and let $\tilde{\boldsymbol{W}} \in \mathbb{C}^{M\times P}$ be unpatterned
- Let $\boldsymbol{F}$ be a function producing all the patterned matrices within $\mathcal{W}$ with independent input variables $\boldsymbol{X}$, $\boldsymbol{Z}$, and $\boldsymbol{Z}^*$
- Define $\boldsymbol{H}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*) \triangleq \boldsymbol{G}(\boldsymbol{W},\boldsymbol{W}^*) = \boldsymbol{G}(\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*),\boldsymbol{F}^*(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*))$. We propose a method for how to derivate $\boldsymbol{H}$ wrt. $\boldsymbol{X}$, $\boldsymbol{Z}$, and $\boldsymbol{Z}^*$
- By using manifolds, we can also find $\mathcal{D}_{\boldsymbol{W}}\boldsymbol{G}$ of size $RS \times (KL + 2NQ)$

Pattern Producing Function $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$

The pattern producing function $\boldsymbol{F} : \mathbb{R}^{K \times L} \times \mathbb{C}^{N \times Q} \times \mathbb{C}^{N \times Q} \to \mathcal{W}$ must be a **diffeomorphism**:

- $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$ must be a bijection [onto & one-to-one]
- There should not be any redundancy in the number of input variables of $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$ such that $\dim_{\mathbb{R}}\{\mathcal{W}\} = KL + 2NQ$
- $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$ is smooth such that it is differentiable wrt. $\boldsymbol{X}$, $\boldsymbol{Z}$, and $\boldsymbol{Z}^*$
- The inverse function $\boldsymbol{F}^{-1}(\boldsymbol{W})$ is smooth
- The size of $\mathcal{D}_{\boldsymbol{W}} \boldsymbol{F}^{-1}$ and $\mathcal{D}_{\boldsymbol{W}^*} \boldsymbol{F}^{-1}$ is $(KL + 2NQ) \times (KL + 2NQ)$

Derivative of $\boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*) = \boldsymbol{G}(\boldsymbol{W}, \boldsymbol{W}^*)$

Using the chain rule on:
$\boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*) = \boldsymbol{G}(\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*), \boldsymbol{F}^*(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*))$ leads to :

$$\mathcal{D}_{\boldsymbol{X}} \boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*) = \left(\mathcal{D}_{\tilde{\boldsymbol{W}}} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{X}} \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$$
$$+ \left(\mathcal{D}_{\tilde{\boldsymbol{W}}^*} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{X}} \boldsymbol{F}^*(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*),$$
$$\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*) = \left(\mathcal{D}_{\tilde{\boldsymbol{W}}} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$$
$$+ \left(\mathcal{D}_{\tilde{\boldsymbol{W}}^*} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F}^*(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*),$$
$$\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*) = \left(\mathcal{D}_{\tilde{\boldsymbol{W}}} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*)$$
$$+ \left(\mathcal{D}_{\tilde{\boldsymbol{W}}^*} \boldsymbol{G}(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}^*)|_{\tilde{\boldsymbol{W}}=\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{Z}^*)} \right) \mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F}^*(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}^*).$$

Derivative of $\boldsymbol{G}(\boldsymbol{W}, \boldsymbol{W}^*)$

- Must first identify $\mathcal{D}_{\boldsymbol{W}} \boldsymbol{F}^{-1}$ and $\mathcal{D}_{\boldsymbol{W}^*} \boldsymbol{F}^{-1}$ and it is necessary to choose a basis for the tangent spaces
- The size of $\mathcal{D}_{\boldsymbol{W}} \boldsymbol{F}^{-1}$ and $\mathcal{D}_{\boldsymbol{W}^*} \boldsymbol{F}^{-1}$ are each $(KL + 2QN) \times (KL + 2NQ)$
- Several examples shown in [Hjørungnes & Palomar 2008b] for how to chose the basis such that $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are the identity map
- The chain rule is used:

$$\mathcal{D}_{\boldsymbol{W}} \boldsymbol{G} = \left[\mathcal{D}_{\boldsymbol{X}} \boldsymbol{H}, \mathcal{D}_{\boldsymbol{Z}} \boldsymbol{H}, \mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{H} \right] \mathcal{D}_{\boldsymbol{W}} \boldsymbol{F}^{-1},$$
$$\mathcal{D}_{\boldsymbol{W}^*} \boldsymbol{G} = \left[\mathcal{D}_{\boldsymbol{X}} \boldsymbol{H}, \mathcal{D}_{\boldsymbol{Z}} \boldsymbol{H}, \mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{H} \right] \mathcal{D}_{\boldsymbol{W}^*} \boldsymbol{F}^{-1}$$

Example: Matrix Function for Unpatterned $\boldsymbol{Z}$

Let $\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}^*) = \boldsymbol{Z}^{-1} \in \mathbb{C}^{N \times N}$, then

$$d\boldsymbol{F} = d\boldsymbol{Z}^{-1} = -\boldsymbol{Z}^{-1}(d\boldsymbol{Z})\boldsymbol{Z}^{-1}$$

Using the $\operatorname{vec}(\cdot)$ operator leads to:

$$d \operatorname{vec} \boldsymbol{F} = -\left(\boldsymbol{Z}^{-T} \otimes \boldsymbol{Z}^{-1}\right) d \operatorname{vec}(\boldsymbol{Z})$$

where

$$\operatorname{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}) = \left(\boldsymbol{C}^T \otimes \boldsymbol{A}\right) \operatorname{vec}(\boldsymbol{B})$$

was utilized. Identifying the derivatives:

$$\mathcal{D}_{\boldsymbol{Z}} \boldsymbol{F} = -\boldsymbol{Z}^{-T} \otimes \boldsymbol{Z}^{-1},$$
$$\mathcal{D}_{\boldsymbol{Z}^*} \boldsymbol{F} = \boldsymbol{0}_{N^2 \times N^2}$$

Useful Matrices $\boldsymbol{L}_d$, $\boldsymbol{L}_l$, and $\boldsymbol{L}_u$

Definition

If $\boldsymbol{A} \in \mathbb{C}^{N\times N}$, then where exist unique constant matrices $\boldsymbol{L}_d$, $\boldsymbol{L}_l$, and $\boldsymbol{L}_u$ such that

$$\operatorname{vec}(\boldsymbol{A}) = \boldsymbol{L}_d \operatorname{vec}_d(\boldsymbol{A}) + \boldsymbol{L}_l \operatorname{vec}_l(\boldsymbol{A}) + \boldsymbol{L}_u \operatorname{vec}_u(\boldsymbol{A}),$$

where $\operatorname{vec}_d(\cdot)$, $\operatorname{vec}_l(\cdot)$, and $\operatorname{vec}_u(\cdot)$ return the diagonal elements, strictly below diagonal, and strictly above diagonal elements, respectively.

- $\boldsymbol{L}_d$ has size $N^2 \times N$.
- $\boldsymbol{L}_l$ has size $N^2 \times \frac{N(N-1)}{2}$.
- $\boldsymbol{L}_u$ has size $N^2 \times \frac{N(N-1)}{2}$.
- $[\boldsymbol{L}_d, \boldsymbol{L}_l, \boldsymbol{L}_u]$ has size $N^2 \times N^2$ is a permutation matrix

Symmetrical Matrices: $\boldsymbol{W}^T = \boldsymbol{W}$

For symmetrical $\boldsymbol{W} \in \mathbb{R}^{N \times N}$, the following pattern producing function $\boldsymbol{F} : \mathbb{R}^{N \times 1} \times \mathbb{R}^{\frac{N(N-1)}{2} \times 1} \to \mathbb{R}^{N \times N}$ denoted $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{W}$ can be used

$$\operatorname{vec}(\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})) = \operatorname{vec}(\boldsymbol{W}) = \boldsymbol{L}_d \boldsymbol{x} + (\boldsymbol{L}_l + \boldsymbol{L}_u) \boldsymbol{y},$$

where

- $\boldsymbol{x} = \operatorname{vec}_d(\boldsymbol{W}) \in \mathbb{R}^{N \times 1}$ contains the main diagonal elements
- $\boldsymbol{y} = \operatorname{vec}_l(\boldsymbol{W}) = \operatorname{vec}_u(\boldsymbol{W}) \in \mathbb{R}^{\frac{N(N-1)}{2} \times 1}$ contains the elements strictly below and also strictly above the main diagonal

Hermitian and Skew Hermitian Patterned Matrices

- Hermitian: Let $\boldsymbol{W}^H = \boldsymbol{H}$. Then

$$\operatorname{vec}(\boldsymbol{W}) = \operatorname{vec}(\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}^*)) = \boldsymbol{L}_d \boldsymbol{x} + \boldsymbol{L}_l \boldsymbol{z} + \boldsymbol{L}_u \boldsymbol{z}^*,$$

  where $\boldsymbol{x} \in \mathbb{R}^{N \times 1}$, $\boldsymbol{z} \in \mathbb{C}^{\frac{N(N-1)}{2} \times 1}$, while $\boldsymbol{L}_d$, $\boldsymbol{L}_l$, and $\boldsymbol{L}_u$ take care of the main diagonal, below main diagonal, and above main diagonal elements, respectively.
- Skew Hermitian: Let $\boldsymbol{W}^H = -\boldsymbol{W} \in \mathbb{C}^{N \times N}$, then

$$\operatorname{vec}(\boldsymbol{W}) = \operatorname{vec}(\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}^*)) = \boldsymbol{L}_d \jmath \boldsymbol{x} + \boldsymbol{L}_l \boldsymbol{z} - \boldsymbol{L}_u \boldsymbol{z}^*,$$

  where $\boldsymbol{x} \in \mathbb{R}^{N \times 1}$, $\boldsymbol{z} \in \mathbb{C}^{\frac{N(N-1)}{2} \times 1}$.

Autocorrelation Matrix

Let $\boldsymbol{W} \in \mathbb{C}^{N \times N}$ be an autocorrelation matrix. Then

- $\boldsymbol{W}$ is Hermitian
- $\boldsymbol{W}$ is positive semidefinite

Pattern producing function found by Cholesky decomposition:

$$\boldsymbol{W} = \boldsymbol{L}\boldsymbol{L}^H,$$

where $\boldsymbol{L} \in \mathbb{C}^{N \times N}$ is a lower triangular matrix with real elements on its main diagonal:

$$\operatorname{vec}(\boldsymbol{L}) = \boldsymbol{L}_d \boldsymbol{x} + \boldsymbol{L}_l \boldsymbol{z}$$

where $\boldsymbol{x} \in \mathbb{R}^{N \times 1}$ and $\boldsymbol{z} \in \mathbb{C}^{\frac{N(N-1)}{2} \times 1}$

Conclusions

- Overview give of the theory for finding derivatives of functions which depend on unpatterned and patterned complex-valued matrices proposed
- Connections to complex-valued manifolds were made to increase the insight of the theory
- Applicable in many areas of telecommunications where a complex-valued matrix should be designed
- More information:
  - Email: `arehj@unik.no`
  - Home-page: `www.unik.no/~arehj`
  - Papers available on request

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