Joint Multiple Cell-Site Processing in Wyner-like Fading Models

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Introduction

- The main focus of information theoretic efforts is to provide ultimate performance bounds on different cellular models, and to identify efficient communications strategies coming close to these bounds

- Communications strategies: Concepts of
  - Modulation/Coding
  - Channel-Accessing/Scheduling
  - Protocols (cross-layer)
  - Cooperation/Processing

- *Joint processing* of signals related to different users is evidently the most appealing approach in either the uplink or downlink

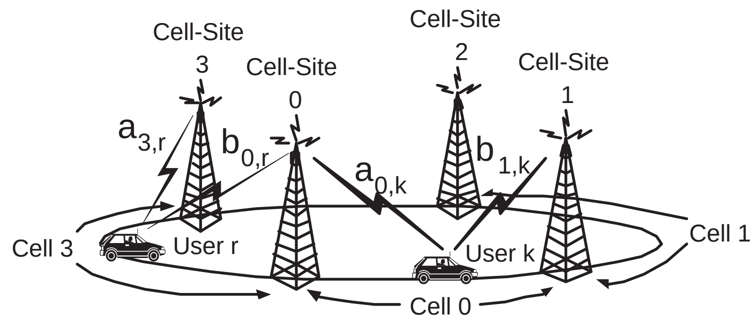
Introduction *(Cont'd)*

- The *uplink* channel is a *multiple access* channel (MAC):
  - *Joint processing of the received signals* at the cell-sites
  - Commonly no user cooperation is assumed
  - User cooperation is also studied lately [Shamai-Somekh-Simeone-Sanderovich-Zaidel-Poor JWCC '07] (and references therein).
- The *downlink* channel is a *broadcast* channel (BC):
  - *Transmitter based joint preprocessing* (complexity considerations)
  - Commonly *single-user detection* at the mobile receiver is assumed
  - User cooperation is also of interest [Shamai-Somekh-Simeone-Sanderovich-Zaidel-Poor JWCC '07] (and references therein).
- The MAC/BC duality principle in different frameworks provides the firm information theoretic connections between these seemingly basically different models [Vishwanath-Jindal-Goldsmith '03], [Viswanath-Tse '03], [Jindal-Vishwanath-Goldsmith '04], [Yu-Lan '04]

Introduction *(Cont'd)*

- Examples for multi-cell uplink analyses:
  - "Older references": [Hanly-Whiting '93], [Wyner '94], [Shamai-Wyner '97], [Somekh-Shamai '00], [Zaidel-Shamai-Verdú '01], [Zaidel-Shamai-Verdú '02], [Somekh-Zaidel-Shamai '04], [Grant-Hanly-Evans-Muller '04], [Aktas-Bacha-Evans-Hanly '04]
  - Recent references: See reference list in [Shamai-Somekh-Zaidel JWCC '04], [Shamai-Somekh-Simeone-Sanderovich-Zaidel-Poor JWCC '07], [Somekh-Zaidel-Shamai IT Dec. '07], [Somekh-Simeone-Bar-Ness-Haimovich-Shamai '07].
- Examples for multi-cell downlink analyses:
  - "Older references": [Shamai-Zaidel '01], [Jafar-Goldsmith '02], [Jafar-Foschini-Goldsmith '04], [Dai-Molisch-Poor '04], [Zhang-Dai-Zhuo-2005], [Foschini-et-al-2005].
  - Recent references: See reference list in [Shamai-Somekh-Zaidel JWCC '04], [Shamai-Somekh-Simeone-Sanderovich-Zaidel-Poor JWCC '07], [Somekh-Zaidel-Shamai IT Dec. '07], [Somekh-Simeone-Bar-Ness-Haimovich-Shamai '07].

General System Model



- A "Wyner-type" multi-cell model with M cells ordered on a *circle*
- Motivation: symmetry properties, more amenable to analytical analysis, equivalent to linear models for $M \gg 1$
- A fully synchronous, optimally coded system is assumed, with cell-sites located at the cells' *boundaries*
- There are K users in each cell, and a single receive/transmit antenna at each cell-site
- Each user "sees" only the two nearest cell-sites
- Models a practical "soft-handoff" scenario at the cells' boundaries

Outline

- Joint multi-cell processing in the uplink is analyzed focusing on the average per-cell sum-rate capacity and spectral efficiency.
- Analytical expressions for the sum-rate capacities under various non-fading/fading conditions are derived for:
  - Intra-cell TDMA scheduling
  - A “Wide-Band” (WB) scheme (all and many $(K \gg 1)$ users are active simultaneously utilizing all bandwidth for coding)
  - **Asymptotic results for K users in the fading regime.**
  - **Full characterization of the randomly activated users scenario in term of per user and outage/ergodic performance.**
- For the downlink channel, assuming *individual per-cell power constraints*:
  - Sum-rate capacity is derived for non-fading channels.
  - Flat-fading channels are analyzed through upper and lower bounds.
  - The impact of uplink scheduling is demonstrated.
- Outlook and concluding remarks.

Uplink System Model

- The discrete-time vector representation of the signals received at the M cell-sites is given, for an arbitrary time index, by

$$\boldsymbol{y}_{ul[M\times 1]} = \boldsymbol{H}_M \boldsymbol{x}_{ul} + \boldsymbol{z}_{ul}$$

  where

  - $\boldsymbol{H}_{M[M\times KM]}$ - Channel transfer matrix
  - $\boldsymbol{x}_{ul[KM\times 1]} \sim \mathcal{N}_c(\mathbf{0}, P\boldsymbol{I}_{KM})$ - Transmitted symbols vector
  - $\boldsymbol{z}_{ul[M\times 1]} \sim \mathcal{N}_c(\mathbf{0}, \boldsymbol{I}_M)$ - Circularly symmetric complex AWGN vector

- The total intra-cell transmit power is denoted by $\bar{P} \triangleq KP$

Uplink System Model *(Cont'd)*

- The $M \times KM$ channel transfer matrix $\boldsymbol{H}_M$ is given by:

$$\boldsymbol{H}_M = \begin{pmatrix} \boldsymbol{a}_0 & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{b}_0 \\ \boldsymbol{b}_1 & \boldsymbol{a}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b}_2 & \boldsymbol{a}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{b}_{M-1} & \boldsymbol{a}_{M-1} \end{pmatrix}$$

- $\{\boldsymbol{a}_{m\,[1\times K]},\ \boldsymbol{b}_{m\,[1\times K]}\}$ - complex channel fades of the K users of the mth and $[(m-1) \bmod M]$th cells, respectively, received at the mth cell-site
- i.i.d. fading among different users (ergodic with respect to the time index)
- The moments and kurtosis of the fading distribution are denoted by

$$m_1 \triangleq E\{a_{m,k}\} = E\{b_{m,k}\} \quad ; \quad m_2 \triangleq E\{|a_{m,k}|^2\} = E\{|b_{m,k}|^2\} \quad ;$$
$$m_4 \triangleq E\{|a_{m,k}|^4\} = E\{|b_{m,k}|^4\} \quad ; \quad \mathcal{K} \triangleq m_4/m_2^2 \quad \forall m, k$$

Uplink Average Per-Cell Sum-Rate Capacity

- Underlying assumptions:
  - Full channel state information is available to the joint multi-cell receiver only
  - Users cannot cooperate their transmissions in any way
- The ergodic average per-cell sum-rate capacity of the uplink channel is given by

$$C_{\mathrm{ul}}(\bar{P}) = \frac{1}{M} E_{H_M} \left\{ \log\det \left(\boldsymbol{I}_M + \frac{\bar{P}}{K} \boldsymbol{H}_M \boldsymbol{H}_M^{\dagger} \right) \right\}$$

- The corresponding spectral efficiency [bits/sec/Hz] is obtained through the relation

$$\bar{P} = \mathsf{C}_{\mathrm{ul}} \left(\frac{E_b^t}{N_0} \right) \frac{E_b^t}{N_0}$$

where E_b^t/N_0 is the transmit E_b/N_0 and $C_{\mathrm{ul}}(\bar{P}) = \mathsf{C}_{\mathrm{ul}}(E_b^t/N_0)$

Uplink Average Per-Cell Sum-Rate Capacity *(Cont'd)*

- The uplink average per-cell sum-rate capacity is governed by the $M \times M$ matrix $\frac{1}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger$, whose non-zero entries are explicitly expressed by

$$\begin{aligned} \left[\frac{1}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger\right]_{m,m} &= \frac{1}{K}\sum_{k=1}^{K}\left|a_{m,k}\right|^2 + \frac{1}{K}\sum_{k=1}^{K}\left|b_{m,k}\right|^2 \\ \left[\frac{1}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger\right]_{m,\widehat{m-1}} &= \frac{1}{K}\sum_{k=1}^{K}a^*_{\widehat{m-1},k}b_{m,k} \qquad , \\ \left[\frac{1}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger\right]_{m,\widehat{m+1}} &= \frac{1}{K}\sum_{k=1}^{K}a_{m,k}b^*_{\widehat{m+1},k} \end{aligned}$$

where

$$\widehat{m+1} \triangleq (m+1) \bmod M \quad ; \quad \widehat{m-1} \triangleq (m-1) \bmod M$$

Extreme SNR Characterization

- The low-SNR regime is characterized through:

$$\mathsf{C}_{ul}(E_b^t/N_0) \approx \mathcal{S}_0/3|_{\text{dB}} \left(E_b^t/N_0|_{\text{dB}} - E_b^t/N_{0\,\min}|_{\text{dB}}\right)$$

  - $E_b^t/N_{0\,\min} \triangleq \log_e 2/\dot{C}_{ul}(0)$ (Scalar AWGN Ch.: $E_b^t/N_{0\,\min} = \log_e 2$)
  - $\mathcal{S}_0$ (slope) $\triangleq -2\left[\dot{C}_{ul}(0)\right]^2/\ddot{C}_{ul}(0)$ (Scalar AWGN Ch.: $\mathcal{S}_0 = 2$)

  ($3|_{\text{dB}} = 10\log_{10} 2$, $\dot{C}_{ul}(0)$ are $\ddot{C}_{ul}(0)$ the derivative of the capacity in nats/dimension at $\bar{P} = 0$)

- The high-SNR regime is characterized through the affine capacity approximation:

$$C_{ul}(\bar{P}) \underset{\bar{P}\gg 1}{\approx} \mathcal{S}_\infty \left(\frac{\bar{P}|_{\text{dB}}}{3|_{\text{dB}}} - \mathcal{L}_\infty\right) = \frac{\mathcal{S}_\infty}{3|_{\text{dB}}}\left(\bar{P}|_{\text{dB}} - 3|_{\text{dB}}\mathcal{L}_\infty\right)$$

  - $\mathcal{S}_\infty$ (multiplexing gain) $\triangleq \lim_{\bar{P}\to\infty} \bar{P}\dot{C}_{\text{ul}}(\bar{P})$ (Scalar AWGN Ch.: $\mathcal{S}_\infty = 1$)

    ($\dot{C}_{ul}(\bar{P})$ is the derivative of the capacity in nats/dimension)
  - $\mathcal{L}_\infty$ (power offset) $\triangleq \lim_{\bar{P}\to\infty}(\log_2 \bar{P} - C_{ul}(\bar{P})/\mathcal{S}_\infty)$ ($C_{ul}(\bar{P})$ in bits/dimension)

    (Scalar AWGN Ch.: $\mathcal{L}_\infty = 0$)

Uplink - No Fading

- In the absence of fading $a_{m,k} = b_{m,k} = 1, \forall m, k$
- The matrix $\frac{\bar{P}}{K}\boldsymbol{H}_M\boldsymbol{H}_M^{\dagger}$ becomes a circulant matrix with non-zero row entries $\left(\bar{P},\ 2\bar{P},\ \bar{P}\right)$
- The average per-cell sum-rate capacity depends only on the total intra-cell transmit power $\bar{P}$:
  - Intra-cell TDMA - Single active user per cell, transmitting $1/K$ fraction of the time, signaling with power $\bar{P} = KP$
  - Wide-Band (WB) - All K users are simultaneously active, occupying the whole bandwidth, signaling with power P
- **Proposition 1** *The average per-cell sum-rate capacity, in the absence of fading, is given for $M \rightarrow \infty$ by*

$$C_{\textit{ul-nf}}(\bar{P}) = \log\left(\frac{1 + 2\bar{P} + \sqrt{1 + 4\bar{P}}}{2}\right)$$

Uplink - No Fading *(Cont'd)*

- **Proposition 2** *The uplink channel extreme-SNR regimes for non-fading channels are characterized for* $M \to \infty$ *by*

$$\frac{E_b^t}{N_0}_{\min} = \frac{\log_e 2}{2};$$
$$S_0 = \frac{4}{3};$$
$$S_\infty = 1;$$
$$\mathcal{L}_\infty = 0.$$

- Extension to $a_{m,k} = 1$, $b_{m,k} = \alpha$, $\forall m, k$ is straight forward [Lifand-Goldsmith '05]

$$C_{ul-nf}(\bar{P}) = \log\left(\frac{1 + (1+\alpha^2)\bar{P} + \sqrt{1 + 2(1+\alpha^2)\bar{P} + (1-\alpha^2)^2\bar{P}^2}}{2}\right).$$

Uplink - Rayleigh Fading - Intra-Cell TDMA

- The focus is on the large number of cells regime ($M \to \infty$)
- The setup is equivalent to a two tap time varying ISI channel [Narula '97]
- **Proposition 3** *The limiting average per-cell sum-rate capacity with intra-cell TDMA scheduling, as $M \to \infty$, is given by*

$$C_{\textit{tdma}}(\bar{P}) = \int_1^\infty \log x \frac{\log_e(x) e^{-\frac{x}{\bar{P}}}}{\mathrm{Ei}\left(\frac{1}{\bar{P}}\right)\bar{P}}\, dx \;,$$

where $\mathrm{Ei}(x) = \int_1^\infty \frac{\exp(-xt)}{t} dt$ *is the exponential integral function.*

- Following [Narula '97] it can be shown that $C_{\text{tdma}}(\bar{P}) \le C_{\text{ul-nf}}(\bar{P})$
- **Proposition 4** *The uplink channel extreme-SNR regimes for Rayleigh-fading channels and intra-cell TDMA scheduling are characterized by*

$$\mathcal{S}_0 = 1 \quad ; \quad E_b^t/N_{0\,\min} = \frac{\log_e 2}{2}$$
$$\mathcal{S}_\infty = 1 \quad ; \quad \mathcal{L}_\infty \approx 0.84 \,.$$

Uplink - Continuous Fading - Intra-Cell TDMA

- We assume an absolute continuity condition on the fading
- **Proposition 5** *The limiting average per-cell sum-rate capacity with intra-cell TDMA scheduling, as* $M \to \infty$ *exists, we denote by* $C_{\text{tdma}}\left(\bar{P}\right)$ *its value. Moreover,*

$$\max\left(E\{\log(1+\bar{P}\,|a_m|^2)\}, E\{\log(1+\bar{P}\,|b_m|^2)\}\right) \leq C_{\text{tdma}}\left(\bar{P}\right)$$
$$\leq E\{\log(1+\bar{P}(|a_m|^2+|b_m|^2))\}.$$

- **Proposition 6** *The uplink channel high-SNR regime for continuous-fading channels and intra-cell TDMA scheduling is characterized by*

$$\mathcal{S}_\infty = 1 \quad ; \quad \mathcal{L}_\infty = -2\max\left(E\{\log|a_m|\}, E\{\log|b_m|\}\right)\ .$$

- The lower bound of Proposition 5 is tight in the high-SNR regime.
- The hypothesis of i.i.d. fading among different users and absolute continuity of the fading can be relaxed.

Proof of Proposition 6

- The techniques: certain Large Random Hermitian Jacobi Matrices.
  - Usual techniques in Large Random Matrices use the large number of random coefficients (quadratic in size of the matrix)
  - Here, the number of random coefficients is linear in the size of the matrix.
  - $H_M H_M^\dagger$ is similar to a random Schrödinger operator. We use the Thouless formula that was "rediscovered" in [Narula '97].
- The Thouless formula:
  - The eigenvalues of $H_M H_M^\dagger$ are related to its eigenvectors.
  - Let $(x_1, \ldots, x_M)$ be an eigenvector, $1/M \det\left(I_M + \bar{P} H_M H_M^\dagger\right)$ is related to its exponential growth, that is $1/M \log|x_M|$.
  - We give a proof similar to [Narula '97].

Proof of Proposition 6 *(Cont'd)*

- Define $G_M = I_M + \bar{P} H_M H_M^\dagger$.
- [Narula '97]: to compute $\det G_M$, write $G_M = L_M U_M$, where L_M is lower triangular and U_M is upper triangular.
- We use the following decomposition: $G_M U_M = L_M$.
- Denote

$$G_M = \begin{pmatrix} d_1 & c_2^\dagger & 0 & 0 \\ c_2 & d_2 & c_3^\dagger & 0 \\ 0 & \ddots & \ddots & c_M^\dagger \\ 0 & 0 & c_M & d_M \end{pmatrix}$$

Proof of Proposition 6 *(Cont'd)*

The decomposition $G_M U_M = L_M$ is as follows:

$$\begin{pmatrix} d_1 & c_2^\dagger & 0 & 0 \\ c_2 & d_2 & c_3^\dagger & 0 \\ 0 & \ddots & \ddots & c_M^\dagger \\ 0 & 0 & c_M & d_M \end{pmatrix} \begin{pmatrix} x_1 & x_1 & \cdots & x_1 \\ 0 & x_2 & \cdots & x_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_M \end{pmatrix} = \begin{pmatrix} -c_2^\dagger x_2 & 0 & \cdots & 0 \\ \ddots & -c_3^\dagger x_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & -c_{M+1}^\dagger x_{M+1} \end{pmatrix}.$$

- $c_i = \bar{P} a_i b_{i+1}^\dagger, \quad d_i = 1 + \bar{P}\left(|a_i|^2 + |b_i|^2\right).$
- $x_0 = 0, \quad x_1 = 1$
- $c_i x_{i-1} + d_i x_i + c_{i+1}^\dagger x_{i+1} = 0.$
- x_i is governed by a random second-order linear recurrence.
- $\frac{1}{M} \log|\det G_M| = \frac{1}{M} \sum_{i=2}^{M+1} \log|c_i| + \frac{1}{M} \log|x_{M+1}|.$
- It is enough to study the sequence x_i.

Proof of Proposition 6 *(Cont'd)*

Study of a Markov chain:

- The study of the sequence with random second-order linear recurrence is equivalent to the study of a Markov chain.
- That Markov chain is closely related to the one studied in [Narula '97].
- Since the fading is absolutely continuous, we can use the theory of Harris chains, which are Markov chains in a continuous state-space with adequate regularity properties.

Proof of Proposition 6 *(Cont'd)*

An alternative method, the Lyapunov exponents:

- The classical way to deal with random second-order linear recurrence is product of two-by-two random matrices:

$$\begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c_i/c_{i+1} & -d_i/c_{i+1} \end{pmatrix} \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}.$$

- The relevant quantity is the Lyapunov exponent of the two-by-two random matrices:

$$\lim_{M\to\infty} \frac{1}{M} \log\det \left[\begin{pmatrix} 0 & 1 \\ -c_M/c_{M+1} & -d_M/c_{M+1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -c_1/c_2 & -d_1/c_2 \end{pmatrix} \right],$$

  which exists by ergodic sub-additivity.

- This method can be generalized to more general Wyner-type models with larger interference range.

Uplink - Continuous Fading - WB: Finite $K > 1$

- Proposition 5 is verified in the WB scheme.
- **Proposition 7** *The uplink channel high-SNR regime for continuous-fading channels in the WB scheme is characterized by*

$$\mathcal{S}_\infty = 1 \quad ; \quad \mathcal{L}_\infty = -E\left\{\log\left(\frac{e + |\boldsymbol{b}_m|^2}{K}\right)\right\} \; ,$$

where the random variables e and $\boldsymbol{b}_m$ are independent, and the distribution of e is the unique invariant probability of the Markov chain defined by:

$$e_{n+1} = |\boldsymbol{a}_n|^2 \left(\frac{e_n + |\boldsymbol{b}_{n-1}|^2 \sin^2(\boldsymbol{a}_n, \boldsymbol{b}_{n-1})}{e_n + |\boldsymbol{b}_{n-1}|^2}\right) \; .$$

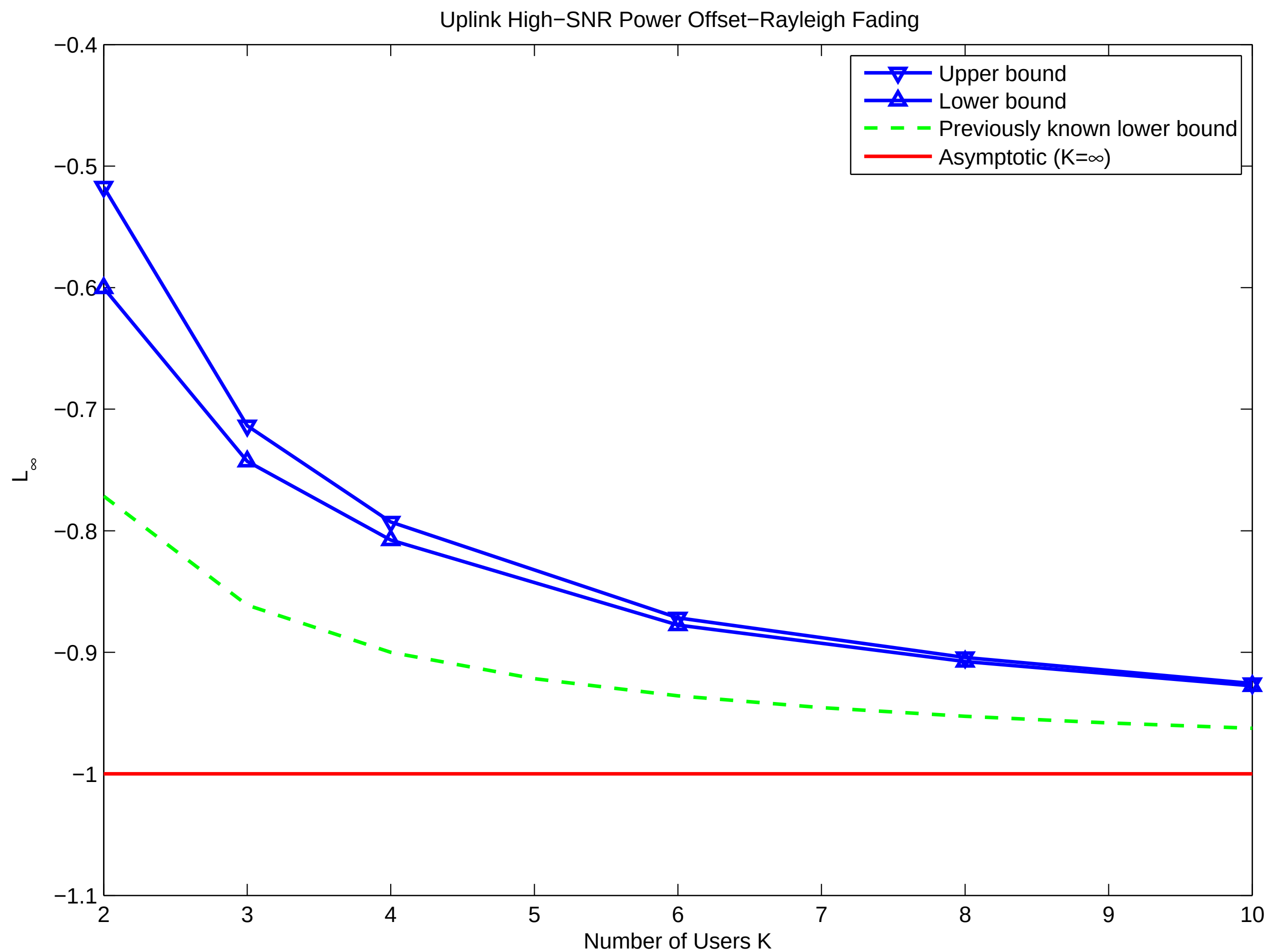
Uplink - Continuous Fading - WB: Finite $K > 1$ *(Cont'd)*

- The proof gives bounds on $\mathcal{L}_\infty$ tighter than previously known bounds of [Liang-Goldsmith '06].
- For $n \in \mathcal{N}$,

$$-E\left\{\log\left(\frac{e_n(\infty)+|\boldsymbol{b}_m|^2}{K}\right)\right\} \leq \mathcal{L}_\infty \leq -E\left\{\log\left(\frac{e_n(0)+|\boldsymbol{b}_m|^2}{K}\right)\right\},$$

  where $e_n(a)$ denotes the Markov chain after n steps starting from a.
- With $n = 2$ the bounds are already very tight (see following graph).

Uplink High-SNR Power Offset - Rayleigh Fading



- The impact of K on the High-SNR Power Offset in Rayleigh fading channels. Bounds computed with $n = 2$.

Uplink - WB - Fading $(K \gg 1)$

- The focus is on the large number of users per cell limit, $K \gg 1$, while keeping $\bar{P} = KP$ constant
- Applying the SLLN, the $M \times M$ matrix $\frac{\bar{P}}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger$ becomes a circulant matrix with non-zero row entries $\left(|m_1|^2\,\bar{P},\ 2m_2\bar{P},\ |m_1|^2\,\bar{P}\right)$
- **Proposition 8** *The average per-cell sum-rate capacity of the WB scheme ($K \gg 1$), in the presence of fading, is given for $M \to \infty$ by*

$$C_{\textit{ul-wb}}(\bar{P}) = \log\left(\frac{1 + 2\bar{P}m_2 + \sqrt{1 + 4\bar{P}m_2 + 4\bar{P}^2(m_2^2 - |m_1|^4)}}{2}\right)$$

- The result upper bounds the corresponding sum-rate capacity for any finite K [Somekh-Shamai '00]
- $m_1 = 0$ is optimum, and assuming $m_2 = 1$ (e.g., Rayleigh fading):

$$C_{\text{ul-wb}}(\bar{P}) = \log(1 + 2\bar{P})$$

Uplink - WB - Fading *(Cont'd)*

- **Proposition 9** *The uplink average per-cell sum-rate capacity in the low-SNR regime with the WB scheme is characterized* $\forall K > 0$, *and* $\forall M \geq 3$, *by*

$$\frac{E_b^t}{N_{0\,\min}} = \frac{\log_e 2}{2m_2} \quad ; \quad \mathcal{S}_0 = \frac{2}{\frac{\mathcal{K}}{2K} + \frac{|m_1|^4}{2m_2^2} + 1} \, .$$

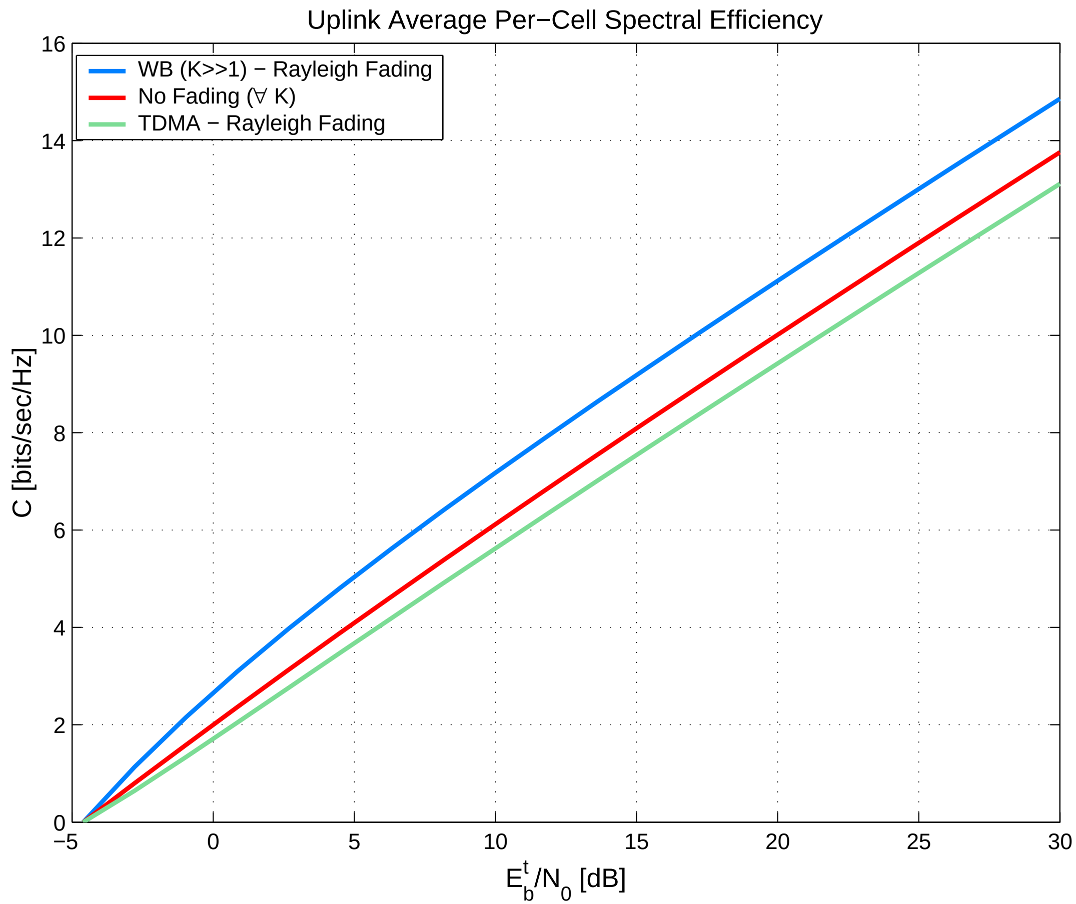
- **Proposition 10** *The uplink average per-cell sum-rate capacity in the high-SNR regime with the WB scheme is characterized for* $K \gg 1$, *and* $M \to \infty$, *by*

$$\mathcal{S}_\infty = 1; \quad \mathcal{L}_\infty \quad = -\log_2 \left(m_2 + \sqrt{m_2^2 - |m_1|^4} \right) \, .$$

- **Proposition 11** *For Rayleigh fading,* $K \gg 1$, *and* $\forall M \geq 3$, *the uplink average per-cell sum-rate capacity in extreme-SNR regimes with the WB scheme is characterized by*

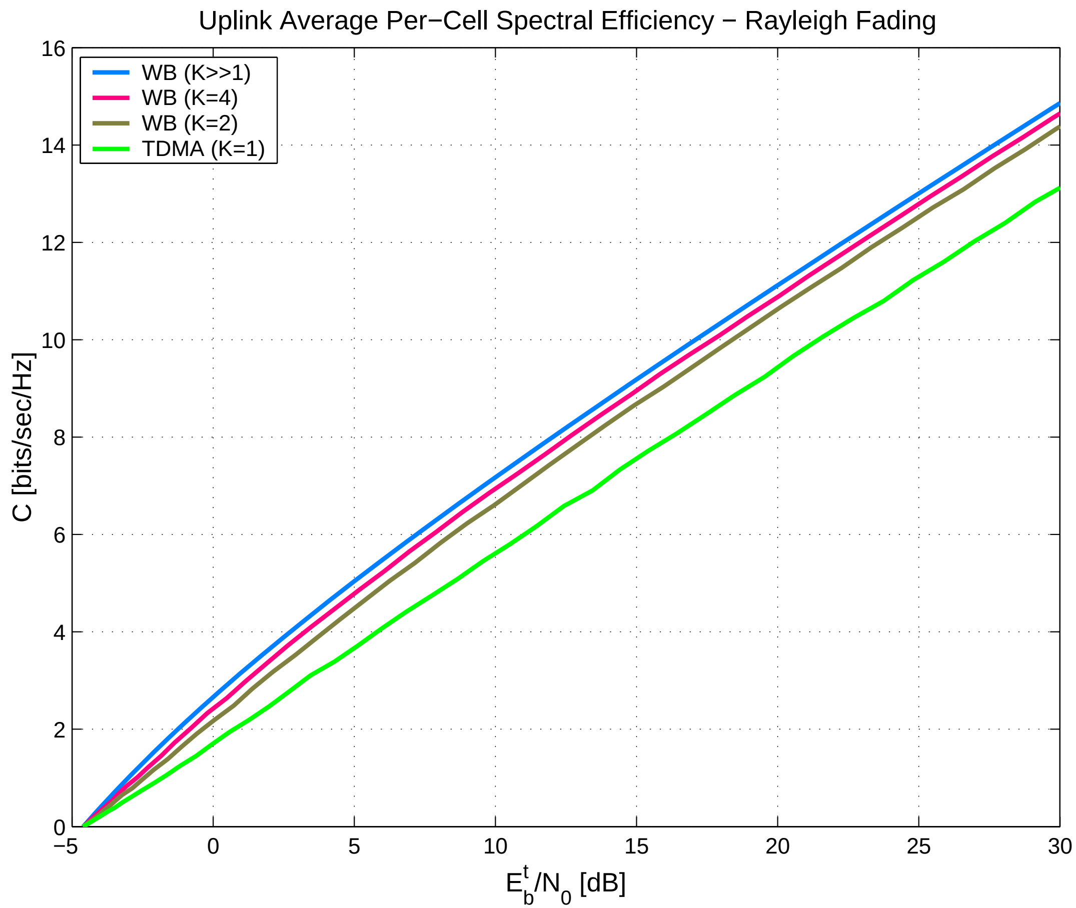
$$E_b^t / N_{0_{min}} = \log_e 2/2 \quad ; \quad \mathcal{S}_0 = 2 \quad ; \quad \mathcal{S}_\infty \quad = 1 \quad ; \quad \mathcal{L}_\infty = -1 \, .$$

Uplink Average Per-Cell Spectral Efficiency



- Uplink average per-cell spectral efficiency plotted as a function of E_b^t/N_0

Uplink Average Per-Cell Spectral Efficiency *(Cont'd)*



- The impact of K on the average per-cell spectral efficiency in Rayleigh fading channels

Uplink - Activity Controlled Users - Intra-Cell TDMA - The Non-ergodic Regime

- If the channel is random but fixed, the per-cell sum-rate capacity of the uplink channel is given by

$$C_{\text{ul}}(\bar{P}) = \frac{1}{M}\left\{\log\det\left(\boldsymbol{I}_M + \frac{\bar{P}}{K}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger\right)\right\},$$

  which is a random variable

- Simple model of activity controlled independent users: $P((a_i, b_{i+1}) = (1, \alpha)) = p$ (active user) and $P((a_i, b_{i+1}) = (0, 0)) = 1 - p$ (silent user).
- All users broadcast at a given rate R_{target}
- The multi-cell processor is fully aware of the CSI. In order to decode the message of user n: identify the first silent user on the left (resp. right) of user n : n_0 (resp. n_1).
- The section between $n_0 + 1$ and $n_1 - 1$ is isolated from the rest and can be decoded independently.

Uplink - Activity Controlled Users - Intra-Cell TDMA - The Non-ergodic Regime

- Denote by L the number of consecutive active users around user n
- L is distributed, for $l \geq 1$ as,

$$P(L = l) = l(1-p)^2 p^{(l-1)}.$$

- **Proposition 12** *The message of the consecutive active users around user n can be reliably decoded if and only if*

$$R_{target} \leq r_L \triangleq \frac{1}{L} \log \frac{\eta_1^{L+1} - \eta_2^{L+1}}{\eta_1 - \eta_2},$$

where

$$\eta_1 = \frac{1 + \bar{P}(1+\alpha^2) + \sqrt{(1+\bar{P}(1+\alpha)^2)(1+\bar{P}(1-\alpha)^2)}}{2},$$

$$\eta_2 = \frac{1 + \bar{P}(1+\alpha^2) - \sqrt{(1+\bar{P}(1+\alpha)^2)(1+\bar{P}(1-\alpha)^2)}}{2}.$$

Uplink - Activity Controlled Users - Intra-Cell TDMA - The Non-ergodic Regime

- The outage capacity is given by

$$P_{outage} = Prob(C_{\text{ul}}(\bar{P}) < R_{target}) = p^i(i(1-p)+1),$$

$$\text{for} \quad r_{i+1} < R_{target} \le r_i.$$

$P_{outage} = 0$ if $R_{target} < r_\infty$ and $P_{outage} = 1$ if $R_{target} > r_1$.

$$r_1 = \log\left(1 + \bar{P}(1+\alpha^2)\right),$$

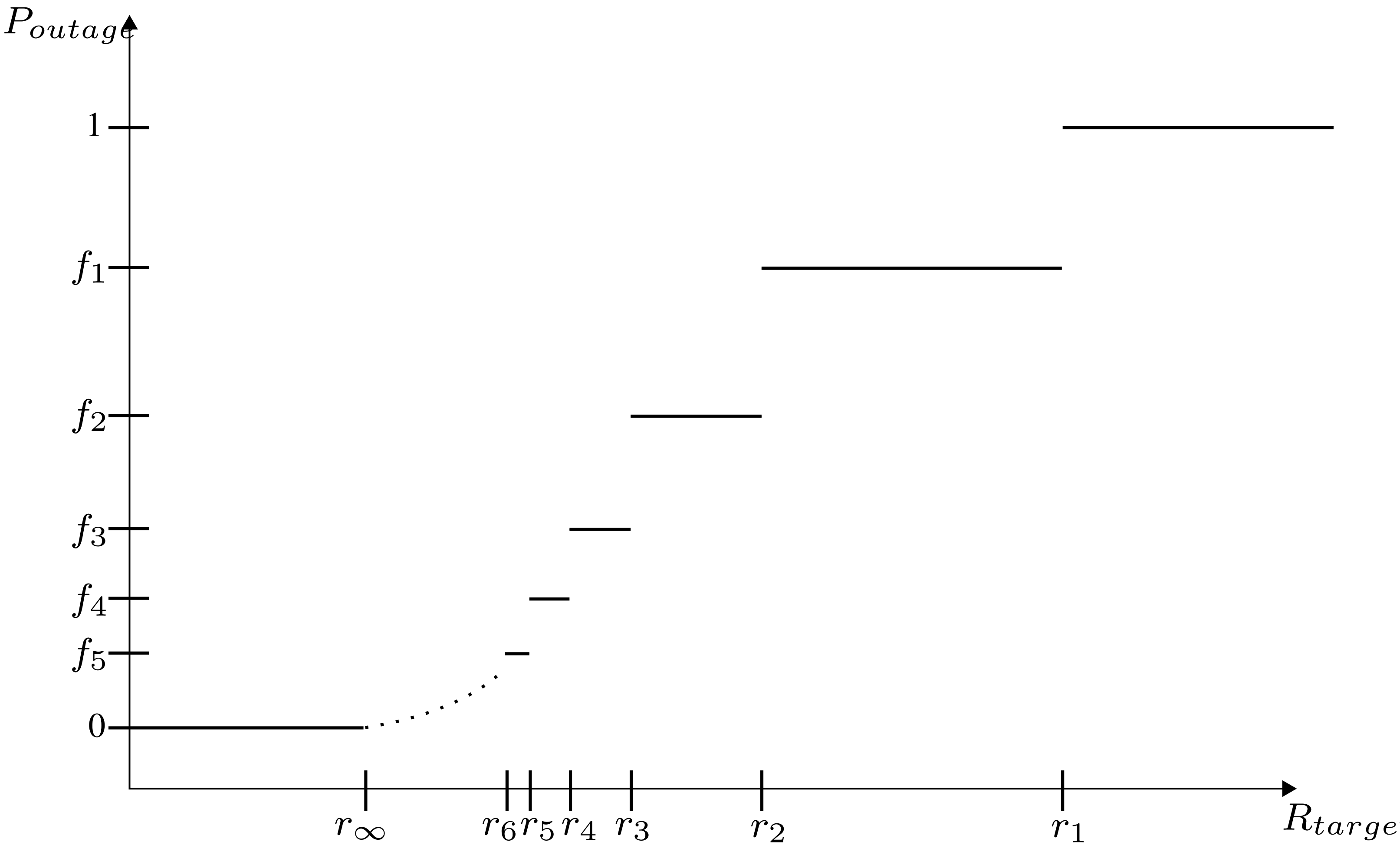
$$r_\infty = \log\left(\frac{1 + (1+\alpha^2)\bar{P} + \sqrt{1 + 2(1+\alpha^2)\bar{P} + (1-\alpha^2)^2\bar{P}^2}}{2}\right).$$

- *Ergodic* capacity (average over long term activity of users):

$$C_{ul-ergo}(\bar{P}) = \sum_{l=1}^{\infty}(1-p)^2 p^{(l-1)} \log\frac{\eta_1^{l+1} - \eta_2^{l+1}}{\eta_1 - \eta_2}.$$

Uplink - Activity Controlled Users - Outage probability

We define $f_i \triangleq p^i(i(1-p)+1)$.



Outage probability versus rate.

Local decoding

- Local decoding: To decode the message of user n, the multi-cell processor is only allowed to use the messages received at the antennas in a *decoding window* around user n.
- Remember that the string between $n_0 + 1$ and $n_1 - 1$ is isolated from the rest of the users. Therefore, if the decoding window includes $n_0 + 1$ and $n_1 - 1$, local decoding is as good as global decoding.
- As the size of the decoding window grows, the probability that $n_0 + 1$ and/or $n_1 - 1$ are not in the decoding window converges exponentially fast to 0.
- This property can be extended to longer range inter-cell interference.
- For p small enough, this property can be extended to multi-dimensional models.

Local decoding - a sub-optimal method

- In the case that $n_0 + 1$ and/or $n_1 - 1$ are not in the decoding window, we present a sub-optimal decoding method
- If, say, $n_0 + 1$ is not in the decoding window, there is a user on the left of the window whose signal is received at only one antenna. We consider this user as interfering noise.
- **Proposition 13** *The message of user n can be decoded using this method if and only if*

$$R \leq \frac{1}{L'} \log \left(v_{L'+1} - (\alpha\rho)^2 \left(\frac{\omega_1}{1+\rho\alpha^2} + \frac{\omega_2}{1+\rho} \right) v_{L'} + \frac{(\alpha\rho)^4}{(1+\rho)(1+\rho\alpha^2)} \omega_1 \omega_2 v_{L'-1} \right),$$

where

$$v_l = \frac{\eta_1^l - \eta_2^l}{\eta_1 - \eta_2},$$

L' *is the number of consecutive active users around user* n *in the decoding window and* $\omega_1 = 1$ *(resp.* ω_2*) if* $n_0 + 1$ *(resp.* $n_0 - 1$*) is in the decoding window, and 0 otherwise.*

Downlink System Model

- The received $MK \times 1$ signal vector, is given by

$$\boldsymbol{y}_{dl} = \boldsymbol{H}_M^{\dagger} \boldsymbol{x}_{dl} + \boldsymbol{z}_{dl}$$

  - $\boldsymbol{H}_{M\,[M \times KM]}$ - Channel transfer matrix
  - $\boldsymbol{x}_{dl\,[M \times 1]}$ - The vector of signals transmitted by the M cell-sites. An *equal individual per-cell-site power constraint* is assumed: $\left[E\left\{\boldsymbol{x}_{dl}\boldsymbol{x}_{dl}^{\dagger}\right\}\right]_{(m,m)} \leq \bar{P}\ \forall m$
  - $\boldsymbol{z}_{dl\,[MK \times 1]} \sim \mathcal{N}_c(\mathbf{0}, \boldsymbol{I}_{MK})$ - Circularly symmetric AWGN vector
- Full CSI is available to the joint multi-cell transmitter only
- The mobile receivers are assumed to be cognizant of their own CSI, and of the employed transmission strategy

Yu-Lan Lagrangian Duality

- **Theorem 1** *[Yu-Lan '04] The sum capacity of the Gaussian multi-antenna broadcast channel is the same as the sum capacity of a dual MAC with a sum power constraint and with a diagonal and "uncertain" noise:*

$$C_{sum} = \min_{\mathbf{\Lambda}} \max_{\boldsymbol{\mathcal{D}}} \log \frac{\det\left(\boldsymbol{H}\boldsymbol{\mathcal{D}}\boldsymbol{H}^{\dagger} + \mathbf{\Lambda}\right)}{\det\left(\mathbf{\Lambda}\right)} ,$$

  such that $\boldsymbol{\mathcal{D}}$ *and* $\mathbf{\Lambda}$ *are* $K \times K$ *and* $N \times N$ *nonnegative diagonal matrices, satisfying* $\mathrm{trace}(\boldsymbol{\mathcal{D}}) \leq 1$ *and* $\sum_n P_n \left[\mathbf{\Lambda}\right]_{(n,n)} \leq 1$*, respectively.*
- P_n - transmitted power by antenna n
- Theorem 1 can be directly applied to the downlink channel of the circular cell-array.
- With total average power constraint $\frac{1}{n}\sum_n P_n \leq \overline{P}$, $\mathbf{\Lambda}_{nn} = 1/\overline{P}$ and no minimum is effective.

Dual Uplink System Model

- The dual uplink channel is described by

$$\tilde{\boldsymbol{y}}_{ul[M\times 1]} = \boldsymbol{H}_M \tilde{\boldsymbol{x}}_{ul} + \tilde{\boldsymbol{z}}_{ul} \ ,$$

- $\boldsymbol{H}_{M[M\times KM]}$ - Channel transfer matrix
- $\tilde{\boldsymbol{x}}_{ul[KM\times 1]}$ - The vector of transmitted symbols with constraints:
  - $E\left\{\tilde{\boldsymbol{x}}_{ul}\tilde{\boldsymbol{x}}_{ul}^{\dagger}\right\} = \boldsymbol{\mathcal{D}}_M = \mathrm{diag}(\boldsymbol{D}_0, \boldsymbol{D}_1, \cdots, \boldsymbol{D}_{M-1})$
  - $\mathrm{trace}\left(\boldsymbol{\mathcal{D}}_M\right) \leq 1$
  - $\boldsymbol{D}_{m[K\times K]}$ - Nonnegative diagonal matrix, representing the power constraints of the mth cell's users
- $\tilde{\boldsymbol{z}}_{ul[M\times 1]} \sim \mathcal{N}_c(0, \boldsymbol{\Lambda}_M)$ - The circularly symmetric zero mean AWGN
  - Equal individual power constraints $\Rightarrow \bar{P}\,\mathrm{trace}\left(\boldsymbol{\Lambda}_M\right) \leq 1$

Downlink - No-Fading [Somekh-Zaidel-Shamai '07]

- For non-fading channels $a_{m,k}, b_{m,k} = 1, \forall m, k$
- The channel transfer matrix becomes "block-circulant"
- **Proposition 14** *For non-fading channels the average per-cell downlink sum-rate capacity is given for $M \longrightarrow \infty$ by*

$$C_{\textit{dl-nf}}(\bar{P}) = \log\left(\frac{1 + 2\bar{P} + \sqrt{1 + 4\bar{P}}}{2}\right).$$

- **Proposition 15** *Extreme-SNR regimes for non-fading channels:*

$$\frac{E_b^t}{N_0}_{\min} = \frac{\log_e 2}{2} \quad ; \quad \mathcal{S}_0 = \frac{4}{3} \quad ; \quad \mathcal{S}_\infty = 1 \quad ; \quad \mathcal{L}_\infty = 0 \quad .$$

- The above results also hold for an *overall* power constraint
- Generalization to $b_{m,k} = \alpha$ straight forward [Jing-Tse-Hou-Soriaga-Smee-Padovani '07]

$$C_{dl-nf} = \log\left(\frac{1 + (1+\alpha^2)\bar{P} + \sqrt{1 + 2(1+\alpha^2)\bar{P} + (1-\alpha^2)^2\bar{P}^2}}{2}\right).$$

Downlink - Rayleigh Fading [Somekh-Zaidel-Shamai ’07]

- **Proposition 16** *For $K \gg 1$ the downlink average per-cell sum-rate capacity for Rayleigh fading satisfies (ignoring little orders of $\log_e K$)*

$$\log\left(1+\bar{P}\left((1-\epsilon)\log_e K+2\right)\right) \le C_{\mathit{dl}}(\bar{P}) \le \log\left(1+2\bar{P}\log_e K\right) ,$$

for an appropriately chosen $\epsilon \in (0,1)$.

  - $\log_e K$-multi-user diversity.

- Lower bound: scheduling by a Threshold Crossing (TC) policy
  - The rate of the TC policy with $K \gg 1$ upper bounds the achievable rate of the TC policy with any K
- Upper bound: iid noise, max fading values incorporated with Hadamard and Jensen inequalities

Downlink - Rayleigh Fading [Somekh-Zaidel-Shamai '07] *(Cont'd)*

- **Proposition 17** *The downlink channel extreme-SNR regimes for Rayleigh fading are characterized by* ($K \gg 1$)

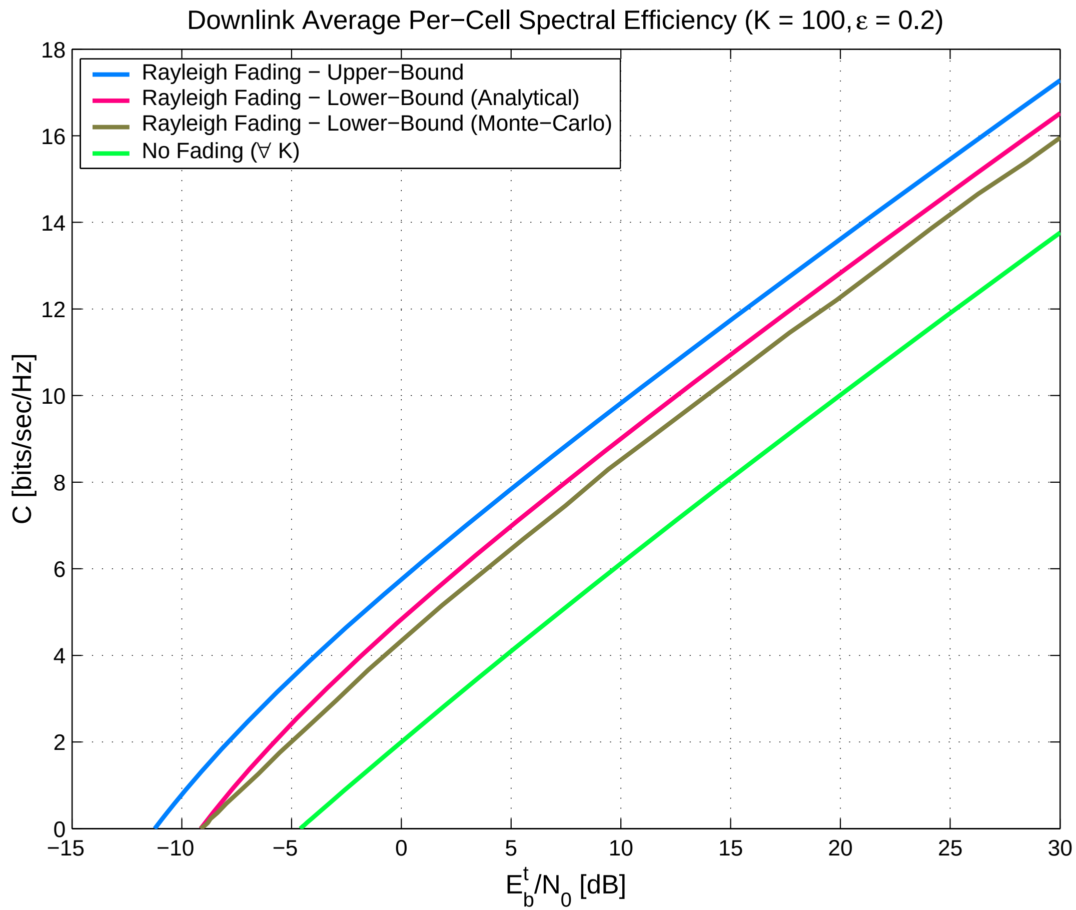
$$\frac{\log_e 2}{2\log_e K} \leq \frac{E_b^t}{N_0}_{\min} \leq \frac{\log_e 2}{(1-\epsilon)\log_e K + 2} \; ;$$

$$\mathcal{S}_0 = 2 \; ;$$

$$\mathcal{S}_\infty = 1 \; ;$$

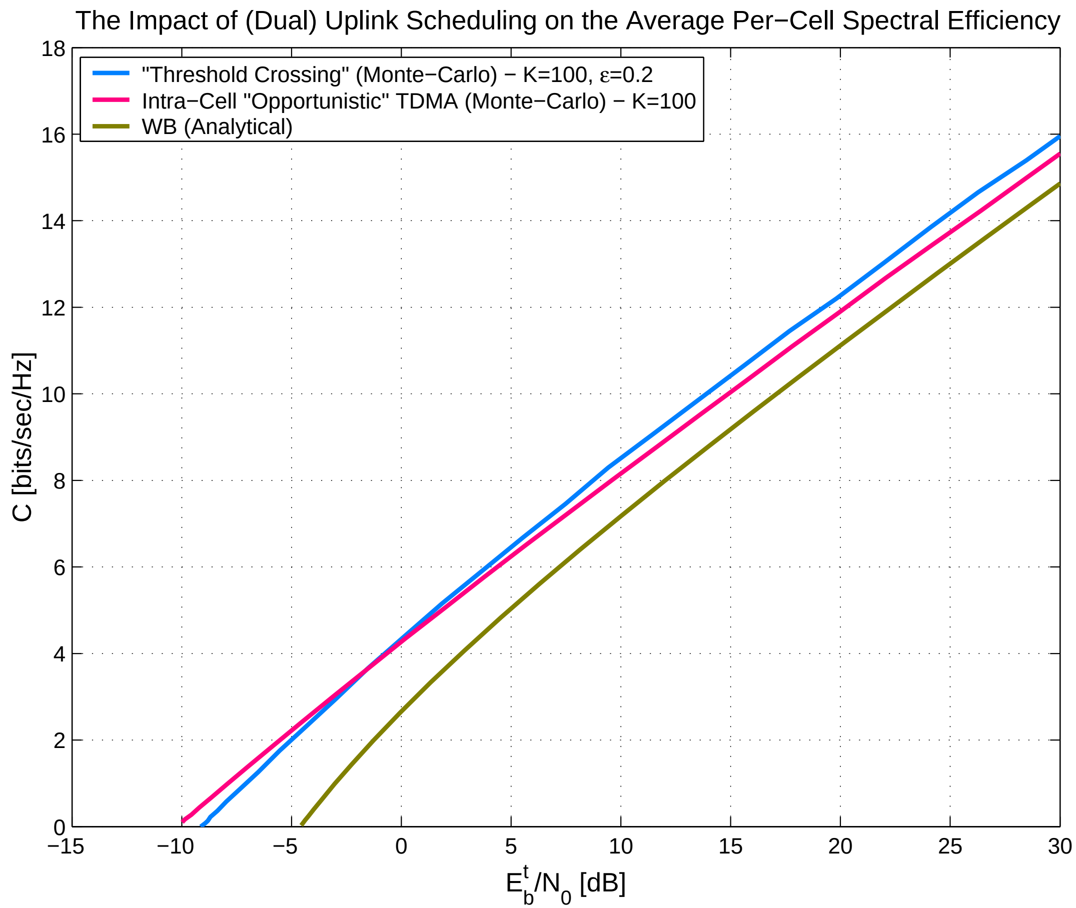
$$-1 - \log_2 \log_e K \leq \mathcal{L}_\infty \leq -\log_2\left((1-\epsilon)\log_e K + 2\right) \; .$$

Downlink Average Per-Cell Spectral Efficiency



- Downlink average pre-cell spectral efficiency ($K = 100$, $\epsilon = 0.2$)

The Impact of (Dual) Uplink Scheduling



- The impact of (dual) uplink scheduling on the average per-cell spectral efficiency ($K = 100$)

Outlook: Downlink and Fading: Finite (not too large) K

- Duality for capacity region, sum-rate:

$$C_{sum} = \min_{\mathbf{\Lambda}} \max_{\boldsymbol{\mathcal{D}}} \log \frac{\det\left(\boldsymbol{H}\boldsymbol{\mathcal{D}}\boldsymbol{H}^{\dagger} + \mathbf{\Lambda}\right)}{\det\left(\mathbf{\Lambda}\right)} ,$$

  such that $\boldsymbol{\mathcal{D}}$ and $\mathbf{\Lambda}$ are $K \times K$ and $N \times N$ nonnegative diagonal matrices, satisfying $\mathrm{trace}(\boldsymbol{\mathcal{D}}) \leq 1$ and $\sum_n P_n \left[\mathbf{\Lambda}\right]_{(n,n)} \leq 1$, respectively.
- Jacobi type (finite diagonal) large random matrices.
  Elements of $\boldsymbol{\mathcal{D}}$ depend on H.
- Lower bound for total power constraint. $\mathbf{\Lambda} = \frac{1}{P}\boldsymbol{I}$. Let $\boldsymbol{\mathcal{D}}$ be $\boldsymbol{I}$ (no maximization) and apply the uplink results.
- Scheduling users by activity factor (no information to transmit to users) breaks the system into separate downlinks.
  Generalization of the analysis of the uplink.

Concluding Remarks

- The impact of joint multiple cell-site processing was demonstrated here through a simple analytically tractable circular (Wyner-like) multi-cell model
- The model represents a practical "soft handoff" scenario
- Analytical results for general fading are obtained using large random Jacobi matrices theory
- Joint (cooperative) multi-cell processing (without spreading or time sharing) eliminates out-of-cell interference, traditionally a limiting factor
- Uplink channel:
  - In the absence of fading, a WB scheme and intra-cell TDMA scheduling are equivalent, and the average per-cell sum-rate capacity approaches the sum-capacity of *a single isolated cell* in the high-SNR regime
  - In flat-fading channels the WB scheme outperforms intra-cell TDMA scheduling
  - Flat-fading enhances performance already for a small K (for $K > 2$ and $K > 1$ in the low-SNR and high-SNR regimes respectively)
  - Fading produces the array diversity gain factor of 2 for $K \gg 1$
  - Analytical treatment for activity controlled users in the non-ergodic regime

Concluding Remarks *(Cont'd)*

- Downlink channel:
  - In the absence of fading the downlink and uplink are equivalent
  - For Rayleigh fading and $K \gg 1$, a multi-user diversity gain factor of $\log_e K$ is observed, in addition to the array diversity gain factor of 2
  - The gain is due to cooperation, incorporating the available CSI in the transmitting and receiving ends
  - Restricting joint processing to N cells can be shown to reduce the number of degrees of freedom by no more than a factor of $N/(N+1)$
- The uplink-downlink duality principle guarantees the same uplink multiuser diversity features with proper scheduling
- Can random matrices theory give general answers in the Wyner-type downlink by application of duality (with per-cell power constraint) ?
- Joint multi-cell processing is a key tool in enhancing performance of cellular systems

Spare Slides

Downlink Average Per-Cell Sum-Rate Capacity

- Using Theorem 1 the average per-cell sum-rate capacity is given by [Yu-Lan ‘04]:

$$C_{dl}(\bar{P}) = E_{H_M}\left\{\frac{1}{M}\min_{\boldsymbol{\Lambda}_M}\max_{\boldsymbol{\mathcal{D}}_M}\log\frac{\det\left(\boldsymbol{H}_M\boldsymbol{\mathcal{D}}_M\boldsymbol{H}_M^{\dagger}+\boldsymbol{\Lambda}_M\right)}{\det\left(\boldsymbol{\Lambda}_M\right)}\right\}$$

- The optimization is over all nonnegative diagonal matrices:
  - $\boldsymbol{\mathcal{D}}_M$ $[MK \times MK]$, s.t. $\text{trace}(\boldsymbol{\mathcal{D}}_M) \leq 1$
  - $\boldsymbol{\Lambda}_M$ $[M \times M]$, s.t. $\text{trace}\left(\boldsymbol{\Lambda}_M\right) \leq 1/\bar{P}$
- The corresponding spectral efficiency [bits/sec/Hz] is obtained through the relation

$$\bar{P} = \mathsf{C}_{dl}(\frac{E_b^t}{N_0})\,\frac{E_b^t}{N_0}$$

where E_b^t/N_0 is the transmit E_b/N_0 and $C_{dl}(\bar{P}) = \mathsf{C}_{dl}(E_b^t/N_0)$.

Downlink Average Per-Cell Sum-Rate Capacity *(Cont'd)*

- The non-zero entries of $\boldsymbol{H}_M \boldsymbol{\mathcal{D}}_M \boldsymbol{H}_M^\dagger$ may be explicitly expressed as

$$\left[\boldsymbol{H}_M \boldsymbol{\mathcal{D}}_M \boldsymbol{H}_M^\dagger\right]_{m,m} = \sum_{k=1}^{K} d_{m,k} \left|a_{m,k}\right|^2 + \sum_{k=1}^{K} d_{\widehat{m-1},k} \left|b_{m,k}\right|^2$$

$$\left[\boldsymbol{H}_M \boldsymbol{\mathcal{D}}_M \boldsymbol{H}_M^\dagger\right]_{m,\widehat{m-1}} = \sum_{k=1}^{K} d_{\widehat{m-1},k} a^*_{\widehat{m-1},k} b_{m,k}$$

$$\left[\boldsymbol{H}_M \boldsymbol{\mathcal{D}}_M \boldsymbol{H}_M^\dagger\right]_{m,\widehat{m+1}} = \sum_{k=1}^{K} d_{m,k} a^*_{m,k} b_{\widehat{m+1},k}$$

where

$$\widehat{m+1} \triangleq (m+1) \bmod M \quad ; \quad \widehat{m-1} \triangleq (m-1) \bmod M \; .$$

Proof of Proposition 14

- Three observations:
  - The minmax optimization problem is a function of the *total* intra-cell transmit power in the dual uplink $\Rightarrow$ We particularize to intra-cell TDMA ($K = 1$)
  - The minmax optimization problem is convex in $\mathbf{\Lambda}_M$ and concave in $\boldsymbol{\mathcal{D}}_M$ [Diggavi-Cover '01]
  - For any circulant matrix $\boldsymbol{A}_{M\,[M\times M]}$, and diagonal matrix $\boldsymbol{B}_{M\,[M\times M]}$, the quantity $\det\left(\boldsymbol{I}_M + \boldsymbol{A}_M\boldsymbol{B}_M\boldsymbol{A}_M^\dagger\right)$ is invariant to cyclic shifts of the diagonal entries of $\boldsymbol{B}_M$.
- Upper Bound:
  - Choose $\mathbf{\Lambda}_M = \mathbf{\Lambda}_M^0 = \frac{1}{M\bar{P}}\boldsymbol{I}_M$
  - Let $\boldsymbol{\mathcal{D}}_M^\star$ be a solution to the remaining maximization problem

Proof of Proposition 14 *(Cont'd)*

- Upper Bound *(Cont'd)*:
  - The corresponding upper bound is unaffected by any cyclic shift of the diagonal entries of $\boldsymbol{\mathcal{D}}_M^\star$
  - A "mixing" matrix of all M cyclic shifts $\frac{1}{M}\sum_{m=0}^{M-1}\boldsymbol{\mathcal{D}}_M^\star(m)$ is also a solution to the maximization problem
  - $\boldsymbol{\mathcal{D}}_M^\star = \frac{1}{M}\boldsymbol{I}_M$ is thus a maximum achieving solution
    $\Rightarrow$ yields and upper bound
- Lower Bound:
  - Choose $\boldsymbol{\mathcal{D}}_M = \boldsymbol{\mathcal{D}}_M^0 = \frac{1}{M}\boldsymbol{I}_M$
  - Using similar arguments it follows that $\boldsymbol{\Lambda}_M^\star = \frac{1}{M\bar{P}}\boldsymbol{I}_M$ is a solution to the remaining minimization problem
- The two bounds agree with $\frac{1}{M}\log\det\left(\boldsymbol{I}_M + \bar{P}\boldsymbol{H}_M\boldsymbol{H}_M^\dagger\right)$
- The proof is completed by applying well known results on the eigenvalues of circulant matrices [Gray '72]

Local Cell-Site Cooperation in the Downlink

- Underlying assumptions:
  - Joint pre-processing for a cluster of no more than N cells
  - Multiple reduced complexity transmitters are used (each with full CSI regarding its own cluster)
  - It is assumed that $M \gg N$
- **Proposition 18** *The downlink average per-cell sum-rate capacity, with pre-processing restricted to N cells, is lower bounded for non-fading channels by*

$$C_{rp,\mathit{dl}}(\bar{P}) \geq \frac{1}{N+1} \sum_{n=0}^{N-1} \log\left(1 + 2\bar{P}\left(1 + \cos\left(2\pi \frac{n}{N}\right)\right)\right) .$$

The high-SNR slope ("pre-log", "multiplexing gain") is lower bounded by

$$\mathcal{S}_{\infty rp} \geq \begin{cases} \frac{N}{N+1} & N \text{ odd}, \\ \frac{N-1}{N+1} & N \text{ even}. \end{cases}$$

Restricted Cell-Site Cooperation in the Downlink *(Cont'd)*

- **Proposition 19** *The downlink average per-cell sum-rate capacity, with pre-processing restricted to N cells, is lower bounded for Rayleigh fading by*

$$C_{rp,\mathsf{dl}}(\bar{P}) \geq \frac{N}{N+1} \log\left(1 + \bar{P}\left((1-\epsilon)\log_e K + 2\right)\right) \ , \quad 0 < \epsilon < 1 \, .$$

The high-SNR slope ("pre-log", "multiplexing gain") is lower bounded by

$$\mathcal{S}_{\infty\, rp} \geq \frac{N}{N+1} \, .$$

- It can be concluded that restricting joint pre-processing to no more than N cells, reduces the number of degrees of freedom by no more than a factor of $N/(N+1)$
- The relatively low performance degradation due to restricted processing is a result of the particular system model (each user only "sees" two cell-sites)
- Even when no CSI whatsoever is available, one looses no more than $1/2$ of the total degrees of freedom (inter-cell time sharing). In this case 2-fold diversity factor can be obtained via two-cell-site cooperation in both uplink (each user is received by two cell-sites), and downlink (Alamouti)

Proof of Propositions 18 & 19

- Step 1:
  - Assume no signals are transmitted to users located in cells indexed by integer multiples of $N+1 \Rightarrow$ Lower bound
  - This generates clusters of N cells with $N+1$ cell-sites
  - No inter-cluster interference
  - Each cluster can be viewed as a *linear* cell-array
  - Consider the dual uplink following [Yu-Lan '04]
- Step 2:
  - Reduce the number of observables in the dual uplink by adding the signal received at the $(N+1)$th cell-site to the one received at the first cell-site
  - This yields a lower bound to the sum-rate capacity (data processing inequality)

Proof of Propositions 18 & 19 *(Cont'd)*

- Step 3:
  - Observe that the resulting sum-rate capacity equals the capacity with full joint pre-processing for a *circular* array of N cells
- Step 4:
  - Multiply the result by $N/(N+1)$ to account for the fact that users of the $(N+1)$th cell and its multiples are ignored

Proof of Proposition 16

- Lower (Achievable Rate) Bound:
  - Choose a particular *dual uplink* input covariance matrix complying to a "Threshold Crossing" (TC) scheme
  - Only users received at both cell-sites with fade power levels exceeding some constant L (i.e., $|a_{m,k}|^2, \left|b_{\widehat{m+1},k}\right|^2 \geq L$), are allowed to transmit
  - As $K \to \infty$ the number of active users per cell crystalizes to $K_0 \triangleq Ke^{-2L}$, transmitting at power $1/(K_0 M)$
  - L should be chosen so that $K_0 \to \infty$ as $K \to \infty$
  - Let $K_0 = Ke^{-2L} = K^{\epsilon} \Rightarrow L = \frac{1-\epsilon}{2} \log_e K$, where $0 < \epsilon < 1$
  - $\boldsymbol{\Lambda}_M = \frac{1}{M\bar{P}} \boldsymbol{I}_M$ is the solution to the remaining minimization problem (Jensen)
  - This achievable rate, obtained for $K \gg 1$, upper bounds the achievable rate of the TC policy with any K

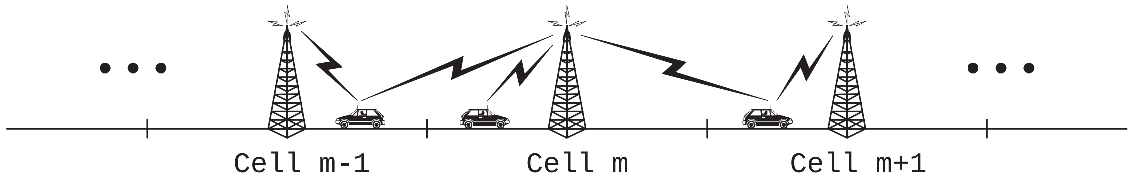
Proof of Proposition 16 *(Cont'd)*

- Upper Bound:
  - Choose a particular noise covariance matrix $\mathbf{\Lambda}_M^0 = \frac{1}{M\bar{P}}\boldsymbol{I}_M$
  - Apply the Hadamard inequality to the resulting $\max \log \det(\cdot)$ problem
  - Bound the channel fades by the strongest fading gain (over all intra-cell users) received at each cell-site
  - Observe that the maximum of K i.i.d. $\chi^2(2)$ distributed random variables behaves like $f(K) \triangleq \log_e K + O(\log_e \log_e K)$ for $K \gg 1$ [Sharif-Hassibi '05]
  - Apply Jensen's inequality to the resulting bound

Proof of Proposition 16 *(Cont'd)*

- Crystallization Argument (Lower Bound):
  - The number of users scheduled for transmission in each cell (in an arbitrary fading block) is a Binomial r.v. with probability of "success" $p = e^{-2L} = \frac{1}{K^{(1-\epsilon)}}$
  - For $K \gg 1$ the Binomial distribution is well approximated by a Normal distribution $\mathcal{N}(m, \sigma^2)$ where $m = Kp = K^\epsilon$ and $\sigma^2 = Kp(1-p)$
  - But $\frac{\sigma^2}{m^2} = \frac{1-p}{Kp} = \frac{1}{K^\epsilon} - \frac{1}{K} \underset{K\to\infty}{\longrightarrow} 0, \forall \epsilon > 0$
    $\Rightarrow$ The probability mass is concentrated around the mean value m
  - Since $m = K^\epsilon \underset{K\to\infty}{\longrightarrow} \infty$, it is concluded that the number of users scheduled for transmission is approximately $K^\epsilon \gg 1$ with high probability

Wyner '94



- A fully synchronous optimally coded system, with cells ordered in an *infinite* linear array (the two-dimensional hexagonal array was also considered)
- Only adjacent cell interference is present and characterized by a single parameter $0 \leq \alpha \leq 1$
- Non-fading channels
- A "wideband" transmission scheme: All bandwidth is available for coding (as opposed to *random* spreading)
- Joint processing of the signals received at all system cell-sites

Wyner '94 *(Cont'd)*

- Received signal at the nth cell-site:

$$y_n = \alpha \sum_{k=1}^{K} x_{n-1,k} + \sum_{k=1}^{K} x_{n,k} + \alpha \sum_{k=1}^{K} x_{n+1,k} + z_n$$

- $\{x_{n,k}\} \sim \mathcal{N}(0, \bar{P})$, $\{z_n\} \sim \mathcal{N}(0, 1)$
- Maximum reliable equal rate per user with optimum joint processing:

$$C_{\text{opt}} = \frac{1}{2K} \int_0^1 \log\left(1 + (1 + 2\alpha \cos 2\pi\theta)^2 K \bar{P}\right) \, d\theta$$

- Maximum reliable equal rate per user with MMSE processing:

$$C_{\text{ms}} = -\frac{1}{K} \log\left(\int_0^1 \frac{1}{1 + (1 + 2\alpha \cos 2\pi\theta)^2 K \bar{P}} \, d\theta\right)$$

Wyner ’94 *(Cont’d)*

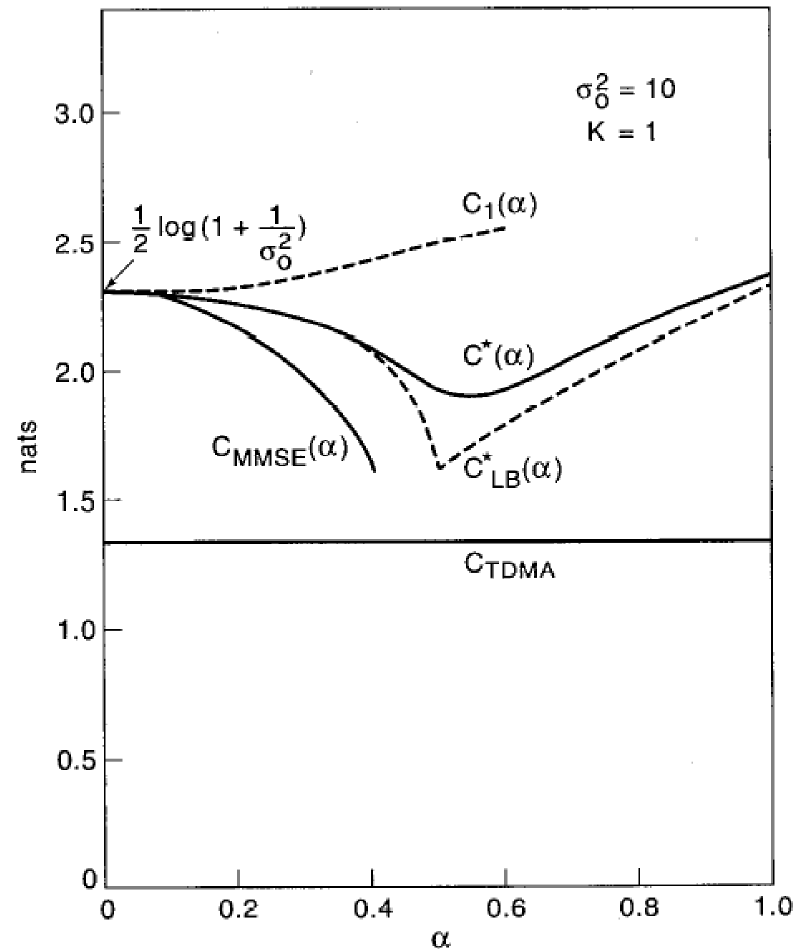


Fig. 1. Plots of $C^*(\alpha)$, $C_1(\alpha)$, C_{TDMA}, $C_{\text{MMSE}}(\alpha)$, and $C^*_{\text{LB}}(\alpha)$ versus α for $\sigma_0^2 = 0.01$, $K = 1$ (linear array). $C^*_{\text{LB}}(\alpha)$ is the lower bound of (2.5).

- A. D. Wyner, “Shannon-theoretic approach to a Gaussian cellular multiple-access channel,” *IEEE Trans. Inform. Theory*, vol. 40, pp. 1713–1727, Nov. 1994.

Somekh-Shamai ’00

- Wyner’s *infinite* linear array model
- Flat-fading channels (i.i.d., strictly stationary and ergodic, unit power fading processes)
- Received signal at the nth cell-site:

$$y_n = \alpha \sum_{k=1}^{K} b_{n,k} x_{n-1,k} + \sum_{k=1}^{K} a_{n,k} x_{n,k} + \alpha \sum_{k=1}^{K} c_{n,k} x_{n+1,k} + z_n$$

- $\{a_{n,k}\}$, $\{b_{n,k}\}$, $\{c_{n,k}\}$: independent fading coefficients
- $\{x_{n,k}\} \sim \mathcal{N}_c(0, \bar{P})$, $\{z_n\} \sim \mathcal{N}_c(0, 1)$
- Information theoretic upper and lower bounds for the maximum reliable equal rate per user are obtained

Somekh-Shamai '00 *(Cont'd)*

- Maximum reliable asymptotic ($K \gg 1$) equal rate per user with optimum joint processing (upper bound for any finite K):

$$C_{\text{opt}} = \frac{1}{K}\int_0^1 \log\Bigg(1 + K\bar{P}\Bigg(\sigma_a^2(1+2\alpha^2) + (1-\sigma_a^2)(1+2\alpha\cos(2\pi\theta))^2\Bigg)\Bigg)\,d\theta$$

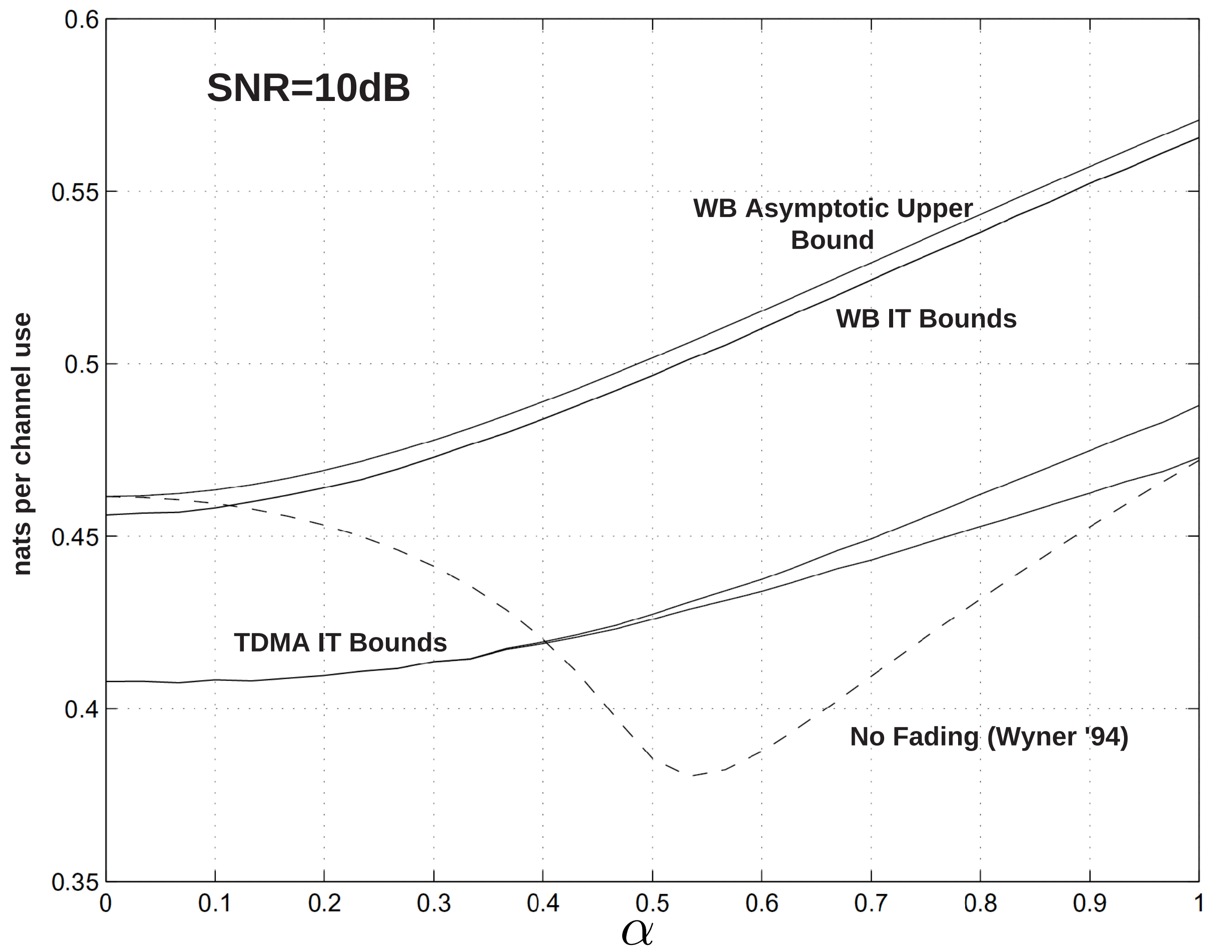
where $\sigma_a^2 = \sigma_b^2 = \sigma_c^2 = E\left\{|a|^2\right\} - |E\{a\}|^2$

- For any zero mean fading process (e.g., Rayleigh fading):

$$C_{\text{opt}} = \frac{1}{K}\log\left(1 + K\bar{P}(1+2\alpha^2)\right)$$

- The result is identical to the single isolated cell setup with a scaled transmit power of $\bar{P}(1+2\alpha^2)$

Somekh-Shamai ’00 *(Cont’d)*

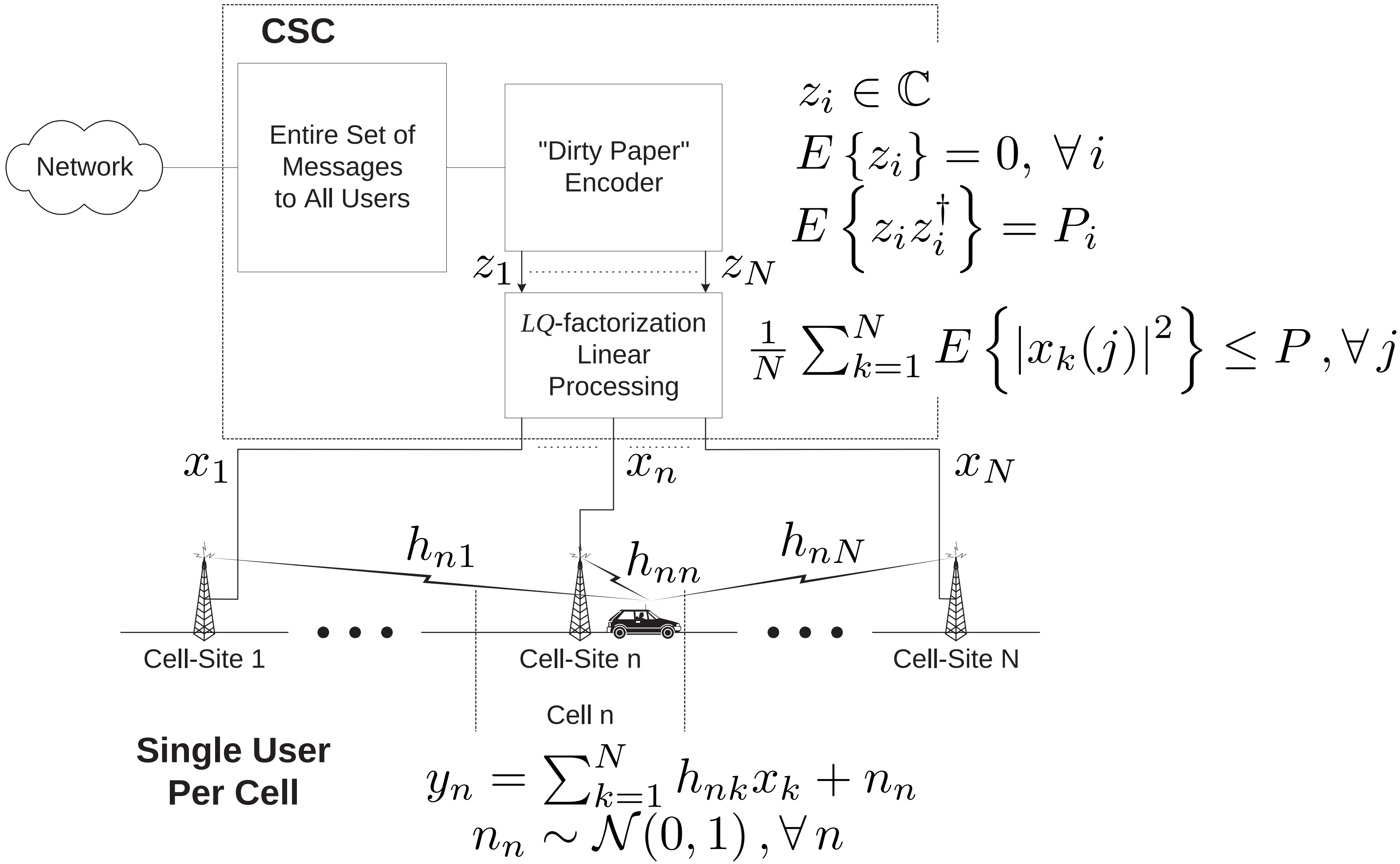


- O. Somekh and S. Shamai (Shitz), “Shannon-theoretic approach to Gaussian cellular multi-access channel with fading,” *IEEE Trans. Inform. Theory*, vol. 46, pp. 1401–1425, July 2000.

Shamai-Zaidel ’01

- Wyner’s infinite linear cell-array model (applied to the downlink channel)
- Single active user per cell (TDMA, orthogonal CDMA)
- Single antennas at both cell-sites and user terminals
- No cooperation of user receivers
- All cell-sites are connected to a central cell-site controller with full CSI, that knows the messages to be transmitted to all users
- A joint LQ-factorization based linear pre-processing is employed, combined with a dirty-paper encoding scheme (LP-DP), putting the whole complexity burden on the transmitting end
- An *overall* system average power constraint $\bar{P}$ is assumed

Shamai-Zaidel '01 *(Cont'd)*



- Schematic description of the cellular system model
- S. Shamai (Shitz) and B. M. Zaidel, "Enhancing the cellular downlink capacity via co-processing at the transmitting end," in *Proc. of IEEE VTC 2001 Spring*, vol. 3, (Rhodes, Greece), pp. 1745–1749, May 6–9, 2001.

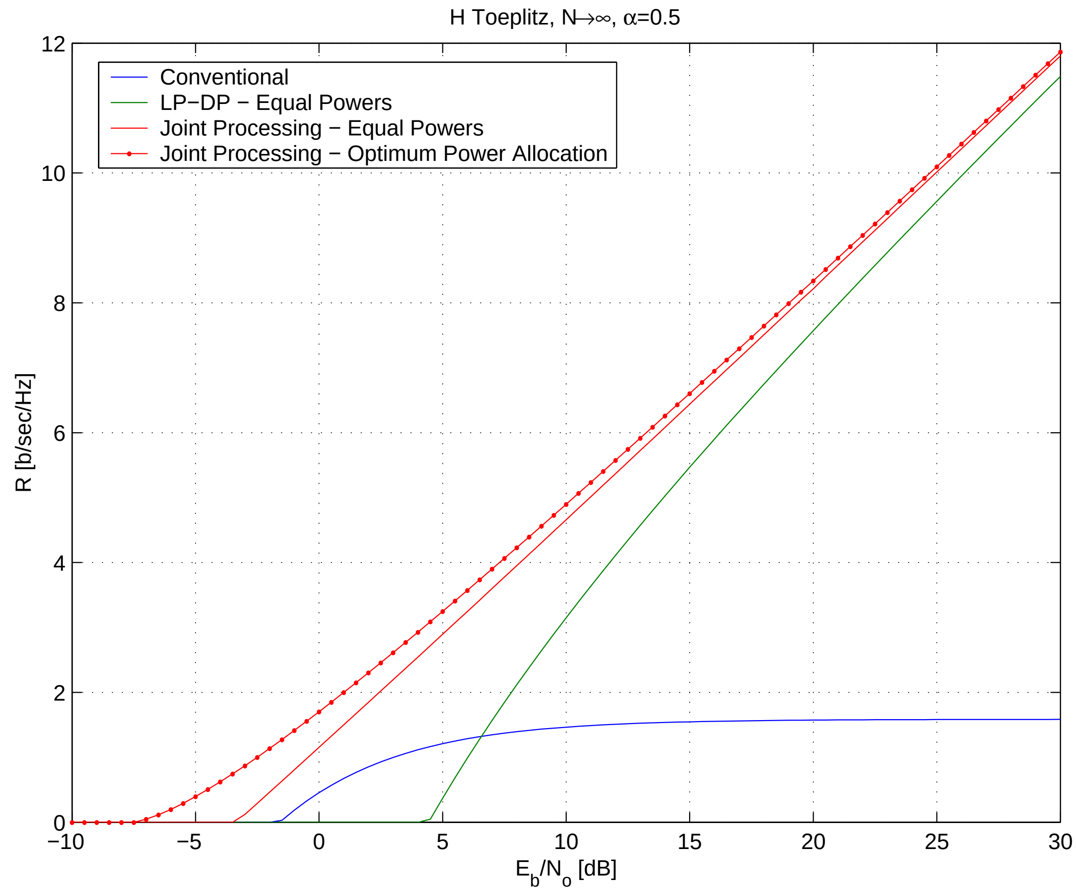
Shamai-Zaidel '01 *(Cont'd)*

- For non-fading channels equal power allocation is optimal, and the average attainable rate (per user) is given by

$$\bar{R}^{opt} = \begin{cases} \log\left(1 + \left(\frac{1+\sqrt{1-4\alpha^2}}{2}\right)^2 \bar{P}\right) & 0 \le \alpha \le \frac{1}{2} \\ \log\left(1 + \alpha^2 \bar{P}\right) & \alpha \ge \frac{1}{2} \end{cases}$$

- The average attainable rate of the LP-DP scheme approaches the rate of the optimum joint decoder in the high-SNR region.
- Numerical results for Rayleigh flat-fading show a similar behavior

Shamai-Zaidel ’01 *(Cont’d)*



- Average attainable rates without fading, for $\alpha = \frac{1}{2}$ and $N \to \infty$
- S. Shamai (Shitz) and B. M. Zaidel, “Enhancing the cellular downlink capacity via co-processing at the transmitting end,” in *Proc. of IEEE VTC 2001 Spring*, vol. 3, (Rhodes, Greece), pp. 1745–1749, May 6–9, 2001.

MIMO BC-MAC Capacity Regions

[Weingarten-Steinberg-Shamai '04]

- DPC capacity region:

$$\mathcal{C}^{\text{DPC}}(P, \boldsymbol{H}_{1,\ldots,m}) = \text{conv}\Big\{ \bigcup_{\pi\in\Pi} \bigcup_{\boldsymbol{B}_1,\ldots,\boldsymbol{B}_m} \big(R_1^{\text{DPC}}(\pi, \boldsymbol{B}_{1,\ldots,m}, \boldsymbol{H}_{1,\ldots,m}), \ldots, R_m^{\text{DPC}}(\pi, \boldsymbol{B}_{1,\ldots,m}, \boldsymbol{H}_{1,\ldots,m}) \big) \Big\}$$

  - Π is the set of all permutations
  - $\pi \in \Pi$ - reverse decoding order (π_{m+1-j} - jth user to be decoded)
  - $R_i^{\text{DPC}}(\pi, \boldsymbol{B}_{1,\ldots,m}, \boldsymbol{H}_{1,\ldots,m}) = \log \frac{\left|\boldsymbol{H}_{\pi_i}(\sum_{j=1}^{i} \boldsymbol{B}_{\pi_j})\boldsymbol{H}_{\pi_i}^\dagger + \boldsymbol{I}_{r_{\pi_i}\times r_{\pi_i}}\right|}{\left|\boldsymbol{H}_{\pi_i}(\sum_{j=1}^{i-1} \boldsymbol{B}_{\pi_j})\boldsymbol{H}_{\pi_i}^\dagger + \boldsymbol{I}_{r_{\pi_i}\times r_{\pi_i}}\right|}$
  - r_i - number of receive antennas of user i

MIMO BC-MAC Capacity Regions *(Cont'd)*

[Weingarten-Steinberg-Shamai '04]

- MAC capacity region with individual power constraints:

$$\mathcal{C}^{\text{MAC}}(\boldsymbol{P}_{1,\ldots,m}, \boldsymbol{H}^{\dagger}_{1,\ldots,m}) =$$
$$\text{conv}\left\{ \bigcup_{\pi\in\Pi} \left\{ (R_1, \ldots, R_m) : R_i = R_i^{\text{MAC}}(\pi, \boldsymbol{P}_{1,\ldots,m}, \boldsymbol{H}^{\dagger}_{1,\ldots,m}) \, \forall i \right\} \right\}$$

  - $\boldsymbol{P}_{1,\ldots,m}$ are the individual power constraints
  - π is the decoding order (π_j - jth user to be decoded)
  - $R_i^{\text{MAC}}(\pi, \boldsymbol{P}_{1,\ldots,m}, \boldsymbol{H}^{\dagger}_{1,\ldots,m}) = \log \left| \boldsymbol{I} + \frac{\boldsymbol{H}^{\dagger}_{\pi_i} \boldsymbol{P}_{\pi_i} \boldsymbol{H}_{\pi_i}}{\sum_{j=i+1}^{m} \boldsymbol{H}^{\dagger}_{\pi_j} \boldsymbol{P}_{\pi_j} \boldsymbol{H}_{\pi_j} + \boldsymbol{I}} \right|$

- MAC capacity region with a total power constraint P:

$$\mathcal{C}^{\text{Union}}(P, \boldsymbol{H}^{\dagger}_{1,\ldots,m}) = \bigcup_{\text{Tr}(\sum_{i=1}^{m} \boldsymbol{P}_i) \leq P} \mathcal{C}^{\text{MAC}}(\boldsymbol{P}_{1,\ldots,m}, \boldsymbol{H}^{\dagger}_{1,\ldots,m})$$

MIMO BC-MAC Capacity Regions *(Cont'd)*

[Weingarten-Steinberg-Shamai '04]

- Main duality result:

$$\begin{aligned} \mathcal{C}^{\mathrm{BC}}(P, \boldsymbol{H}_{1,\ldots,m}) \\ &= \mathcal{C}^{\mathrm{DPC}}(P, \boldsymbol{H}_{1,\ldots,m}) \\ &= \mathcal{C}^{\mathrm{Union}}(P, \boldsymbol{H}^{\dagger}_{1,\ldots,m}) \end{aligned}$$

- Fundamental Concept: Enhanced Channel $\Rightarrow$ mitigates the Minkowski inequality penalty in the vector EPI inequality
- Result does not rely on BC-MAC duality

DPC Achievable Region of the MIMO BC

[G. Caire, SPWAC 2005]

- Let $\boldsymbol{S} \in \mathbb{S}_+$ be an input covariance constraint. The region $\mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{H}_{1,\ldots,K}, \boldsymbol{N}_{1,\ldots,K})$ defined by

$$\text{coh} \bigcup_{\pi} \bigcup_{\sum_k \boldsymbol{B}_k \leq \boldsymbol{S}} \left\{ \boldsymbol{R} \, : \, R_{\pi_k} \leq \log \frac{\det \left(\boldsymbol{N}_{\pi_k} + \boldsymbol{H}_{\pi_k} \left(\sum_{i \leq k} \boldsymbol{B}_{\pi_i} \right) \boldsymbol{H}_{\pi_k}^{\dagger} \right)}{\det \left(\boldsymbol{N}_{\pi_k} + \boldsymbol{H}_{\pi_k} \left(\sum_{i < k} \boldsymbol{B}_{\pi_i} \right) \boldsymbol{H}_{\pi_k}^{\dagger} \right)} \right\}$$

  is achievable by DPC

- We can think of the above region as achieved by individual Gaussian coding with input covariance matrices $\boldsymbol{B}_k$, where while coding for user π_k we precode to take into account all users π_i with $i > k$
- Successive precoding order: $\pi_K, \pi_{K-1}, \ldots, \pi_1$
- $\mathcal{R}_{dpc}(\mathcal{E}; \boldsymbol{H}_{1,\ldots,K}, \boldsymbol{N}_{1,\ldots,K}) = \bigcup_{\text{Tr}\, \boldsymbol{S} \leq \mathcal{E}} \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{H}_{1,\ldots,K}, \boldsymbol{N}_{1,\ldots,K})$

Weingarten-Steinberg-Shamai: Main Ideas of a Brilliant Proof

[G. Caire, SPWAC 2005]

- Proving optimality for a given covariance constraint $\boldsymbol{S}$ is enough
- Proving optimality for squared invertible $\boldsymbol{H}_k$ is enough
- In this case, the general MIMO BC is equivalent to an aligned MIMO BC given by:

$$\boldsymbol{y}_k = \boldsymbol{x} + \boldsymbol{w}_k$$

  with fixed noise marginals $\boldsymbol{w}_k \sim \mathcal{N}_c(\boldsymbol{0}, \boldsymbol{N}_k)$

- Step 1: For every point $\boldsymbol{R} \notin \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$, there exists an enhanced aligned degraded MIMO BC whose DPC region outerbounds the original capacity region and does not contain $\boldsymbol{R}$
- Step 2: The capacity region of an aligned degraded MIMO BC coincides with its DPC region

Step 1

[G. Caire, SPWAC 2005]

- Since $\mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$ is convex, for any $\boldsymbol{R} \notin \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$ there exist a supporting hyperplane that separates $\boldsymbol{R}$ and $\mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$
- Given a supporting hyperplane touching $\mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$ in $\boldsymbol{R}^*$, achieved by DPC in encoding order π, there exist noise covariance matrices

$$\boldsymbol{N}'_{\pi_1} \leq \boldsymbol{N}'_{\pi_2} \leq \cdots \leq \boldsymbol{N}'_{\pi_K}$$

(degraded condition) and

$$\boldsymbol{N}'_k \leq \boldsymbol{N}_k, \; k = 1, \ldots, K$$

(enhanced condition) with the same supporting hyperplane at $\boldsymbol{R}^*$

Shortcoming of Direct Bergman's Proof in the Vector Case

[G. Caire, SPWAC 2005]

- Let $(R_1, R_2) \notin \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,2})$. Then there exist $\boldsymbol{B}_1, \boldsymbol{B}_2 = \boldsymbol{S} - \boldsymbol{B}_1$ such that

$$R_1 \geq R_{1,dpc} = \log\det\left(\tilde{\boldsymbol{N}}_1 + \boldsymbol{B}_1\right) - \log\det(\tilde{\boldsymbol{N}}_1)\ ,$$
$$R_2 \geq R_{2,dpc} = \log\det\left(\tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2 + \boldsymbol{S}\right) - \log\det\left(\tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2 + \boldsymbol{B}_1\right)$$

- From Fano's inequality (neglecting vanishing terms), two-user case

$$R_1 \leq \frac{1}{n} I(W_1; \boldsymbol{Y}_1 | W_2), \quad R_2 \leq \frac{1}{n} I(W_2; \boldsymbol{Y}_2)$$

which yields

$$\frac{1}{n} h(\boldsymbol{Y}_1 | W_2) \geq \log\det(\pi e(\tilde{\boldsymbol{N}}_1 + \boldsymbol{B}_1))$$

Shortcoming of Direct Bergman's Proof in the Vector Case *(Cont'd)*

[G. Caire, SPWAC 2005]

- In order to lowerbound $\frac{1}{n}h(\boldsymbol{Y}_2) = \frac{1}{n}I(\boldsymbol{Y}_2; W_2) + \frac{1}{n}h(\boldsymbol{Y}|W_2)$ $\geq R_2 + \frac{1}{n}h(\boldsymbol{Y}_2|W_2)$, we seek a lowerbound to $\frac{1}{n}h(\boldsymbol{Y}_2|W_2)$ via the EPI:

$$\frac{1}{Mn}h(\boldsymbol{Y}_2|W_2) = \frac{1}{Mn}h(\boldsymbol{Y}_1 + \tilde{\boldsymbol{Z}}|W_2) \geq \log\left(\det\left(\pi e(\boldsymbol{B}_1 + \tilde{\boldsymbol{N}}_1)\right)^{1/M} + \det(\pi e \tilde{\boldsymbol{N}}_2)^{1/M}\right)$$

- Unfortunately, Minkowski's inequality goes in the wrong direction:

$$\det(\boldsymbol{K}_1)^{1/M} + \det(\boldsymbol{K}_2)^{1/M} \leq \det(\boldsymbol{K}_1 + \boldsymbol{K}_2)^{1/M}$$

- This is not a problem for $M = 1$ (scalar case), but it prevents from going on in the vector case !
- If there exists $\alpha > 0$ such that $\alpha(\boldsymbol{B}_1 + \tilde{\boldsymbol{N}}_1) = \tilde{\boldsymbol{N}}_2$ (proportionality condition), then Minkowski's inequality holds with equality, and we have

$$\frac{1}{n}h(\boldsymbol{Y}_2|W_2) \geq \log\det(\pi e(\boldsymbol{B}_1 + \tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2))$$

Step 1 *(Cont'd)*

[G. Caire, SPWAC 2005]

- Under the parallel condition (always true when scalar) we have

$$
\begin{aligned}
\frac{1}{n}h(\boldsymbol{Y}_2) &\geq R_2 + \frac{1}{n}h(\boldsymbol{Y}_2|W_2) \\
&> R_{2,dpc} + \frac{1}{n}h(\boldsymbol{Y}_2|W_2) \\
&= \log\det\left(\pi e(\tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2 + \boldsymbol{S})\right) \\
&\quad - \log\det\left(\pi e(\tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2 + \boldsymbol{B}_1)\right) + \frac{1}{n}h(\boldsymbol{Y}_2|W_2) \\
&\geq \log\det\left(\pi e(\tilde{\boldsymbol{N}}_1 + \tilde{\boldsymbol{N}}_2 + \boldsymbol{S})\right)
\end{aligned}
$$

contradiction !!!

- This shows that any achievable rate must lie inside the Gaussian DPC region

Step 2

[G. Caire, SPWAC 2005]

- Consider an aligned degraded MIMO BC, defined by the noise increment covariances $\tilde{\boldsymbol{N}}_k = \boldsymbol{N}_k - \boldsymbol{N}_{k-1} \geq \boldsymbol{0}$, $k = 1, \ldots, K$, $\boldsymbol{N}_0 = \boldsymbol{0}$
- Let $\boldsymbol{R} \in \partial \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}_{1,\ldots,K})$ and let $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_K$ denote the input covariances achieving $\boldsymbol{R}$ (they exist and satisfy $\sum_k \boldsymbol{B}_k = \boldsymbol{S}$)
- There exist $\tilde{\boldsymbol{N}}'_k$ such that $\sum_{i=1}^{k} \tilde{\boldsymbol{N}}'_k \leq \tilde{\boldsymbol{N}}_k$ (enhanced channel), $\alpha_k \left(\sum_{i=1}^{k} \boldsymbol{B}_i + \sum_{i=1}^{k} \tilde{\boldsymbol{N}}'_k \right) = \tilde{\boldsymbol{N}}'_{k+1}$ (proprtioanlly) and $\boldsymbol{R} \in \partial \mathcal{R}_{dpc}(\boldsymbol{S}; \boldsymbol{N}'_{1,\ldots,K})$ achieved by the same input covariances (rate preservation)
- We can apply Bergman's converse proof to the enhanced channel and show optimality of Gaussian DPC coding