Scaling Laws of Multiple-Antenna Group Broadcast Channels

Tareq Y. Al-Naffouri

Electrical Engineering Department
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia

Joint work with Amir Dana and Babak Hassibi

Introduction to broadcast channels

- Multiple antennas add tremendous value to point to point systems
- Research shifted recently to the role of multiple antennas in pointmultiuser systems
  - (Uplink) Multiple Access (MAC)
  - (Downlink) Broadcast (BC)
- Broadcast scenarios (point to multi-point) are especially important because downlink scheduling is the major bottleneck for broadband wireless networks

Three main questions in a broadcast scenario

Q1) Quantify the maximum sum rate possible to all users

A1) Capacity region is achieved using dirty paper coding (DPC) (Caire and Shamai '02, Viswanath and Tse '02, Vishwanath et al. '02, Yu and Cioffi '02, Weingarten et. al. '06)

Q2) Quantify the asymptotic behavior in regimes of interest

A2) Regimes include

- Large number of users (Masoud and Hassibi '05,)
- Large number of Antennas (Masoud '05)
- High and low SNR (Jindal & Goldsmith)

Q3) How do scheduling schemes performs under various non-idealities

A3) **(i)** Time correlation (Kountouris and Gesbert '05)

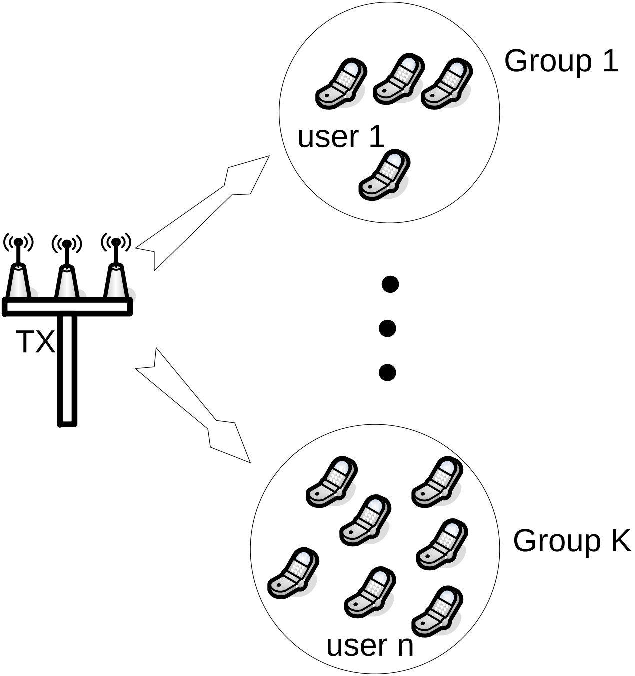
(ii) Frequency correlation (Fakhereddin et. al. '06)

(iii) Channel estimation error (Vikali et. al. '06)

(iv) Spatial correlation (D. Park and S Y. Park '05, Al-Naffouri et. al. '06)

Group broadcast scenario

- Broadcast problem: users interested in *independent* information
- Group Broadcast: Groups of users, each group of users interested in the same information
  - e.g. DAB/DVB with limited shows; users classified according to shows they are interested in
  - Single group: multicast problem (Khitsi et. al. 06, Jindal and Luo 06)
  - Multiple-groups each consisting of one user: broadcast problem
- Would like to answer **Q2)**: Asymptotic behavior in various regimes (large number of users and antennas)


Group 1
user 1
TX
Group K
user n

System model

- Base station equipped with M antennas
- n users each equipped with a single receive antenna.
- n single-antenna users with received signal

$$y_i = h_i^* s + \nu_i$$

  - Input satisfies $E[s^* s] \leq P$
  - Noise is white Gaussian $\nu \sim CN(0, I_M)$
  - User channels are independent and distributed as $CN(0, I_M)$
- Users are partitioned into K groups of $\frac{n}{K}$ users each; each group is interested in the same data.

Group broadcast capacity: Formal expression

- When there is one user only

$$C_{\text{one user}} = E \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \log\det\left(1 + \|h\|_B^2\right)$$

- Single group broadcast

$$\boxed{C_{\text{single group}} = E \max\nolimits_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min\nolimits_i \log\det\left(1 + \|h_i\|_B^2\right)}$$

- Group broadcast eventually limited by the worst user

Group broadcast capacity: Formal expression (2)

- Multiple groups broadcast: K power matrices $B_1, \ldots, B_K$, one for each group.
- Matrices should maximize sum-rate under total power constraint

$$C_{\text{multiplegroups}} = E \max_{B_k \geq 0 \ \sum_{k=1}^{K} \operatorname{Tr}(B_k) \leq P} \log\det\left(1 + \sum_{k=1}^{K} \|h_k\|_{B_k}^2\right)$$

- With K *user* groups, we need to take care of the "worst" user of each group

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Study behavior of C_{GB} for large number of users n and antennas M
  - Large n and fixed M
  - Large M and fixed n
  - Large M and n with $M = \beta n$
  - Large M and n with $M = \log n$

Capacity bounding techniques (1)

Upper bounds

1. K times rate of one group

$$
\begin{aligned}
C_{GB} &\leq KC_{\text{single group}} \\
&= K\log(1 + \max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2)
\end{aligned}
$$

2. MAC-BC duality

   - Maximum sum rate for K users, chosen one from each group

$$
C_{K\text{ users}} = \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^K b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)
$$

- Rate has to appeal to every user in every group

$$C_{K \text{ users}} \leq \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^{K} b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)$$

- Get rid of the determinant using AM-GM inequality $\det(A) \leq \left(\frac{\mathrm{tr}(A)}{M}\right)^M$ to write

$$C_{GB} \leq M \log\left(1 + \frac{P}{M} \max_k \min_{h_{i_1}} \cdots \min_{h_{i_K}} \{\|h_{i_1}\|^2, \cdots, \|h_{i_K}\|^2\}\right)$$

Capacity bounding techniques (2)

Lower Bounds

1. Time sharing

$$C_{GB} \geq \frac{1}{K} \sum_{k=1}^{K} \log\det \left(1 + \max_{B_k \geq 0 \ \mathrm{Tr}(B_k)=P} \ \min_{h_{i_k}} \|h_{i_k}\|_{B_k} \right)$$

2. Treating interference as noise

$$C_{GB} \geq K \log \left(\frac{\frac{1}{K}\frac{P}{M} \min_i \|h_i\|^2}{1 + \frac{K-1}{K}\frac{P}{M} \min_i \|h_i\|^2} \right)$$

Need to study scaling of the weighted max − min norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \ \min_i \|h_i\|_B^2$$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's

Bounds on the max-min weighted Euclidean norm

Here we obtain upper and lower bounds on the weighted Euclidean norm for fixed M and n

Lower Bounds

1. max-min norm is greater than min norm

$$\max_{\mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2 \geq \frac{P}{M} \min_i \|h_i\|^2$$

2. h_i belongs to a finite set $\{h_1, \cdots, h_{\frac{n}{K}}\}$

$$\max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\frac{n}{K}} \min_i \|h_i\|^2$$

So

$$\max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\min\{M, \frac{n}{K}\}} \min_i \|h_i\|^2$$

3. Diagonal values and eigenvalues: Define $H = [h_1 \cdots h_{\frac{n}{K}}]$, then

$$\lambda_{min}(H^* H) \leq \min_i \|h_i\|^2 \leq \lambda_{max}(H^* H)$$

Upper Bounds

1. max-min is less than min-max

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \leq P \min_i \|h_i\|_B^2$$

2. Replace minimization with averaging (Jindal and Luo '06)

$$\begin{aligned} \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 &\leq \max_B \frac{1}{\frac{n}{K}} \sum_{i=1}^{\frac{n}{K}} \|h_i\|_B^2 \\ &\leq P \lambda_{max}(H^* H) \end{aligned}$$

Study boils down to studying the scaling of

1) min norm $\min_i \|h_i\|^2$ 2) eigenvalues of $H^* H$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's
- Find the asymptotics of $\min_i \|h_i\|^2$

Scaling of the Euclidean norm

In the rest of the presentation, we study the scaling of the minimum Euclidean norm $\min_i \|h_i\|^2$ for

- Large n and fixed M
- Large M and fixed n
- Large M and n with $M = \beta n$
- Large M and n with $M = \log n$

Scaling of the minimum of iid variables

- Let $x_1, x_2, \cdots, x_n$ be nonnegative iid r. v.'s with CDF $F(x)$, and CF $\phi(x)$.
- Need to find scaling law of $x_{\min}(n) = \{x_1, x_2, \cdots, x_n\}$
- CDF of the mimimum is given by

$$F_{\min}(x) = 1 - (1 - F(x))^n$$

- Can show $n^{\frac{1}{i_0}} x_{\min}(n)$ converges in distribution to y with CDF

$$F_y(y) = 1 - \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} y^{i_0}\right)$$

- We thus say that

$$\boxed{x_{\min} \text{ converges to } \frac{E}{n^{\frac{1}{i_0}}}}$$

where E is the expectation that arises from the distribution (1)

$$E = \int_0^\infty \exp\left(-\frac{F^{(i_0)}(0)}{i_0!}x^{i_0}\right)$$

$$= \frac{C_{i_0}}{F^{(i_0)}(0)^{\frac{1}{i_0}}} \quad C_{i_0} = \frac{\Gamma(\frac{1}{i_0})(i_0!)^{\frac{1}{i_0}}}{i_0}$$

- The constant i_0 is the least i_0 for which $F^{(i_0)}(0) \neq 0$
- Can find i_0 and $F^{(i_0)}(0)$ using initial value theorem

$$\boxed{\lim_{x\to 0} F^{(i_0)}(x) = \lim_{s\to\infty} s^{i_0}\phi(s)}$$

- Note that there is no restriction on distribution $F(x)$

Scaling for large n, fixed M

- Scaling law for $\min_{h_i} \|h_i\|^2$, $h_i \sim CN(0, R)$.
- CDF of $\|h_i\|^2$ will have different forms depending on eigenvalues of R
- Characteristic function given by

$$\phi(s) = \prod_{l=1}^{M} \frac{1}{1 + \lambda_l s}$$

- It is easy to see that

$$F^{(i_0)}(0) = \lim_{s \to \infty} s^i \phi(s) = \begin{cases} 0 & \text{for } i < M \\ \frac{1}{\det(R)} & \text{for } i = M \end{cases}$$

- We thus conclude that

$$\boxed{\min_i \|h_i\|^2 \text{ scales as } C_M \det(R)^{\frac{1}{M}} \frac{1}{n^{\frac{1}{M}}}} \quad C_M = \frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M}$$

Scaling for large M, fixed n

- By the law of large numbers

$$\min_i \|h_i\|^2 = M$$

which implies

$$\boxed{\max \min \|h_i\|_B^2 \leq PM}$$

- Applying the law of large numbers to

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\min\{M, \frac{n}{K}\}} \min_i \|h_i\|$$

implies

$$\boxed{\max \min \|h_i\|_B^2 \geq P \tfrac{K}{n} M}$$

Scaling for large M and n, $M = \beta n$

We consider the regime: $M, n \to \infty$ with $M = \beta n$

- Use $\lambda_{min}(H_i^* H_i) \leq \min_i \|h_i\|^2$ to show

$$\min_i \frac{\|h_i\|^2}{M} \geq (1 - \sqrt{K\beta})^2$$

  which implies

$$\boxed{\max \min \frac{\|h_i\|_B^2}{M} \geq P(1 - \sqrt{K\beta})^2}$$

- Use $\max\ \min_i \|h_i\|_B^2 \leq P\frac{K}{n}\lambda_{max}(H^* H)$ to show

$$\boxed{\max \min \frac{\|h_i\|_B^2}{M} \leq P(1 + \frac{1}{\sqrt{K\beta}})^2}$$

Behavior of the min Euclidean Norm

The behavior of $\min_i \|h_i\|^2$ looks like

Regime	**Asymptotic Value**	**Method**
large n	$\frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M} \frac{1}{n^{\frac{1}{M}}}$	min of iid variables
large M	M	Law of large numbers
$M = \beta \frac{n}{K}$	$\geq (1 - \sqrt{K\beta})^2$ $\leq (1 + \sqrt{K\beta})^2$	Random Matrix theory
$M = \log n$	$\mathcal{H} \in [1 - \epsilon_l, 1]$ $\epsilon \simeq .8414$	Chernof Bound

Behavior of the max min Euclidean Norm

The behavior of $\max_B \min_i \|h_i\|^2$ looks like

Regime	**Lower Bound**	**Upper Bound**
large n	$\frac{C_M}{M} \frac{1}{n^{\frac{1}{M}}}$	$C_M \frac{1}{n^{\frac{1}{M}}}$
large M	$P \frac{K}{n} M$	PM
$M = \beta \frac{n}{K}$	$P(1 - \sqrt{K\beta})^2$	$P(1 + \frac{1}{\sqrt{K\beta}})^2$
$M = \log n$	$P\mathcal{H} \;\; , \mathcal{H} \in [1 - \epsilon_l, 1] \;\; \epsilon \simeq .8414$	constant

$$C_M = \frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M}$$

GB scaling for large n, fixed M

- Group broadcast capacity scales as

$$\boxed{C_{GB} = \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

where

$$\frac{1}{M} \leq \alpha \leq 1$$

- For spatially correlated case, the capacity incurs a $\det(R)^{\frac{1}{M}}$ hit

$$\boxed{C_{GB} = \alpha \det(R)^{\frac{1}{M}} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

- Unfortunate result: sum-rate decreases with the number of users.
- Counter this: increase the resources (i.e., number of antennas M).

GB scaling for large M, fixed n

- Upper bound: K times rate of single group

$$C_{GB} \quad \leq \quad K \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \log(1 + \min_i \|h_i\|_B^2)$$

$$\boxed{C_{GB} \leq K \log(1 + PM) \quad \text{(law of large numbers)}}$$

- Lower bound: Use time sharing

$$C_{GB} \geq \log(1 + \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B)$$

$$\boxed{C_{GB} \geq \log(1 + P \frac{K}{n} M)}$$

GB scaling with M and n, $M = \beta n$

- Number of users and antennas grow to infinity while their ratio remains constant $\frac{M}{n} = \frac{\beta}{K}$.
- Lower bound: Use time sharing

$$\boxed{C \geq \log\left(1 + P(1 - \sqrt{K\beta})^2\right)}$$

- To obtain an upper bound, we start with the bound

$$C_{GB} \leq K \log(1 + \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2)$$

to show

$$\boxed{C_{GB} \leq K \log(1 + P(1 + \tfrac{1}{\sqrt{\beta}})^2)}$$

- If we allow the number of antennas to grow linearly with the number of users, we can guarantee a constant sum rate.

Can we have constant rate with sublinear growth?

- But is it still possible to do so without straining the resources as much?
- We showed that for large n

$$
\begin{aligned}
C &= \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}} \qquad \frac{C_M}{M} \simeq 1 \\
&= \alpha P \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}
\end{aligned}
$$

- To guarantee a constant rate, intuition requires to set $M = \log n$

GB scaling with M and n, $M = \log n$

- Use the Chernof bound, we show that

$$\lim_{M=\log n, n\to\infty} \min_i \frac{\|h_i\|^2}{M} = \mathcal{H} \in [1-\epsilon_l, 1] \quad \text{w.p.1}$$

  where $\epsilon_l \simeq .8414$.

- Capacity is lower-bounded by a constant

$$\boxed{C \geq \log(1 + P\mathcal{H})} \tag{1}$$

- Capacity is also upper bounded by a constant because it is for $M = \beta n$

Conclusion

- Studied the scaling law of the group broadcast problem
- Capacity decreases as $n^{-\frac{1}{M}}$ with number of users
- Can guarantee a constant rate if we allow M to grow as $\log n$
- As a by-product (or a prerequisite), we studied the scaling of
  - Minimum Euclidean norm $\min_i \|h_i\|^2$
  - Max min Euclidean norm $\max_B \min_i \|h_i\|_B^2$

  in various regimes
- Several results apply for general distributions on h_i