Network Calculus
A General Framework for Interference Management and Resource Allocation

Martin Schubert

Fraunhofer Institute for Telecommunications HHI, Berlin, Germany
Fraunhofer German-Sino Lab for Mobile Communications (MCI)
Heinrich Hertz Chair for Mobile Communications
Technical University of Berlin
Outline

1. Interference Functions
2. Application to SIR-Based Utility Sets
3. Application to Game Theory
4. Structure of Interference Functions and Utility Sets
5. Conclusions
Interference in Multiuser Wireless Networks

- evolution of wireless networks:
  - high-rate services
  - densely populated user environments
- interference between users puts a limit on how many users per cell can be served at a certain data rate
- countermeasure: adaptive signal processing and resource allocation/scheduling strategies
"...letting transmitted signals interfere with each other (in a controlled way) increases capacity provided that the receivers take into account the multiaccess interference."

[Verdu, “Fifty Years of Shannon Theory”]

\[
\begin{align*}
\text{capacity user 1} & \quad \log(1 + p_1 \| h_1 \| / \sigma_n^2) \\
\text{capacity user 2} & \quad \log(1 + p_2 \| h_2 \| / \sigma_n^2)
\end{align*}
\]
objective: modelling of performance tradeoffs caused by interference

in the past, results were mainly derived for special system layouts (e.g. MIMO MAC), are there more general principles?

generally difficult due to complicated interdependencies between system functionalities ("cross-layer problem")

chosen approach: abstract framework ("network calculus")

- focus on core properties
- rigorous, allows to handle problems analytically
- provides intuition and roadmap for implementation
Some Interference Models

- classical linear model

\[ I_k(p, \sigma_n^2) = v^T p + \sigma_n^2 \]

- interference in a multiuser MIMO channel with optimum antenna combining

\[ I_k(p, \sigma_n^2) = \frac{1}{h_k^H \left( \sigma_n^2 I + \sum_{l \neq k} p_l h_l h_l^H \right)^{-1} h_k} \]

- generalization: adaptive receive strategy \( z_k \)

\[ I_k(p, \sigma_n^2) = \min_{z_k \in \mathbb{Z}_k} \left( p^T v(z_k) + \sigma_n^2 n_k(z_k) \right), \quad k = 1, 2, \ldots, K \]

Interference \( v(z_k) \)

Noise \( \sigma_n^2 n_k(z_k) \)
Axiomatic Approach: Interference Functions

Definition

A function $\mathcal{I} : \mathbb{R}_+^K \rightarrow \mathbb{R}_+$ is said to be an interference function if the following axioms are fulfilled:

- **A1** (positivity) $\mathcal{I}(p) > 0$ if $p > 0$
- **A2** (scale invariance) $\mathcal{I}(\alpha p) = \alpha \mathcal{I}(p)$ $\forall \alpha \in \mathbb{R}_+$
- **A3** (monotonicity) $\mathcal{I}(p) \geq \mathcal{I}(p')$ if $p \geq p'$
Another Example: Robust Nullsteering Beamforming

interference can be reduced by nullsteering beamforming:

- assume that the interference direction is only known up to an uncertainty $c$ from a region $C$
- the beamformer $\mathbf{u}$ minimizes the worst-case interference power:

$$
I(\mathbf{p}) = \min_{\|\mathbf{u}\|=1} \left( \max_{c \in C} \sum_l p_l |\mathbf{u}^H \mathbf{h}_l(c)|^2 \right)
$$

$\Rightarrow$ this is also an interference function (A1–A3 fulfilled)
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QoS Model for Wireless Systems

- signal-to-interference ratio

\[
\text{SIR}(p) = \frac{p_k}{I_k(p)}
\]

- the QoS is a strictly monotonic function of the SIR

\[
\text{QoS}(p) = \phi(\text{SIR}(p))
\]

examples:

- \(\phi(x) = x\) \quad \text{SIR}
- \(\phi(x) = \log(x)\) \quad \text{SIR in dB}
- \(\phi(x) = \frac{1}{1 + x}\) \quad \text{Min. Mean Squared Error (MMSE)}
- \(\phi(x) = x^{-\alpha}\) \quad \text{BER slope, diversity order } \alpha
- \(\phi(x) = \log(1 + x)\) \quad \text{capacity for Gaussian signals}
- \ldots
for multiuser systems, the transmission strategy is typically a tradeoff between efficiency and fairness requirements.
Fixed-Point Iteration

For standard interference functions it was shown [Yates’95]

If target SIR $\gamma = [\gamma_1, \ldots, \gamma_K]$ are feasible, i.e., $C(\gamma) \leq 1$, under a sum-power constraint, then for an arbitrary initialization $p^{(0)} \geq 0$, the iteration

$$p_k^{(n+1)} = \gamma_k \cdot I_k(p^{(n)}), \quad k = 1, 2, \ldots, K$$

converges to the optimum of the power minimization problem

$$\inf_{p \geq 0} \sum_{k=1}^{K} p_k \quad \text{s.t.} \quad \frac{p_k}{I_k(p)} \geq \gamma_k, \quad \forall k,$$
The fixed-point iteration has the following properties:

- component-wise monotonicity
- optimum achieved iff
  \[ p_k^{(n+1)} = \gamma_k I_k(p^{(n)}), \forall k \]
- optimizer \( \lim_{n \to \infty} p^{(n)} \) is unique

convergence to the optimal power levels

\[ C(\gamma) = 0.81 \]
consider concave interference functions

\[ I_k(p) = \min_{z_k \in \mathcal{Z}_k} \left( p^T v(z_k) + n_k(z_k) \right), \quad k = 1, 2, \ldots, K \]

Interference Noise

the parameter \( z_k \) can be interpreted as a receive strategy

for \( K \) users, we have an interference coupling matrix

\[ V(z) = [v_1(z_1), \ldots, v_K(z_K)]^T \]
By exploiting the special structure of concave interference functions, a new iteration is obtained:

Alternating optimization of receive strategies $z^{(n)}$ and power allocation $p^{(n)}$

1. $z_k^{(n)} = \arg \min_{z_k \in Z_k} \left[ V(z)p^{(n)} + n(z) \right]_k$, $k \in \{1, 2, \ldots, K\}$
2. $p^{(n+1)} = (I - \Gamma V(z^{(n)}))^{-1} \cdot \Gamma N(z^{(n)})$
Convergence Behavior

![Graph showing convergence behavior with improved algorithm that exploits concavity fixed-point iteration](image)

- Improved algorithm that exploits concavity fixed-point iteration
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Cooperative Bargaining

*K* players try to reach an unanimous agreement on utilities

\[ u = [u_1, \ldots, u_k] \]

- the utility region \( \mathcal{U} \subset \mathbb{R}^K_+ \)
  is convex, comprehensive, closed, bounded
- depending on the chosen strategy, the solution outcome \( \varphi \) results
- if the bargaining fails, the disagreement outcome \( d \) results
Standard Properties of Utility Sets

- downward-comprehensive

  for all $u \in U$ and $u' \in \mathbb{R}^K_{++}$

  $u' \leq u \implies u' \in U$

- closed (contains its boundary)
- convex
- upper-bounded
Theorem

Every compact comprehensive utility set from $\mathbb{R}^K_{++}$ can be expressed as a sub-level set

$$\mathcal{U} = \{ u \in \mathbb{R}^K_{++} : C(u) \leq 1 \}$$

depending on an interference function $C(u)$.

The sub-level set $\mathcal{U}$ is convex if and only if $C(u)$ is a convex interference function.
Example: The SIR Feasible Set

- example: $K$ interference functions $I_1,\ldots,I_K$ and weighting factors $\gamma = [\gamma_1,\ldots,\gamma_K]$ (e.g. SIR requirements). The optimum of the weighted SIR balancing problem is

$$C(\gamma) = \inf_{p > 0} \left( \max_{1 \leq k \leq K} \frac{\gamma_k I_k(p)}{p_k} \right)$$

- SIR feasible region

$$S = \{ \gamma : C(\gamma) \leq 1 \}$$

- $C(\gamma)$ satisfies A1–A3
- every SIR region is a level set of an interference function
WPO  Weak Pareto Optimality. The players should not be able to collectively improve upon the solution outcome.

IIA  Independence of Irrelevant Alternatives. If the feasible set shrinks but the solution outcome remains feasible, then the outcome is also the solution of the smaller set.

SYM  Symmetry: If the region is symmetric, then the outcome does not depend on the identities of the users.

STC  Scale Transformation Covariance. The outcome is component-wise scale-invariant.
For convex comprehensive sets the unique Nash bargaining solution fulfilling the axioms WPO, IIA, SYM, STC is obtained as the solution of

\[
\max_{\{u \in U : u > d\}} \prod_{k=1}^{K} (u_k - d_k)
\]

Often, the choice of the zero of the utility scales does not matter, so we can choose \(d = 0\)
the product optimization approach is equivalent to proportional fairness [Kelly'98]

$$\hat{u} = \arg \max_{u \in \mathcal{U}} \prod_{k=1}^{K} u_k = \arg \max_{u \in \mathcal{U}} \log \prod_{k=1}^{K} u_k = \arg \max_{u \in \mathcal{U}} \sum_{k=1}^{K} \log u_k$$

if the region $\mathcal{U}$ is convex closed comprehensive and bounded, then symmetric Nash bargaining and proportional fairness are equivalent
Bargaining over SIR Feasible Sets

- for wireless systems, an important performance measure is the signal-to-interference ratio

\[ \text{SIR}_k(p) = \frac{p_k}{I_k(p)} \]

← useful power
← interference (+noise) power

- indicator of feasibility: \( C(\gamma) = \inf_{p > 0} \left( \max_k \frac{\gamma_k I_k(p)}{p_k} \right) \)

- the SIR region

\[ S = \{ \gamma \in \mathbb{R}_+^K : C(\gamma) \leq 1 \} \]

is generally not convex, so results from classical bargaining theory cannot be applied directly

\[ \gamma_2 \]

\[ \gamma_1 \]

\[ \text{feasible} \quad C(\gamma) \leq 1 \]

\[ \text{infeasible} \quad C(\gamma) > 1 \]
if the underlying interference functions are log-convex, then
the SIR region is log-convex

SIR region has special properties which can be exploited for
bargaining (closed, comprehensive, log-convex)
If the region $\mathcal{U}$ is strictly convex after a log-transformation ("log-convex"), then the Nash axioms WPO, IIA, SYM, STC characterize a single-valued solution outcome

$$\varphi(\mathcal{U}) = \arg \max_{\mathbf{u} \in \mathcal{U}} \sum_{k=1}^{K} \log u_k.$$
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Theorem

Let $\mathcal{I}$ be an arbitrary interference function, then

$$\mathcal{I}(\mathbf{p}) = \min_{\hat{\mathbf{p}} \in \mathcal{L}(\mathcal{I})} \max_k \frac{p_k}{\hat{p}_k} = \max_{\hat{\mathbf{p}} \in \mathcal{L}(\mathcal{I})} \min_k \frac{p_k}{\hat{p}_k}$$

- $\mathcal{I}(\mathbf{p})$ can always be represented as the optimum of a weighted max-min (or min-max) optimization problem
- The weights $\hat{\mathbf{p}}$ are elements of convex/concave level sets

$$\mathcal{L}(\mathcal{I}) = \{ \hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \leq 1 \}$$

$$\overline{\mathcal{L}}(\mathcal{I}) = \{ \hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \geq 1 \}$$
the set $L(I)$ is closed bounded and monotonic decreasing
\[ \hat{p} \leq \hat{p}', \quad \hat{p}' \in L(I) \implies \hat{p} \in L(I) \]

the set $\overline{L}(I)$ is closed and monotonic increasing
\[ \hat{p} \geq \hat{p}', \quad \hat{p}' \in \overline{L}(I) \implies \hat{p} \in \overline{L}(I) \]

every interference function can be interpreted as an optimum of a utility/cost resource allocation problem
Concave Interference Functions

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>We say that $\mathcal{I} : \mathbb{R}<em>+^K \mapsto \mathbb{R}</em>+$ is a <strong>concave</strong> interference function if it fulfills the axioms:</td>
</tr>
</tbody>
</table>

| A1  | (non-negativeness) | $\mathcal{I}(p) \geq 0$ |
| A2  | (scale invariance) | $\mathcal{I}(\alpha p) = \alpha \mathcal{I}(p)$ $\forall \alpha \in \mathbb{R}_+$ |
| A3  | (monotonicity)     | $\mathcal{I}(p) \geq \mathcal{I}(p')$ if $p \geq p'$ |
| C1  | (concavity)        | $\mathcal{I}(p)$ is concave on $\mathbb{R}_+^K$ |
Examples for Concave Interference Functions

- **beamforming:**
  \[
  I_k(p) = \frac{1}{h_k^H \left( \sigma_n^2 I + \sum_{l \neq k} p_l h_l h_l^H \right)^{-1} h_k}
  \]

- **generalization:** receive strategy \( z_k \)
  \[
  I_k(p, \sigma_n^2) = \min_{z_k \in \mathbb{Z}_k} \left( p^T v(z_k) + \sigma_n^2 n_k(z_k) \right), \quad k = 1, 2, \ldots, K
  \]
  \( \text{Interference} \quad \text{Noise} \)
Representation of Concave Interference Functions

Theorem

Let \( I(p) \) be an arbitrary concave interference function, then

\[
I(p) = \min_{w \in \mathcal{N}_0(I)} \sum_{k=1}^{K} w_k p_k, \quad \text{for all } p > 0.
\]

where

\[
\mathcal{N}_0(I) = \{w \in \mathbb{R}_+^K : I^*(w) = 0\}
\]

and \( I^*(w) = \inf_{p > 0} \left( \sum_{l=1}^{K} w_l p_l - I(p) \right) \) is the conjugate of \( I \).
Interpretation of Concave Interference Functions

\[ I(p) = \min_{w \in N_0(I)} \sum_{k=1}^{K} w_k p_k \]

- the set \( N_0(I) \) is closed, convex, and upward-comprehensive
- any concave interference function can be interpreted as the solution of a loss/cost minimization problem
Convex Interference Functions

**Definition**

We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is a **convex** interference function if it fulfills the axioms:

- **A1** (non-negativeness) $\mathcal{I}(p) \geq 0$
- **A2** (scale invariance) $\mathcal{I}(\alpha p) = \alpha \mathcal{I}(p)$ $\forall \alpha \in \mathbb{R}_+$
- **A3** (monotonicity) $\mathcal{I}(p) \geq \mathcal{I}(p')$ if $p \geq p'$
- **C2** (convexity) $\mathcal{I}(p)$ is convex on $\mathbb{R}_+^K$
Example: Robustness

- An example is the worst-case model

\[ \mathcal{I}_k(p) = \max_{c_k \in C_k} p^T v(c_k), \quad \forall k, \]

where the parameter \( c_k \) models an ‘uncertainty’ (e.g. caused by channel estimation errors or system imperfections).

- the optimization is over a compact uncertainty region \( C_k \)

- \( \mathcal{I}_k(p) \) is a convex interference function
Representation of Convex Interference Functions

**Theorem**

Let $\mathcal{I}(\mathbf{p})$ be an arbitrary convex interference function, then

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k , \quad \text{for all } \mathbf{p} > 0.$$  

where

$$\mathcal{W}_0(\mathcal{I}) = \{ \mathbf{w} \in \mathbb{R}_+^K : \bar{\mathcal{I}}^*(\mathbf{w}) = 0 \}$$

and $\bar{\mathcal{I}}^*(\mathbf{w}) = \sup_{\mathbf{p} > 0} \left( \sum_{l=1}^{K} w_l p_l - \mathcal{I}(\mathbf{p}) \right)$ is the conjugate of $\mathcal{I}$. 
Interpretation of Convex Interference Functions

\[ I(p) = \max_{w \in \mathcal{W}_0(I)} \sum_{k=1}^{K} w_k \cdot p_k \]

- the set \( \mathcal{W}_0(I) \) is closed, convex, and downward-comprehensive
- any convex interference function can be interpreted as the solution of a utility maximization problem
Definition

We say that \( \mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+ \) is a log-convex interference function if it fulfills the axioms:

A1 (non-negativeness) \( \mathcal{I}(p) \geq 0 \)

A2 (scale invariance) \( \mathcal{I}(\alpha p) = \alpha \mathcal{I}(p) \quad \forall \alpha \in \mathbb{R}_+ \)

A3 (monotonicity) \( \mathcal{I}(p) \geq \mathcal{I}(p') \) if \( p \geq p' \)

C3 (log-convexity) \( \mathcal{I}_k(e^s) \) is log-convex on \( \mathbb{R}^K \)
Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be log-convex interference functions, then the SIR-balancing optimum

$$C(\gamma) = \inf_{p > 0} \left( \max_{1 \leq k \leq K} \frac{\gamma_k I_k(p)}{p_k} \right)$$

is a log-convex interference function, i.e., $C(\exp q)$ is a log-convex (thus convex) function.

- the SIR feasible set $S = \{\gamma : C(\gamma) \leq 1\}$ is convex on a logarithmic scale
- this “hidden convexity” can be exploited for designing resource allocation algorithms
**Theorem**

Every log-convex interference function $\mathcal{I}(\mathbf{p})$, with $\mathbf{p} > 0$, can be represented as

$$
\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{L}(\mathcal{I})} \left( f_{\mathcal{I}}(\mathbf{w}) \cdot \prod_{l=1}^{K} (p_l)^{w_l} \right).
$$

where

$$
f_{\mathcal{I}}(\mathbf{w}) = \inf_{\mathbf{p} > 0} \frac{\mathcal{I}(\mathbf{p})}{\prod_{l=1}^{K} (p_l)^{w_l}}, \quad \mathbf{w} \in \mathbb{R}_+^K, \quad \sum_k w_k = 1
$$

$$
\mathcal{L}(\mathcal{I}) = \{ \mathbf{w} \in \mathbb{R}_+^K : f_{\mathcal{I}}(\mathbf{w}) > 0 \}.
$$
every convex function $I(p)$ can be expressed as

$$I(p) = \max_{w \in \mathcal{W}_0} \sum_k w_k p_k$$

$\log \sum_k w_k e^{s_k}$ is convex

$\Rightarrow \log \max_{w \in \mathcal{W}_0} \sum_k w_k e^{s_k}$ is convex

$\Rightarrow I(e^s)$ is log-convex

if $I(p)$ is convex then $I(e^s)$ is log-convex

(but the converse is not true)
Categories of Interference Functions

- General interference functions
- Log-convex interference functions
- Convex interference functions
- Concave interference functions

Diagram:

```
  general interference functions
     □ convex interference functions □ concave interference functions
        □ log-convex interference functions □
```
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Conclusions

- the framework of interference functions is applicable to different areas in wireless communications:
  - physical layer design
  - medium access control
  - resource allocation and utility optimization
- results provide intuition about the behavior of coupled multiuser systems
- useful for characterizing operating points of the system, design of algorithms
- many interesting open questions