Rare event simulation: a Point Process interpretation with application in probability and quantile estimation

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In the context of reliability analysis, complex models representing physical experiments are used to estimate the probability of a failure domain defined by a given threshold not to be overpassed by the model output when the input is random. These models are often very time consuming and no analytical expression is available. Thus the evaluation of a probability of failure given a threshold or the estimation of a quantile given a targeted probability cannot be done by usual computational tools, especially when the failure event is *rare*.

We introduce here a new approach to this problem through a point process framework. Indeed for every real random variable with continuous *cdf* (that can be the output of a complex numerical code with random input), we show that one can build a sequence of simulations whose distribution is related to a Poisson Process with parameter 1. More precisely, given X a random vector (modelling the input with known distribution), ϕ a real valued function (giving the scalar output of the numerical model), q a threshold defining the event $F = \{\phi(X) > q\}$ and p its probability, the number of samples (*ie.* simulations) required to get a realisation of $\phi(X)$ in F follows a Poisson law with parameter $\log(1/p)$; this is to be compared with the usual geometric law with parameter p obtained with naive Monte-Carlo. Then simulating several processes in parallel allows us to derive an estimator for a probability of exceeding a threshold or a quantile [Walter, 2014].

The probability estimator is unbiased, its distribution can be characterised and its coefficient of variation $\delta^2 = -\log p/n + o(1/n)$ is always lower than the one of a naive Monte-Carlo $\delta^2_{\text{MC}} = (1-p)/(np)$. Especially when $p \ll 1$, $\delta^2_{\text{MC}} \approx 1/(np)$ and therefore our approach "add a $\log$" to the $1/p$ coefficient. It is also related to well-known Sequential Monte-Carlo methods or Adaptive Multilevel Splitting algorithms [Cérou et al., 2009, Cérou et al., 2012, Guyader et al., 2011], also called *Subset Simulation* [Au and Beck, 2001]. We then show that our approach brings the optimal algorithm expected from these methods regarding the computational time required to achieve a given precision.

Like with a naive Monte-Carlo the quantile estimator satisfies a Central Limit Theorem and its bias is of order $1/n$, with the same "$\log$ attribute" as for the probability estimator. It is a very efficient parallel alternative to naive Monte-Carlo and a massive gain in quantile estimation can be achieved.

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