Mismatched Decoding

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ISIT 2016 BARCELONA

Call for papers

The 2016 IEEE International Symposium on Information Theory will take place in Barcelona, Spain, from July 10 to 15, 2016. A lively city, known for its style, architecture, culture, gastronomy and nightlife, Barcelona is one of the top tourist destinations in Europe. Interested authors are encouraged to submit previously unpublished contributions from a broad range of topics related to information theory, including but not limited to the following areas:

Topics

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Coding Theory
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Complexity and Computation Theory
Compressed Sensing and Sparsity
Cryptography and Security
Detection and Estimation
Emerging Applications of IT
Information Theory and Statistics
Information Theory in Biology
Network Coding and Applications
Network Information Theory
Pattern Recognition and Learning
Physical Layer Security
Quantum Information and Coding Theory
Sequences
Shannon Theory
Signal Processing
Source Coding and Data Compression
Wireless Communication and Networks

Researchers working in emerging fields of information theory or on novel applications of information theory are especially encouraged to submit original findings.

The submitted work and the published version are limited to 5 pages in the standard IEEE conference format. Submitted papers should be of sufficient detail to allow for review by experts in the field. Authors should refrain from submitting multiple papers on the same topic.

Information about when and where papers can be submitted will be posted on the conference web page. The paper submission deadline is January 24, 2016, at 11:59 PM, Eastern Time (New York, USA). Acceptance notifications will be sent out by April 3, 2016.

We look forward to your participation in ISIT in the centennial year of Claude Shannon's birth.

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Joint work with

- Jonathan Scarlett (EPFL)
- Alfonso Martinez (UPF)

Outline

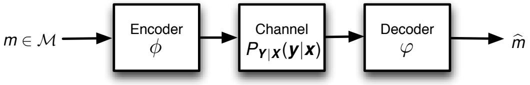
Setup

Rates

Error Exponents

Randomized Decoding

Setup

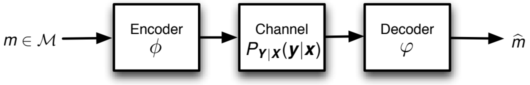


- Code $\mathcal{C}(n, |\mathcal{M}|) = \{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(|\mathcal{M}|)}\}$
  - codewords $\boldsymbol{x}^{(i)} \in \mathcal{X}^n$
  - rate $R = \frac{1}{n} \log |\mathcal{M}|$
- Discrete memoryless channel

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(i)}) = \prod_{k=1}^{n} P_{Y|X}(y_k|x_k^{(i)})$$

Setup Rates Error Exponents Randomized Decoding

Setup



- Error probability

$$P_e(n, |\mathcal{M}|) = \mathbb{P}\{\hat{M} \neq M\}$$

- Maximum-Likelihood decoding

$$\hat{m} = \arg\max_{i=1,\ldots,|\mathcal{M}|} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(i)}) = \arg\max_{i=1,\ldots,|\mathcal{M}|} \prod_{k=1}^{n} P_{Y|X}(y_k|x_k^{(i)})$$

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Setup

- Capacity

$$C = \max_{P_X} I(X;Y)$$

- Random-coding error exponent for distribution Q

$$E_r(R,Q) = \max_{\rho\in[0,1]} E_0(\rho,Q) - \rho R$$

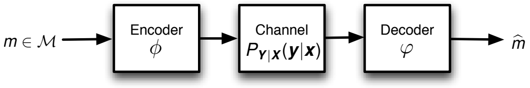
$$E_0(\rho,Q) = -\log \sum_{x,y} Q(x) P_{Y|X}(y|x) \left(\frac{\sum_{\bar{x}} Q(\bar{x}) P_{Y|X}(y|\bar{x})^{\frac{1}{1+\rho}}}{P_{Y|X}(y|x)^{\frac{1}{1+\rho}}} \right)^{\rho}$$

C. Shannon, "A mathematical theory of communication," Bell Syst. Tech. Journal, vol. 27, pp. 379–423, July and Oct. 1948.

R. G. Gallager, *Information Theory and Reliable Communication*, John Wiley & Sons, Inc. New York, NY, USA, 1968.

Setup Rates Error Exponents Randomized Decoding

Setup



- Maximum-Likelihood decoding

$$\hat{m} = \arg\max_{i=1,\dots,|\mathcal{M}|} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(i)}) = \arg\max_{i=1,\dots,|\mathcal{M}|} \prod_{k=1}^{n} P_{Y|X}(y_k|x_k^{(i)})$$

- Maximum-Metric decoding

$$\hat{m} = \arg\max_{i=1,\dots,|\mathcal{M}|} q(\boldsymbol{x}^{(i)}, \boldsymbol{y}) = \arg\max_{i=1,\dots,|\mathcal{M}|} \prod_{k=1}^{n} q(x_k^{(i)}, y_k)$$

Motivation and Examples

- Zero error
- Zero undetected error
- Channel uncertainty $q(x, y) = \widehat{P}_{Y|X}(y|x)$
- Practical constraints
  - nearest-neighbor (non AWGN)
  - metric quantization
- Bitwise decoding

$$q(x, y) = \prod_{j=1}^{m} q_j(b_j(x), y)$$

Rates

$$C_q = ?$$

Rates

i.i.d. Random Coding

$$\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}] = \prod_{k=1}^{n} Q(x_k)$$

- Mismatched decoder

$$\widehat{m} = \arg\max_{i=1,\ldots,|\mathcal{M}|} q(\boldsymbol{x}^{(i)}, \boldsymbol{y})$$

- Can achieve rate

$$\sum_{x,y} Q(x) P_{Y|X}(y|x) \log \frac{q(x,y)}{\sum_{\bar{x}} Q(\bar{x}) q(\bar{x}, y)}$$

$$I(X;Y) = \sum_{x,y} Q(x) P_{Y|X}(y|x) \log \frac{P_{Y|X}(y|x)}{\sum_{\bar{x}} Q(\bar{x}) P_{Y|X}(y|\bar{x})}$$

T. R. M. Fischer, “Some remarks on the role of inaccuracy in Shannon’s theory of information transmission," in Trans. 8th Prague Conf. on Inf. Theory, 1971, pp. 211–226.

Rates

i.i.d. Random Coding

- Mismatched decoder

$$\widehat{m} = \arg\max_{i=1,\ldots,|\mathcal{M}|} q(\boldsymbol{x}^{(i)}, \boldsymbol{y}) = \arg\max_{i=1,\ldots,|\mathcal{M}|} q(\boldsymbol{x}^{(i)}, \boldsymbol{y})^s$$

- Generalized Mutual Information

$$I^{\text{GMI}}(Q) = \sup_{s \geq 0} \sum_{x,y} Q(x) P_{Y|X}(y|x) \log \frac{q(x,y)^s}{\sum_{\bar{x}} Q(\bar{x}) q(\bar{x}, y)^s}$$

G. Kaplan and S. Shamai, “Information rates and error exponents of compound channels with application to antipodal signaling in a fading environment," Arch. Elek. Über., vol. 47, no. 4, pp. 228–239, 1993.

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Rates

Constant Composition Random Coding

- Codewords have the same empirical distribution

$$p_{\boldsymbol{x}}(x) = \frac{\text{Number of symbols } x \text{ in } \boldsymbol{x}}{n}$$

- If metric $q(x, y)$ is replaced by $q(x, y)^s e^{a(x)}$

$$\prod_{k=1}^{n} q(x_k, y_k)^s e^{a(x_k)} = \left(\prod_{k=1}^{n} q(x_k, y_k)\right)^s e^{\sum_{k=1}^{n} a(x_k)}$$

- LM rate

$$I_{\text{LM}}(Q) = \sup_{s \geq 0, a(\cdot)} \sum_{x,y} Q(x) P_{Y|X}(y|x) \log \frac{q(x, y)^s e^{a(x)}}{\sum_{\bar{x}} Q(\bar{x}) q(\bar{x}, y)^s e^{a(\bar{x})}}$$

J. Hui, "Fundamental issues of multiple accessing," Ph.D. dissertation, MIT, 1983.

I. Csiszár and J. Körner, "Graph decomposition: A new key to coding theorems," IEEE Trans. Inf. Theory, Jan. 1981.

Rates

Cost Constrained Random Coding

- Codewords meet a cost function

$$\mathbb{P}\{\boldsymbol{X} = \boldsymbol{x}\} = \frac{1}{\mu_n} \prod_{k=1}^{n} Q(x_k) \mathbb{1}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} a(x_k) - \mathbb{E}_Q[a(X)]\right| \leq \frac{\delta}{n}\right\}$$

- LM rate is also achieved

$$I_{\text{LM}}(Q) = \sup_{s \geq 0, a(\cdot)} \sum_{x,y} Q(x) P_{Y|X}(y|x) \log \frac{q(x, y)^s e^{a(x)}}{\sum_{\bar{x}} Q(\bar{x}) q(\bar{x}, y)^s e^{a(\bar{x})}}$$

A. Ganti, A. Lapidoth, and E. Telatar, "Mismatched decoding revisited: General alphabets, channels with memory, and the wide-band limit," IEEE Trans. Inf. Theory, Nov. 2000.

S. Shamai and I. Sason, "Variations on the Gallager bounds, connections, and applications," IEEE Trans. Inf. Theory, Dec. 2002

Rates

Properties

- Different ensembles achieve different rates
- Ensemble tightness: $\overline{P}_e \to 1$ when
  - $R > I_{\text{GMI}}(Q)$ (i.i.d. random coding)
  - $R > I_{\text{LM}}(Q)$ (constant-composition random coding)
  - $R > I_{\text{LM}}(Q)$ (cost-constrained random coding)
- $\max_{Q(x)} I_{\text{LM}}(Q)$ is positive iff mismatched capacity is positive
- Data Processing Inequality: $I_{\text{LM}}(Q) \le I(X;Y)$ with equality iff

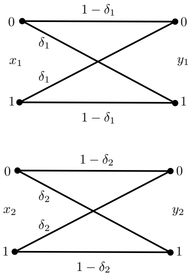
$$\log q(x,y) = \alpha(x) + \beta(y) + c \log P_{Y|X}(y|x)$$

  for some $\alpha(x)$, $\beta(y)$, $c > 0$
- $I_{\text{LM}}(Q)$ can be *non-convex* in Q

N. Merhav, G. Kaplan, A. Lapidoth, and S. Shamai, "On information rates for mismatched decoders," IEEE Trans. Inf. Theory, Nov. 1994.

Rates

Example



decoder assumes δ

- Capacity

$$C = \big(1 - H_2(\delta_1)\big) + \big(1 - H_2(\delta_2)\big)$$

- Random coding with $Q(x) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

$$I_{\text{LM}}(Q) = 2\left(1 - H_2\left(\frac{\delta_1+\delta_2}{2}\right)\right)$$

- Individual random coding on each BSC

$$C = \big(1 - H_2(\delta_1)\big) + \big(1 - H_2(\delta_2)\big)$$

A. Lapidoth, "Mismatched decoding and the multiple-access channel," IEEE Trans. Inf. Theory, vol. 42, no. 5, pp. 1439–1452, Sep. 1996.

Rates

Improvements

- Parallel codebooks (multiple-access channel coding)
- Superposition codebooks (broadcast channel coding)
- Applying LM rate to

$$P^{(2)}_{Y|X}\big((y_1,y_2)|(x_1,x_2)\big) = P_{Y|X}(y_1|x_1)P_{Y|X}(y_2|x_2)$$
$$q^{(2)}\big((x_1,x_2),(y_1,x_2)\big) = q(x_1,y_1)q(x_2,y_2)$$

- Converse?

A. Lapidoth, “Mismatched decoding and the multiple-access channel," IEEE Trans. Inf. Theory, vol. 42, no. 5, pp. 1439–1452, Sep. 1996

A. Somekh-Baruch, “On Achievable Rates and Error Exponents for Channels With Mismatched Decoding", IEEE Trans. Inf. Theory, vol. 61, no. 2, pp. 727–740, Feb. 2015

J. Scarlett, A. Martinez and A. Guillén i Fàbregas, “Multiuser Coding Techniques for Mismatched Decoding," submitted IEEE Trans. Inf. Theory., arXiv:1311.6635

I. Csiszár and P. Narayan, “Channel capacity for a given decoding metric," IEEE Trans. Inf. Theory, vol. 45, no. 1, pp. 35–43, Jan. 1995

A. Somekh-Baruch, “A General Formula for Mismatched Capacity", IEEE Trans. Inf. Theory, vol. 61, no. 9, Sept. 2015

Error Exponents

Definition (Exponent ensemble tightness)

We say that an error exponent $E(R)$ is ensemble tight if

$$\liminf_{n\to\infty} -\frac{1}{n}\log \overline{P}_e(n, e^{nR}) = E(R).$$

Error Exponents

- Upper bound

$$\overline{P}_e \leq \mathbb{E}\left[\min\left\{1, (|\mathcal{M}|-1)\Pr\{q(\boldsymbol{X}', \boldsymbol{Y}) \geq q(\boldsymbol{X}, \boldsymbol{Y}) | \boldsymbol{X}, \boldsymbol{Y}\}\right\}\right]$$

- Lower bound

$$\overline{P}_e \geq \frac{1}{4}\mathbb{E}\left[\min\left\{1, (|\mathcal{M}|-1)\Pr\{q(\boldsymbol{X}', \boldsymbol{Y}) \geq q(\boldsymbol{X}, \boldsymbol{Y}) | \boldsymbol{X}, \boldsymbol{Y}\}\right\}\right]$$

- Ensemble tight exponent given by

$$\liminf_{n\to\infty} -\frac{1}{n}\log \mathbb{E}\left[\min\left\{1, (|\mathcal{M}|-1)\Pr\{q(\boldsymbol{X}', \boldsymbol{Y}) \geq q(\boldsymbol{X}, \boldsymbol{Y}) | \boldsymbol{X}, \boldsymbol{Y}\}\right\}\right]$$

- Method of types can be used to find exponent (primal form)

J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, “Mismatched Decoding: Error Exponents, Second-Order Rates and Saddlepoint Approximations”, IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2647-2666, May 2014.

I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems, 2nd ed. Cambridge University Press, 2011.

R. Gallager, “Fixed composition arguments and lower bounds to error probability," http://web.mit.edu/gallager/www/notes/notes5.pdf.

Error Exponents

Code Ensembles

- i.i.d.

$$\mathbb{P}\{\boldsymbol{X} = \boldsymbol{x}\} = \prod_{k=1}^{n} Q(x_k)$$

- Constant composition

$$\mathbb{P}\{\boldsymbol{X} = \boldsymbol{x}\} = \begin{cases} \frac{1}{|T(Q)|} & \boldsymbol{x} \in T(Q) \\ 0 & \text{otherwise} \end{cases}$$

- Multiple cost constraints, $\phi_l = \mathbb{E}_Q[a_l(X)]$

$$\mathbb{P}\{\boldsymbol{X} = \boldsymbol{x}\} = \frac{1}{\mu_n}\prod_{k=1}^{n} Q(x_i)\mathbb{1}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} a_l(x_k) - \phi_l\right| \leq \frac{\delta}{n}, l = 1, \ldots, L\right\}$$

J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, “Mismatched Decoding: Error Exponents, Second-Order Rates and Saddlepoint Approximations”, IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2647-2666, May 2014.

Error Exponents

Ensemble Tightness

$$E_r(R,Q) = \max_{0\le\rho\le1} E_0(\rho,Q) - \rho R$$

- i.i.d.

$$E_0(\rho,Q) \triangleq \sup_{s\ge0} -\log\sum_{x,y} Q(x)P_{Y|X}(y|x)\left(\frac{\sum_{\overline{x}} Q(\overline{x})q(\overline{x},y)^s}{q(x,y)^s}\right)^\rho$$

- Constant composition

$$E_0(\rho,Q) \triangleq \sup_{s\ge0,a(\cdot)} -\sum_x Q(x)\log\sum_y P_{Y|X}(y|x)\left(\frac{\sum_{\overline{x}} Q(\overline{x})q(\overline{x},y)^s e^{a(\overline{x})}}{q(x,y)^s e^{a(x)}}\right)^\rho$$

- Multiple cost constraints

$$E_0(\rho,Q,\{a_l(\cdot)\})$$

$$\triangleq \sup_{\substack{s\ge0\\ \{r_l\},\{\overline{r}_l\}}} -\log\sum_{x,y} Q(x)P_{Y|X}(y|x)\left(\frac{\sum_{\overline{x}} Q(\overline{x})q(\overline{x},y)^s e^{\sum_{l=1}^L \overline{r}_l(a_l(\overline{x})-\phi_l)}}{q(x,y)^s e^{\sum_{l=1}^L r_l(a_l(x)-\phi_l)}}\right)^\rho$$

J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, “Mismatched Decoding: Error Exponents, Second-Order Rates and Saddlepoint Approximations”, IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2647-2666, May 2014.

Error Exponents

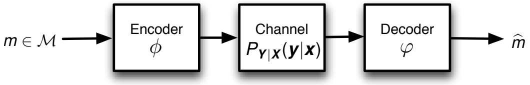
Proposition

$$E_r^{\text{iid}}(R,Q) \le E_r^{\text{cost}}(R,Q,\{a_l\}) \le E_r^{\text{cc}}(R,Q)$$

$$\sup_{a_1(\cdot),a_2(\cdot)} E_r^{\text{cost}}(R,Q,\{a_1,a_2\}) = E_r^{\text{cc}}(R,Q)$$

J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, “Mismatched Decoding: Error Exponents, Second-Order Rates and Saddlepoint Approximations”, IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2647-2666, May 2014.

Randomized Decoding



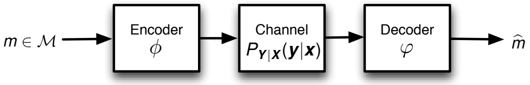
- Maximum-Likelihood decoding

$$\widehat{m} = \arg\max_{i=1,\dots,|\mathcal{M}|} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(i)})$$

- Randomized Likelihood decoding

$$\mathbb{P}[\hat{M} = \hat{m} | \boldsymbol{Y} = \boldsymbol{y}] = \frac{P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(\hat{m})})}{\sum_{m=1}^{M} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}^{(m)})}$$

Randomized Decoding



- Maximum-metric decoding

$$\widehat{m} = \arg\max_{i=1,\dots,|\mathcal{M}|} q(\boldsymbol{x}^{(i)}, \boldsymbol{y})$$

- Randomized mismatched decoding

$$\mathbb{P}[\hat{M} = \hat{m} | \boldsymbol{Y} = \boldsymbol{y}] = \frac{q(\boldsymbol{x}^{(\hat{m})}, \boldsymbol{y})}{\sum_{m=1}^{M} q(\boldsymbol{x}^{(m)}, \boldsymbol{y})}$$

Randomized Decoding

$$\mathbb{P}[\text{no error}] = \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \mathbb{E}\left[\frac{q^n(\boldsymbol{x},\boldsymbol{y})}{q^n(\boldsymbol{x},\boldsymbol{y}) + \sum_{\overline{m}\neq 1} q^n(\boldsymbol{y}|\boldsymbol{X}^{(\overline{m})})}\right]$$

$$= \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \mathbb{E}\left[\left(\frac{1}{\left(1 + \sum_{\overline{m}\neq 1} \frac{q^n(\boldsymbol{X}^{(\overline{m})},\boldsymbol{y})}{q^n(\boldsymbol{x},\boldsymbol{y})}\right)^s}\right)^{\frac{1}{s}}\right]$$

$$\geq \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \left(\frac{1}{\mathbb{E}\left[\left(1 + \sum_{\overline{m}\neq 1} \frac{q^n(\boldsymbol{X}^{(\overline{m})},\boldsymbol{y})}{q^n(\boldsymbol{x},\boldsymbol{y})}\right)^s\right]}\right)^{\frac{1}{s}} \qquad \text{(Jensen)}$$

$$\geq \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \left(\frac{1}{1 + \sum_{\overline{m}\neq 1} \mathbb{E}\left[\left(\frac{q^n(\boldsymbol{X}^{(\overline{m})},\boldsymbol{y})}{q^n(\boldsymbol{x},\boldsymbol{y})}\right)^s\right]}\right)^{\frac{1}{s}} \qquad (\text{if } s \in [0,1])$$

$$= \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \left(\frac{1}{1 + (|\mathcal{M}|-1)\mathbb{E}\left[\left(\frac{q^n(\overline{\boldsymbol{X}},\boldsymbol{y})}{q^n(\boldsymbol{x},\boldsymbol{y})}\right)^s\right]}\right)^{\frac{1}{s}}$$

$$\geq 1 - \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \min\left\{1, \frac{1}{s}(|\mathcal{M}|-1)\frac{\mathbb{E}[q^n(\overline{\boldsymbol{X}},\boldsymbol{y})^s]}{q^n(\boldsymbol{x},\boldsymbol{y})^s}\right\}$$

Randomized Decoding

- Randomized mismatched decoder, $s \in [0,1]$

$$\overline{P}_e \leq \mathbb{E}\left[\min\left\{1, \frac{1}{s}(M-1)\frac{\mathbb{E}[q^n(\overline{\boldsymbol{X}},\boldsymbol{Y})^s \mid \boldsymbol{Y}]}{q^n(\boldsymbol{X},\boldsymbol{Y})^s}\right\}\right]$$

- Maximum metric mismatched decoder, $s \geq 0$

$$\overline{P}_e \leq \mathbb{E}\left[\min\left\{1, (M-1)\frac{\mathbb{E}[q^n(\overline{\boldsymbol{X}},\boldsymbol{Y})^s \mid \boldsymbol{Y}]}{q^n(\boldsymbol{X},\boldsymbol{Y})^s}\right\}\right]$$

Randomized Decoding

- Achievable rates

$$\tilde{I}_{\mathrm{LM}}(Q) = \sup_{s\in[0,1],a(x)} \sum_{x,y} Q(x)P_{Y|X}(y|x)\log\frac{q(x,y)^s e^{a(x)}}{\sum_{\overline{x}} Q(\overline{x})q(\overline{x},y)^s e^{a(\overline{x})}}$$

- Error exponent

$$\widetilde{E}_0(\rho,Q) \triangleq \sup_{s\in[0,1],a(\cdot)} -\sum_x Q(x)\log\sum_y P_{Y|X}(y|x)\left(\frac{\sum_{\overline{x}} Q(\overline{x})q(\overline{x},y)^s e^{a(\overline{x})}}{q(x,y)^s e^{a(x)}}\right)^{\rho}$$

- Ensemble tightness can be proved (primal domain)

J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "The Likelihood Decoder: Error Exponents and Mismatch", 2015 IEEE Int. Symp. Inf. Theory, Hong Kong, June 2015.

Randomized Decoding